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A Sufficient Condition on the Pressure for the Regularity of Weak Solutions to the Navier–Stokes Equations

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Abstract. It is well known that a weak solution (v,p) to the Navier–Stokes equations is regular if v satisfies some suitable extra conditions (see (1.2), (1.3)). However, with the exception of the recent papers [BV4], [BV5] (see also [K], [Be]) not so much attention has been payed to "alternative natural assumptions" that p may fulfill, in order that (v,p) be regular. By "alternative natural assumptions", we mean assumptions that formally follow from the Poisson equation relating pressure and velocity (see (1.4)). The objective of this paper is to prove that (v,p) is regular if $|p|/(1+|v|)$ obeys some conditions that are in formal agreement with this relation.

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1. Introduction and main results

This paper is concerned with the regularity of weak solutions, in the sense of Leray–Hopf [Se], to the Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \mu\Delta v + (v\cdot\nabla)v + \nabla p = f, & \\ \operatorname{div} v = 0 & \text{in } Q_T; \\ v = 0 & \text{on } \Sigma_T; \\ v(x,0) = v_0(x). & \end{cases} \tag{1.1}$$

Here Ω is a regular, open, bounded, connected subset of $\mathbb{R}^n$, $n \geq 3$, Γ is its boundary, $Q_t = \Omega \times [0,t]$, $\Sigma_t = \Gamma \times [0,t]$, and T is an arbitrary positive number.

It is well known (see, e.g. [S] and the references cited therein; see also [BV3] for a more elementary proof) that a weak solution in the sense of Leray–Hopf is regular if*

$$v \in L^r(0,T;L^q(\Omega)) \tag{1.2}$$

with (r,q) satisfying

$$\frac{2}{r} + \frac{n}{q} = 1, \quad q \in]n,+\infty]. \tag{1.3}$$

* We use standard notations for function spaces; see also the end of this section.

However, the well-known equation:

$$-\Delta p = \sum_{i,j=1}^{n} \partial_i \partial_j (v_i v_j), \tag{1.4}$$

relating p and v suggests that, at most, the inequality $|p| \lesssim |v|^2$ holds (but not the reverse inequality, as needed later on). In [BV4], we showed that if

$$\frac{p}{1+|v|} \in L^r(0,T;L^q(\Omega)), \tag{1.5}$$

with

$$\frac{2}{r} + \frac{n}{q} < 1, \tag{1.6}$$

then $v \in L^\infty(Q_T)$, implying regularity [S]. More specifically, in [BV4] we prove that the solution is regular if (1.5) holds just on a subset of Q_T where $|v(x,t)| > k$, for some (arbitrary large) k; see [BV4], eq. (1.6). This fact is of a certain interest, since *a priori*, according to (1.4) there is no relation between regions where $|p|$ is large, and regions where $|v|$ is large.

A question left open in [BV4], is to ascertain if regularity can be obtained under conditions (1.5), (1.3). The objective of this paper is to answer this question in the affermative. The method we shall use is completely different than that used in [BV4], and rests upon very simple L^α-estimates for regular solutions to (1.1). It is worth of noticing that relaxing (1.6) to (1.3) under condition (1.5) requires a different method than that usually adopted under condition (1.2). This because, in the latter case, regularity can be recovered from a suitable linearization of the term $v \cdot \nabla v$ in (1.1), as in [S] and [GaMa]; see also [LUS], Chapter III, Remark 7.3, with the coefficients b_i replaced by v_i. This procedure, of course, would not produce any result under the assumption (1.5).

Before stating our main theorem, we wish to mention the related results proved in [BV5]. Specifically, there we showed regularity results for v, under suitable assumptions on $|p|/(1+|v|)^\theta$, $\theta \in [0,1)$. These results are stated in the framework of Marcinckiewicz spaces $L^q_*(Q_T)$. As is known, such spaces are interpolation spaces satisfying $L^q \hookrightarrow L^q_* \hookrightarrow L^{q-\epsilon}$, for each $\epsilon > 0$. In the particular case $\theta = 0$, we proved that if $p \in L^\gamma_*(Q_T)$, for some $\gamma \in (2, N)$, then $v \in L^\mu_*(Q_T)$, $\mu = N\gamma/(N-\gamma)$. For $\gamma = N/2$, it follows that $\mu = N$. Note that, to present, $N = n+2$ is the smallest exponent known to guarantee the regularity of a weak solution $v \in L^N(Q_T)$.

For an extension of condition (1.2) to values of r belonging to $[1,2]$, we refer to [BV2]. For the case $r = \infty$, we refer to [BV3] and [KoS].

Let $H_\alpha(\Omega)$ be the completion in the $L^\alpha(\Omega)$-norm of the subset of of vector functions w from $C_0^\infty(\Omega)$ satisfying $\operatorname{div} w = 0$. Our main result reads as follows.

Theorem I. *Let be $\alpha > n$. Assume that $v_0 \in H_\alpha(\Omega)$ and that F is regular. Let (v,p) be a weak solution of problem* (1.1) *in Q_T such that* (1.5) *holds for some pair*

(r, q) *satisfying* (1.3). *Then*

$$v \in C(0,T; H_\alpha(\Omega)), \quad |v|^{\alpha/2} \in L^2(0,T; H_0^1(\Omega)). \tag{1.7}$$

Moreover, if $q < +\infty$*, the solution* v *satisfies the estimate given in* (2.14). *If* $q = +\infty$ *a similar estimate holds (see the end of Section 2).*

Remark 1. For simplicity, we assume f regular. However, Theorem I continues to hold if only $f \in L^1(0,T; L^\alpha(\Omega))$; see (2.14).

Remark 2. We could obtain estimates sharper than those proved in (2.2) and (2.14). However, this would produce no improvement on the main result.

Remark 3. In the light of results proved, *e.g.*, in [S], Theorem I ensures, in particular, that if $f \in C^\infty(\Omega \times (0,T])$ and Ω is of class C^∞, then $v, p \in C^\infty(\Omega \times (0,T])$.

The proof of Theorem I will be given in the next section.

We end this section by introducing some notation.* We denote by L^α, $1 \le \alpha \le \infty$, the usual Lebesgue spaces with corresponding norm $\| \; \|_\alpha$. The space $H_\alpha(\Omega)$ was already introduced before Theorem I. For a vector v defined in Ω with values in $\mathbb{R}^n$ we set

$$|\nabla v|^2 = \sum_{i,j=1}^n \left(\frac{\partial v_j}{\partial x_i} \right)^2 .$$

Sometimes, we shall write simply $\|\nabla v\|_\alpha$ instead of $\||\nabla v|\|_\alpha$. As customary, $H_0^1 = H_0^1(\Omega)$ denotes the space of functions defined and square-integrable in Ω together with their first derivatives, and vanishing at Γ. In the sequel we shall use the Sobolev inequality:

$$\|g\|_{2^*}^2 \le c_0 \|\nabla g\|_2^2, \quad \forall g \in H_0^1, \tag{1.8}$$

where $2^* = 2n/(n-2)$.

By $C(0,T;X)$, where X is a Banach space, we denote the functional space consisting of continuous functions on the closed interval $[0,T]$ with values in X. We also consider, with usual definitions and notations, spaces $L^r(0,T;X)$.

2. Proof of Theorem I

The following quantities play a leading role in the sequel (see also [BV1]; note that below N_α is the $1/\alpha$ power of the quantity N_α defined in [BV1], and similarly

* In general, we shall use the same symbol for spaces of scalar and vector functions.

for M_α).

$$\begin{aligned} N_\alpha(v) &= \left(\int_\Omega |\nabla v|^2 |v|^{\alpha-2} dx \right)^{1/\alpha}, \\ M_\alpha(v) &= \left(\int_\Omega |\nabla |v|^{\alpha/2}|^2 dx \right)^{1/\alpha}. \end{aligned} \tag{2.1}$$

Let us start by showing that in order to prove Theorem I it is sufficient to prove in $[0,T]$ the estimate

$$\begin{aligned} &\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \frac{\mu}{2} N_\alpha^\alpha(v) + 2\mu \frac{\alpha-2}{\alpha} M_\alpha^\alpha(v) \\ &\le c_1 + c_2 \|p/(1+|v|)\|_q^r (|\Omega| + \|v\|_\alpha^\alpha) + \|f\|_\alpha \|v\|_\alpha^{\alpha-1} \end{aligned} \tag{2.2}$$

for regular solutions (v,p) of problem (1.1). In fact, with this last result in hand we can argue as follows. Denote by t^* the lower upper bound of the set of values τ for which v satisfies (1.7) with T replaced by τ. By well known results t^* is strictly positive (see, for instance, [v.W], [M], [S], [G.M]), moreover the solution is regular in $]0,t^*[$. Hence our estimate (2.2) holds in this last interval. It follows from this estimate that v belongs to $L^\infty(0,t^*;H_\alpha)$. This implies that v belongs to $C(0,t^*;H_\alpha)$, since $\alpha > n$. Taking $v(t^*) \in H_\alpha$ as "initial data" one shows that, if it were $t^* < T$, it would exist a positive ϵ such that v would belong to $C(0,t^*+\epsilon;H_\alpha)$. Hence it must be $t^* = T$.

It is worth noting that if $n=3$ (and, more in particular, if also $v_0 \in H_0^1 \cap L^\alpha$) then the auxiliary results quoted above can be obtained by elementary techniques.

In the light of the above argument our aim is simply reduced to prove the estimate (2.2) for regular solutions.

Lemma 2.1. *Let (v,p) be a regular solution to problem* (1.1) *in $\Omega \times]0,T]$. Then*

$$\begin{aligned} &\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \frac{\mu}{2} N_\alpha^\alpha(v) + 4\mu \frac{\alpha-2}{\alpha} M_\alpha^\alpha(v) \\ &\le \frac{(\alpha-2)^2}{2\mu} \int_\Omega p^2 |v|^{\alpha-2} dx + \|f\|_\alpha \|v\|_\alpha^{\alpha-1}. \end{aligned} \tag{2.3}$$

Proof. Note, first, that

$$|\nabla |v|^{\alpha/2}| \le \frac{\alpha}{2} |v|^{\frac{\alpha}{2}-1} |\nabla v|, \quad \text{a.e. in } \Omega. \tag{2.4}$$

In order to prove (2.3) we multiply both sides of equation $(1.1)_1$ by $|v|^{\alpha-2}v$, and integrate over Ω. After suitable integration by parts, and by taking into account that $\operatorname{div} v = 0$, we get

$$\begin{aligned} &\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \mu N_\alpha^\alpha(v) + 4\mu \frac{\alpha-2}{\alpha^2} M_\alpha^\alpha(v) \\ &= -\int_\Omega \nabla p \cdot v|v|^{\alpha-2} dx + \int_\Omega f \cdot v |v|^{\alpha-2} dx. \end{aligned} \tag{2.5}$$

On the other hand, one has

$$\begin{aligned} &-\int_\Omega \nabla p \cdot v|v|^{\alpha-2}dx \\ &= (\alpha-2)\sum_{i,j=1}^n \int_\Omega p\frac{\partial v_j}{\partial x_i} v_i v_j |v|^{\alpha-4}dx \\ &= \frac{2(\alpha-2)}{\alpha}\int_\Omega p|v|^{\alpha/2-2}\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}(|v|^{\alpha/2})dx. \end{aligned} \tag{2.6}$$

From (2.5) and $(2.6)_1$, since

$$\left|\sum_{i,j=1}^n v_i v_j \frac{\partial v_j}{\partial x_i}\right| \leq |v|^2|\nabla v|, \tag*{2.7}$$

one gets

$$\begin{aligned} &\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \mu N_\alpha^\alpha(v) + 4\mu\frac{\alpha-2}{\alpha^2}M_\alpha^\alpha(v) \\ &\leq (\alpha-2)\int_\Omega |p|\,|\nabla v|\,|v|^{\alpha-2}dx + \|f\|_\alpha \|v\|_\alpha^{\alpha-1}. \end{aligned} \tag{2.8}$$

Since

$$(\alpha-2)\int_\Omega |p|\,|\nabla v|\,|v|^{\alpha-2}dx \leq \frac{(\alpha-2)^2}{2\mu}\int_\Omega p^2|v|^{\alpha-2}dx + \frac{\mu}{2}N_\alpha^\alpha(v),$$

(2.3) follows. □

Lemma 2.2. *Let $|v|^{\alpha/2}$ belong to H_0^1. Then*

$$\|v\|^\alpha_{\frac{\alpha n}{n-2}} \leq c_0 M_\alpha^\alpha(v), \tag{2.9}$$

where c_0 is the constant in equation (1.8).

Proof. Apply the estimate (1.8) to $g = |v|^{\alpha/2}$. □

Let now (r, q) be the exponents in equation (1.3). Assume that $q < +\infty$ (the case $q = +\infty$ is easier, but requires small modifications). Obviously

$$\int_\Omega p^2|v|^{\alpha-2}dx \leq \int_\Omega \left(\frac{p}{1+|v|}\right)^2 (1+|v|)^{\alpha\frac{q-n}{q}}(1+|v|)^{\frac{\alpha n}{q}}dx. \tag{2.10}$$

Since $2/q + (q-n)/q + (n-2)/q = 1$ Hölder's inequality shows that

$$\int_\Omega p^2|v|^{\alpha-2}dx \leq \|p/(1+|v|)\|_q^2\, \|1+|v|\,\|_\alpha^{\alpha(1-\frac{n}{q})}\|1+|v|\,\|^{\frac{\alpha n}{q}}_{\frac{\alpha n}{n-2}}.$$

Since $(q-n)/q + n/q = 1$ one gets

$$\int_\Omega p^2|v|^{\alpha-2}dx \leq \epsilon^{-\frac{q}{q-n}}\|p/(1+|v|)\|_q^r\, \|1+|v|\,\|_\alpha^\alpha + \epsilon^{q/n}\|1+|v|\,\|^\alpha_{\frac{\alpha n}{n-2}}, \tag{2.11}$$

where $\epsilon > 0$. Hence (2.3) shows that

$$\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \frac{\mu}{2}N_\alpha^\alpha(v) + 4\mu\frac{\alpha-2}{\alpha}M_\alpha^\alpha(v)$$
$$\leq \epsilon^{-\frac{q}{q-n}}\frac{(\alpha-2)^2}{2\mu}\|p/(1+|v|)\|_q^r\, 2^{\alpha-1}(|\Omega| + \|v\|_\alpha^\alpha)$$
$$+\epsilon^{\frac{q}{n}}\frac{(\alpha-2)^2}{2\mu}2^{\alpha-1}\left(|\Omega|^{\frac{n-2}{n}} + \|v\|^\alpha_{\frac{\alpha n}{n-2}}\right) + \|f\|_\alpha\|v\|_\alpha^{\alpha-1}.$$

Finally, by using (2.9) and by fixing ϵ by

$$c_0\epsilon^{\frac{q}{n}}\frac{(\alpha-2)^2}{\mu}2^{\alpha-2} = 2\mu\frac{\alpha-2}{\alpha}$$

one obtains (2.2) where

$$c_1 = \frac{2\mu(\alpha-2)}{\alpha c_0}|\Omega|^{\frac{n-2}{n}}$$

and

$$c_2 = 2^{\alpha-2}\frac{(\alpha-2)^2}{\mu}\left[2^{\alpha-3}\alpha(\alpha-2)c_0/\mu^2\right]^{\frac{n}{q-n}}.$$

Next, from (2.2) and by comparison theorems for O.D.E. one shows that

$$y(t) \leq M(t) \equiv \left(y(0) + \int_0^t h(s)ds\right)e^{\int_0^t k(s)ds}, \tag{2.12}$$

where

$$h(t) \equiv \alpha\big(c_1 + c_2|\Omega|\,\|p/(1+|v|)\|_q^r + \|f\|_\alpha\big)$$

and

$$k(t) \equiv \alpha\big(c_2 + \|p/(1+|v|)\|_q^r + \|f\|_\alpha\big).$$

From (2.2) together with (2.12) it readily follows that

$$\begin{aligned} &y(t) + \frac{\mu\alpha}{2}\int_0^t N_\alpha^\alpha(v)ds + 2\mu(\alpha-2)\int_0^t M_\alpha^\alpha(v)ds \\ &\leq y(0) + \int_0^t h(s)ds + \int_0^t k(s)M(s)ds. \end{aligned} \tag{2.13}$$

Since $M(s) \leq M(t)$ if $s \leq t$ we obtain, in particular,

$$\begin{aligned} &\|v(t)\|_\alpha^\alpha + \frac{\mu\alpha}{2}\int_{Q_t}|\nabla v|^2|v|^{\alpha-2}dxdt + 2\mu(\alpha-2)\int_{Q_t}|\nabla(|v|^{\alpha/2})|^\alpha dxdt \\ &\leq \left(\|v_0\|_\alpha^\alpha + \int_0^t h(s)ds\right)\left[1 + \left(\int_0^t k(s)ds\right)e^{\int_0^t k(s)ds}\right], \end{aligned} \tag{2.14}$$

for each $t \in [0,T]$.

Finally, if $q = +\infty$ one has (instead of (2.10))

$$\int_\Omega p^2 |v|^{\alpha-2} dx \leq \|p/(1+|v|)\|_\infty^2 \|1+|v|\,\|_\alpha^\alpha. \tag{2.15}$$

Hence

$$\begin{aligned}&\frac{1}{\alpha}\frac{d}{dt}\|v\|_\alpha^\alpha + \frac{\mu}{2}N_\alpha^\alpha(v) + 4\mu\frac{\alpha-2}{\alpha}M_\alpha^\alpha(v)\\ &\leq 2^{\alpha-2}\frac{(\alpha-2)^2}{\mu}\|p/(1+|v|)\|_\infty^2(|\Omega| + \|v\|_\alpha^\alpha) + \|f\|_\alpha \|v\|_\alpha^{\alpha-1},\end{aligned} \tag{2.16}$$

which corresponds to (2.2). □

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