HBU-69

Differential and Integral Equations Volume 9, Number 4, July 1996, pp. 729 – 744

ON THE SEMICONDUCTOR DRIFT DIFFUSION EQUATIONS

HUGO BEIRÃO DA VEIGA
Dipartimento di Matematica, Via F. Buonarroti, 2, 56127 Pisa, Italy

(Submitted by: H. Brezis)

Abstract. In [2] we proved global boundedness for the system (1.1); see Theorem 1.1 below. Here we show global existence for weak solutions (Theorem 1.2), existence of a global bounded attractor (Theorem 1.3), and strong continuous dependence on the data (Theorem 1.4) for the weak solutions.

1. Introduction. In this paper we study the following system of nonlinear partial differential equations that describes the transport of holes and electrons in a semiconductor device

$$\begin{cases} \dfrac{\partial p}{\partial t} - \nabla \cdot (D_1 \nabla p + \mu_1 p \nabla u) = R(p,n), \\ \dfrac{\partial n}{\partial t} - \nabla \cdot (D_2 \nabla n - \mu_2 n \nabla u) = R(p,n), \\ -\nabla \cdot (a \nabla u) = f + p - n \qquad \text{in } \mathbb{R}_+ \times \Omega, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} p = \phi(x), \quad n = \psi(x) & \text{on } \mathbb{R}_+ \times D, \\ (D_1 \nabla p + \mu_1 p \nabla u) \cdot \nu = (D_2 \nabla n - \mu_2 n \nabla u) \cdot \nu = 0 & \text{on } \mathbb{R}_+ \times B\,, \end{cases} \tag{1.2}$$

$$\begin{cases} u = U(x) & \text{on } \mathbb{R}_+ \times D\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R}_+ \times B\,, \end{cases} \tag{1.3}$$

and initial condition

$$p(0,x) = p_0(x), \quad n(0,x) = n_0(x) \quad \text{in } \Omega. \tag{1.4}$$

It is worth noting that the solutions p and n must be nonnegative. Here Ω is a bounded Lipschitzian domain in $\mathbb{R}^N$. We assume that the boundary Γ of Ω is the union of two disjoint sets D and B, where B is closed. For convenience we assume that D has not-vanishing $(N-1)$-dimensional measure. We denote by ν the unit outward to Γ. We refer to [20, 18, 15] for more detailed descriptions of the model. The unknowns u, p, and n denote the electrostatic potential, the free hole carrier concentration and the free electron carrier concentration. The solutions $p(x,t)$ and $n(x,t)$ are required to

Received for publication February 1995.
AMS Subject Classifications: 35K60, 35B40, 35Q99.

be nonnegative. We assume that D_1, D_2, μ_1, μ_2 and a (the dielectric permitivity) are positive constants. This leads to the equations (1.1).

In this paper we assume that $\phi, \psi \in H^1(\Omega) \cap L_+^\infty(\Omega)$ and $U \in H^1(\Omega) \cap L^\infty(\Omega)$, where the symbol "+" means the cone of nonnegative functions. We remark that the boundary condition $p = \phi$ on D does not change if we replace ϕ by the variational solution $\bar{\phi}$ of the problem $\Delta\bar{\phi} = 0$ in Ω, $\bar{\phi} = \phi$ on D, $\partial\bar{\phi}/\partial\nu = 0$ on B. Hence, we assume in the sequel that $\phi = \bar{\phi}$. Note that, by the maximum principle, $\sup_\Omega \phi = \sup_D \phi$. We also replace ψ by $\bar{\psi}$.

For convenience, we assume that the net density of ionized impurities f satisfies

$$f \in L^\infty(0, +\infty;\ L^\infty(\Omega)). \tag{1.5}$$

However, in Theorems 1.1 and 1.3 we may replace the above assumption by the assumption (1.5) in reference [2] and by weaker assumptions in Theorem 1.2.

Concerning the initial data we assume that

$$p_0, n_0 \in L_+^2(\Omega). \tag{1.6}$$

As in reference [2], in the Theorems 1.1 and 1.3 we can assume that the recombination term $R(p, n)$ is a locally Lipschitz continuous function on $\mathbb{R}_+ \times \mathbb{R}_+$, such that

$$\lim_{p+n\to+\infty} \frac{R(p,n)^+}{p+n} = 0\,, \tag{1.7}$$

where $z^+ = \max\{z, 0\}$; moreover,

$$\begin{cases} R(p,0) \geq 0\,, & \forall p \geq 0\,, \\ \\ R(0,n) \geq 0\,, & \forall n \geq 0\,. \end{cases} \tag{1.8}$$

However, for convenience, we will assume (unless assumed otherwise) that R is given by the Shockley-Read-Hall recombination term

$$R(p,n) = (1 - pn)/(r_0 + r_1 p + r_2 n), \tag{1.9}$$

in which r_0, r_1, and r_2 are positive constants. Here and there we will make suitable comments about more general conditions that may be assumed.

Before stating our results we introduce some notation. We set $Q_T = (0, T) \times \Omega$, $Q = Q_\infty$. We denote by $\|\cdot\|_r$, $r \in [1, +\infty]$, the canonical norm in $L^r = L^r(\Omega)$ and by $\|\cdot\|_{r,s;T}$, $r, s \in [1, +\infty]$ and $T \in (0, +\infty]$, that in $L^s(0, T; L^r)$. For convenience, we set $\|\cdot\| = \|\cdot\|_2$ and $\|\cdot\|_{r,s} = \|\cdot\|_{r,s;+\infty}$. We denote by $|E|$ the N-dimensional Lebesgue measure of a set E.

For convenience, we use notation like $\|(p,n)\|^2 = \|p\|^2 + \|n\|^2$, $|\nabla(p,n)|^2 = |\nabla p|^2 + |\nabla n|^2$, and so on. We denote by V the Hilbert space $V = \{v \in H^1 : v = 0$ on $D\}$ and by V' its dual space. In order to use here a standard notation, let us set

$H = L^2(\Omega)$. By identifying H with its dual H' one has $V \hookrightarrow H \hookrightarrow V'$, where each space is dense in the next one. The spaces, V, H, and V' are in a typical situation, often considered in studying weak solutions of partial differential equations. We denote by $(\cdot, \cdot)$ the scalar product in H (or in H^N—we use the same notation for scalar and for vector fields) and by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V or, more in general, between the dual of a functional space and the space itself. If v belongs to $L^2_{\text{loc}}(0, +\infty; V)$ we sometimes denote by v' the derivative of v as a distribution in $(0, +\infty)$ with values in V. Since $V \hookrightarrow V'$ it could be that $v' \in L^2_{\text{loc}}(0, +\infty; V')$. For properties connected to this (already classical) setting up we refer the reader to [14], [4].

We set $\mu_0 = \sqrt{\mu_1 \mu_2}$, $\mu_3 = \max\{\mu_1, \mu_2\}$, $\mu_4 = \min\{\mu_1, \mu_2\}$, $\rho = \min\{D_1/\mu_1, D_2/\mu_2\}$, $b = r_0^{-1} \mu_3$. Moreover,

$$M_0 = \max\{\|p_0\|_\infty, \|n_0\|_\infty, \|\phi\|_\infty, \|\psi\|_\infty\}, \quad M = \max\{M_0, 1\}. \tag{1.10}$$

We denote by c_0 a positive constant such that the Poincaré inequality

$$\int v^2 \, dx \le c_0 \int |\nabla v|^2 \, dx, \quad \forall v \in V, \tag{1.11}$$

holds and by c_1 a positive constant such that the Sobolev embedding theorem

$$\Big(\int v^{2^*} \, dx\Big)^{1/2^*} \le c_1 \Big(\int |\nabla v|^2 \, dx\Big)^{1/2}, \quad \forall v \in V, \tag{1.12}$$

holds. If $N \ge 3$, we denote by 2^* the embedding Sobolev exponent $2^* = 2N/(N-2)$ and by $\hat{2}$ its dual exponent $\hat{2} = 2N/(N+2)$. If $N = 2$ (hence $r \in (4, +\infty]$; recall (1.5)) we set $2^* = 4r/(r-4)$ and $\hat{2} = 4r/(4+3r)$. Note that (1.14) also holds for $N = 2$. Moreover $1/2^* + 1/\hat{2} = 1$.

Before going on, we want to point out that we assume the reader to be well acquainted with the formulation of PDE's in weak form. We adopt here classical terminology and notation in order to bring out clearly the underlying ideas. The interpretation of some of the terminology and the justification of some of the calculations (in terms of weak solutions, distributional derivatives, duality pairing, and so on) is done by using well-known standard devices. We refer the reader to [4], [11], [13], [12], [14]; see, in particular, [4], Chapter XVIII, Section 1 and Section 3.

Under the above hypotheses and, moreover, if $p_0, n_0 \in L^\infty_+(\Omega)$, then there is a weak solution (p, n, u) of problem (1.1)–(1.4) in the following class: $p - \phi$ and $n - \psi$ belong to $L^2_{\text{loc}}(O, +\infty; V)$; $u - U$ belong to $L^r_{\text{loc}}(O, +\infty; V)$; p and n are nonnegative almost everywhere in Q and belong to $L^\infty_{\text{loc}}(O, +\infty; L^\infty)$. Moreover, the solution is unique in the above class. Functions p, n, and u in the above class are said to be a weak solution of (1.1)–(1.4) if, for each fixed $v \in V$, one has $a(\nabla u, \nabla v) = (f + p - n, v)$, and also, in the sense of $\mathcal{D}'((0, +\infty))$ (or equivalently, almost everywhere in $(0, +\infty)$)

$$\langle p', v \rangle + (D_1 \nabla p, \nabla v) + \mu_1 (p \nabla u, \nabla v) = (R(p, n), v),$$

and

$$\langle n', v\rangle + (D_2\nabla n, \nabla v) + \mu_2(-n\nabla u, \nabla v) = (R(p,n), v)\,.$$

Moreover, $p(0) = p_0$, $n(0) = n_0$. Note that p and n are continuous on $[0, +\infty)$ with values in $H = L^2(\Omega)$. We may also write the above equations in terms of $y = p - \phi$, $z = n - \psi$ and $w = u - U$.

Let us start by recalling the following result, a particular case of Theorem 1.1 in reference [2].

Theorem 1.1. *Let $p_0, n_0 \in L^\infty_+(\Omega)$. Then the above solution (p, n) of problem* (1.1)–(1.4) *is uniformly bounded in $Q = \mathbb{R}_+ \times \Omega$. More precisely*

$$\sup_Q (p(t,x) + n(t,x)) \leq CM\left(1 + \|f\|_{r,s}^{2(1+\frac{1}{\chi})}\right), \tag{1.13}$$

where

$$\chi = \frac{2}{N} \;\text{ if }\; N \geq 3\;, \quad \chi = \frac{1}{2} \;\text{ if }\; N = 2. \tag{1.14}$$

The constant C depends only on N, c_0, c_1, a, D_1, D_2, μ_1, μ_2, and $|\Omega|$.

If $N = 3$, we denote by V_N the space $V_N = \{v \in V : \nabla v \in L^N(\Omega)\}$. If $N = 2$ we denote by V_N the space $\{v \in V : \nabla v \in L^{N_1}(\Omega)\}$ for some fixed $N_1 > 2$. V'_N denotes the dual space of V_N.

In the sequel we prove the following existence theorem, under the assumption (1.6).

Theorem 1.2. *To each pair of initial data p_0, n_0 in the class* (1.6) *there corresponds* (*at least*) *a weak solution (p, n, u) for the problem* (1.1)–(1.4) *in the class $p - \phi, n - \psi \in L^2_{\text{loc}}(O, +\infty; V)$, $p_t, n_t \in L^2_{\text{loc}}(O, +\infty; V'_N)$, $u - U \in L^2_{\text{loc}}(O, +\infty; V)$. Moreover,* (1.17) *holds. In particular $p, n \in L^\infty_{\text{loc}}(O, +\infty; L^2_+)$.*

In order to prove the existence and the uniqueness of a stronger solution we introduce the following assumption.

Consider the elliptic mixed boundary value problem

$$-\Delta u = g \;\text{ in }\; \Omega, \qquad u = U \;\text{ on }\; D, \qquad \partial u/\partial \nu = 0 \;\text{ on }\; B. \tag{1.15}$$

We assume that there is a functional space Y and a real number q ($q > 2$ if $N = 2$; $q = N$ otherwise) such that if $g \in L^2(\Omega)$ and $U \in Y$ then the variational solution u of problem (1.15) satisfies

$$\|\nabla u\|_q \leq c\,(\|g\| + \|U\|_Y)\,. \tag{1.16}$$

Note that this is an assumption on $\{\Omega, B, D\}$. This assumption never holds if $N > 4$ since $H^2(\Omega)$ is not contained in $L^N(\Omega)$. If $N = 4$ the assumption holds only in some very special cases. In fact it does not hold if Γ is regular in a neighborhood of a point of the boundary of D in Γ. In fact, in this case, it is false in general that $u \in W^{1,4}$ (see counterexample in Shamir's paper, [19]). If $N = 2$ the assumption holds if Ω is a bounded domain with a polygonal boundary (or a regular transformation of such a set).

In this case $q > 2$ can be arbitrarily fixed; moreover, $Y = W^{1-\frac{1}{q},q}(B)$. This follows from results by Lorenzi ([10]). In fact, it is not difficult to reduce the problem (1.15) (in a neighborhood of each point of the boundary of D in Γ) to the problem (1.3) in reference [10], and then to apply Theorem 1 in this last reference in order to prove the above result. Since it is sufficient to have (1.16) for some $q > 2$, it could be possible to use Gröger's results ([6]). If $N = 3$ and if Ω is a bounded convex set with a polyhedral boundary (or a regular transformation of it) then the solution of problem (1.15) belongs to $H^{3/2}(\Omega)$. This follows from Theorem 2.6.3 in Grisvard's book ([9]). For regular boundary points ($\omega = \pi$) the result is still true since it can be reduced (by a reflection argument) to the case of a cut ($\omega = 2\pi$), for which $H^{3/2}$-regularity still holds ([9], Section 2.7, page 83). Recall that, if $N = 3$, then $H^{3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)$. It is worth noting that in [9] the author considers only homogeneous boundary conditions, but this looks inessential there. We also note that $W^{1/3}$-regularity holds under much weaker hypotheses on the angles between faces than that needed to get $H^{3/2}$-regularity. But we do not know about precise statements in the literature. However, $3\pi/2$ should be the correct upper bound to the angles, in order to get (1.16) when $N = 3$.

One has the following result where, for brevity, we assume that $f \in L^\infty(Q)$ and that $R(p,n)$ is given by (1.9). See also [3].

Theorem 1.3. *Let the assumption* (1.16) *hold and let* ϕ, ψ, U, f *and* $R(p,n)$ *be as in Theorem* 1.1*; moreover,* $U \in Y$. *Then, to each pair of initial data* $(p_0, n_0) \in L^2_+(\Omega)$ *there corresponds a unique solution* (p,n) *of problem* (1.1)–(1.4) *in the class* $p - \phi, n - \psi \in L^2_{\mathrm{loc}}(O, +\infty; V)$*;* $p_t, n_t \in L^2_{\mathrm{loc}}(O, +\infty; V')$. *Moreover,* $p, n \in C(o, +\infty; L^2(\Omega))$ *and there is a positive constant* C_0 *(that depends on the norms* $\|\phi\|_\infty$, $\|\psi\|_\infty$, $|||f|||_\infty$ *but not on* p_0, n_0 *and* U*) such that*

$$\|p(t)\|^2 + \|n(t)\|^2 \leq C_0 + ce^{-\nu t}\left(\|p_0\|^2 + \|n_0\|^2\right) \tag{1.17}$$

for each $t \geq 0$. *The positive constant* c *and* ν *are independent of the data* ϕ, ψ, U, f, p_0 *and* n_0.

The above result shows that the set

$$B_0 = \{(\bar{p}, \bar{n}) \in L^2(\Omega) : \bar{p} \geq 0, \bar{n} \geq 0, \ \text{ and } \ \|\bar{p}\|^2 + \|\bar{n}\|^2 \leq C_0\}$$

is a global bounded attractor in the space $L^2(\Omega)$.

Next, we consider the problem of the uniqueness of the solution. Except for some very special situations, uniqueness results is not very interesting if existence in the same class is unlikely. Obviously, the situation in which existence and uniqueness hold in the same class is particularly important. Under the assumptions of Theorem 1.1 uniqueness is easy to prove since p and n belong to $L^\infty(0, +\infty; L^\infty_+)$. This is mainly due to the fact that the initial data p_0 and n_0 belong to $L^\infty(\Omega)$.

We do not know whether uniqueness holds under the assumptions of Theorem 1.2. However it holds under the assumptions of Theorem 1.3, as follows from Theorem 1.4 below.

Due to lack of regularity for solutions of the mixed problem, instead of imposing it artificially we prefer to assume the condition (1.18) below. This assumption is quite natural in view of the regularity result known for solutions of the mixed problem in Lipschitz domains, provided that $N < 4$. If $N = 4$ the assumption holds only in very special cases. See remarks after the related assumption (1.16). The assumption is the following. For each $g \in L^2$ let u be the variational solution of the elliptic problem $-\Delta u = g$ in Ω, $u = 0$ on D, $\partial u/\partial \nu = 0$ on B. Then, if $N = 3$ or $N = 4$ we assume that $\nabla u \in L^N$ and that

$$\|\nabla u\|_N \leq c\|g\|. \tag{1.18}$$

If $N = 2$ we replace N by some $N_1 > N$. Our assumption holds in general if $N = 2$ and seems to hold, when $N = 3$, if angles between faces are smaller than $3\pi/2$.

The next condition concerns the recombination function R, a locally Lipschitz continuous function defined for nonnegative p and n. We assume that

$$|R(p,n) - R(q,m)| \leq c(1 + |p| + |q| + |n| + |m|)^\beta \cdot (|p-q| + |n-m|), \tag{1.19}$$

for all nonnegative p, q, n, m, where $\beta = 4/N$ if $N = 3$ or $N = 4$, and $\beta < 4/N = 2$ if $N = 2$. Obviously, this assumption holds for the Shockley-Read-Hall recombination term (1.9).

In Theorem 1.4 below we consider weak solutions (p, n, u) of problem (1.1)–(1.4) in the class

$$\begin{cases} p, n \in L^\infty(0,T; L^2_+)\,; & p - \phi, n - \psi \in L^2(0,T;V), \\ p_t, n_t \in L^2(0,T;V')\,; & u - U \in L^2(0,T;V). \end{cases} \tag{1.20}$$

Theorem 1.4. *Assume that $N \leq 4$ and let ϕ and ψ be as above. Moreover, let $f \in L^2(0,T;L^N)$ if $N \neq 2$ or $f \in L^2(0,T;L^{N_1})$, for some $N_1 > N$, if $N = 2$. Let (p,n,u) and (q,m,v) be two solutions of problem* (1.1)–(1.4) *in the class* (1.20)*, and denote by (p_0, n_0) and (q_0, m_0) respectively their initial data in the space $L^2_+(\Omega)$. Assume that $\nabla u \in L^\infty(0,T;L^N)$ if $N \neq 2$, or $\nabla u \in L^\infty(0,T;L^{N_1})$, for some $N_1 > 2$, if $N = 2$. Then, under the assumptions* (1.18), (1.19)*, there is a function $h(t)$, integrable on* $[0,T]$*, such that*

$$\|p(t) - q(t)\|^2 + \|n(t) - m(t)\|^2 \leq c e^{h(t)}(\|p_0 - q_0\|^2 + \|n_0 - m_0\|^2). \tag{1.21}$$

In particular, the solution of problem (1.1)–(1.4) *is unique in the above class.*

Concerning the above hypothesis on ∇u, one has the following auxiliary result.

Proposition 1.5. *In Theorem* 1.4*, the assumption $\nabla u \in L^\infty(0,T;L^N)$ if $N \neq 2$, $\nabla u \in L^\infty(o,T;L^{N_1})$ if $N = 2$, is satisfied whenever* (1.16) *holds and f belongs to $L^\infty(0,T;L^2)$.*

Remark. We have begun the study of problem (1.1)–(1.4) in the light of references [7], [5]. After having obtained the results stated here, we have come across the Gajewski and Gröger paper ([8]), where the authors obtain results related to ours. The interested reader is strongly advised to consult the above reference, also in order to compare proofs and results.

2. Proofs. We denote by $c, \hat{c}, \tilde{c}, c_i$, $i = 0, 1, 2, \ldots$, positive constants that depend only on N, Ω, a, D_1, D_2, μ_1, μ_2, and on the particular recombination function R. Different constants will be denoted by the same symbol c, even in the same equation. We set

$$\tilde{M}_0 = \max\{\|\phi\|_\infty, \|\psi\|_\infty\|\}, \tag{2.1}$$

and define $\bar{w} = w_{(k)} = \max\{w - k, 0\}$. By arguing as in the proof of (2.5), (2.8) in reference [2] we show that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int(\mu_2\bar{p}^2 + \mu_1\bar{n}^2)\,dx + \frac{\rho\mu_0^2}{2}\int|\nabla(\bar{p},\bar{n})|^2\,dx \\ &\leq \frac{\mu_0^2}{4a}\int f^2(\bar{p}+\bar{n})\,dx + \frac{\mu_0^2}{2a}k\int f^2\,dx + b\int(\bar{p}+\bar{n})\,dx. \end{aligned} \tag{2.2}$$

We estimate the last term on the left-hand side of equation (2.3) in reference [2] by using the inequality written just after the equation (2.4) in this last reference. Note that the right-hand side of (2.3) in [2] is bounded by the last term in the above equation (2.2). Next,

$$\frac{\mu_0^2}{4a}k\int f^2(\bar{p}+\bar{n})\,dx \leq \frac{\rho\mu_0^2}{8}\|\nabla(\bar{p},\bar{n})\|^2 + c\|f\|_\infty^4,$$

and

$$\frac{\mu_0^2}{2a}\int kf^2\,dx \leq ck^2 + c\|f\|_\infty^4.$$

Hence it readily follows, by setting $k = \hat{M}_0$ in (2.2), that

$$\|\bar{p}(t)\|^2 + \|\bar{n}(t)\|^2 \leq ce^{-\nu t}(\|\bar{p}_0\|^2 + \|\bar{n}_0\|^2) + c(\|f\|_\infty^4 + \hat{M}_0^2 + b^2),$$

where $\nu = \rho\mu_0^2/8c_0$. Consequently, for each $t \geq 0$,

$$\|p(t)\|^2 + \|n(t)\|^2 \leq ce^{-\nu t}(\|p_0\|^2 + \|n_0\|^2) + C_0, \tag{2.3}$$

where

$$C_0 = c_3(\|f\|_\infty^4 + \|\phi\|_\infty^2 + \|\psi\|_\infty^2 + 1). \qquad \square$$

This estimate is satisfied by the solutions in Theorem 1.1. Hence the set B_0 is a global bounded attractor in the sense that bounded sets in $L^2(\Omega)$, consisting of initial data in $L_+^\infty(\Omega)$, are uniformly attracted by B_0. However, by working in the functional space $L^2(\Omega)$, the desirable result is to consider initial data in $L_+^2(\Omega)$ and to be able to prove uniqueness of the solution together with (2.3). The first step in this direction is to prove that to each pair $(p_0, n_0) \in L_+^2(\Omega)$ there corresponds a global weak solution of our problem satisfying (2.3) (Theorem 1.2). The second step (Theorem 1.3) is to prove the existence of a unique, more regular, solution in correspondence with each pair of initial data $(p_0, n_0) \in L^2_+(\Omega)$. The existence part will be proved first. Uniqueness will be proved separately, since it is a consequence of Theorem 1.4.

Set

$$\tilde{p}_0 = \min\{p_0, k\}, \qquad \tilde{n}_0 = \min\{n_0, k\}, \tag{1.$\tilde{4}$}$$

for $k = 1, 2, 3, \ldots$. In proving the results below it would be sufficient to assume that $f \in L^{r,s}(Q)$, $r = 4N/(N+2)$, $s \in [4, +\infty]$. Under this more general assumption we must also introduce the function $\tilde{f} = \max\{f, k\}$. However, for simplicity, we assume here that $f \in L^\infty(Q)$.

By Theorem 1.1, for each fixed k, there is a unique solution $(\tilde{p}, \tilde{n}, \tilde{u})$ of problem (1.1)–(1.3) in correspondence with the above initial data (1.$\tilde{4}$). In particular, $\tilde{p}, \tilde{n} \in L^\infty_+(Q)$, $\tilde{p} - \phi, \tilde{n} - \psi \in L^2_{\text{loc}}(0, +\infty; V)$, $\tilde{p}_t, \tilde{n}_t \in L^2_{\text{loc}}(0, +\infty; V')$. Our aim is to establish estimates for $(\tilde{p}, \tilde{n}, \tilde{u})$ that are independent of k, and then pass to the limit in the equation as $k \to +\infty$. A first estimate of this kind is obtained from (2.3), since $\|\tilde{p}_0\| \leq \|p_0\|$ and $\|\tilde{n}_0\| \leq \|n_0\|$. Hence

$$\|\tilde{p}(t)\|^2 + \|\tilde{n}(t)\|^2 \leq ce^{-\nu t}(\|p_0\|^2 + \|n_0\|^2) + C_0. \tag{2.4}$$

We denote the equations (1.1), (1.2), (1.3), (1.4), when p, n, u, p_0, n_0 are replaced by $\tilde{p}, \tilde{n}, \tilde{u}, \tilde{p}_0, \tilde{n}_0$, by (1.$\tilde{1}$), (1.$\tilde{2}$), (1.$\tilde{3}$), (1.$\tilde{4}$), respectively. The calculations that follow concern the variables with "$\sim$". However, for convenience, we drop the symbol "$\sim$" from the variables $\tilde{p}, \tilde{n}, \tilde{u}, \tilde{p}_0, \tilde{n}_0$. With this convention, multiply the equation $(1.\tilde{1})_1$ by $p - \phi$ (i.e., by $\tilde{p} - \phi$) and integrate over Ω. By taking into account that $\partial(p-\phi)/\partial t$ belongs to $L^2_{\text{loc}}(0, +\infty; V')$, that $p - \phi$ belongs to $L^2_{\text{loc}}(0, +\infty; V)$ and that (in the usual weak sense) $\partial(p-\phi)/\partial\nu = 0$ on B (since $\partial\phi/\partial\nu = 0$ on B) we get

$$\frac{1}{2}\frac{d}{dt}\|p-\phi\|^2 + D_1 \int \nabla p \cdot \nabla(p-\phi)\,dx + \mu_1 \int p\nabla u \cdot \nabla(p-\phi)\,dx = \int R(p,n)(p-\phi)dx.$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|p-\phi\|^2 &+ D_1\|\nabla(p-\phi)\|^2 + \mu_1 \int \nabla\phi \cdot \nabla(p-\phi)\,dx \\ &+ \mu_1 \int (p-\phi)\nabla u \cdot \nabla(p-\phi)\,dx + \mu_1 \int \phi\nabla u \cdot \nabla(p-\phi)dx = \int R(p-\phi)\,dx.\end{aligned}$$

Since $((p-\phi)\nabla u,\ \nabla(p-\phi)) = (2a)^{-1}((f+p-n), (p-\phi)^2)$ we show, by using suitable devices, that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|p-\phi\|^2 &+ \frac{D_1}{2}\|\nabla(p-\phi)\|^2 + \frac{\mu_1}{2a}\int (f+p-n)(p-\phi)^2\,dx \\ &+ \mu_1 \int \phi\nabla u \cdot \nabla(p-\phi)\,dx \leq c\|\nabla\phi\|^2 + \int R(p-\phi)\,dx.\end{aligned} \tag{2.5}$$

Next, since $a(\nabla(u-U), \nabla w) = -a(\nabla U, \nabla w) + (f+p-n, w)$ for each $w \in V$, by setting $w = u - U$, one shows that

$$\frac{a}{4}\|\nabla(u-U)\|^2 \leq \frac{a}{2}\|\nabla U\|^2 + \frac{c_0}{a}\|f+p-n\|^2.$$

Hence

$$\|\nabla u\|^2 \leq 6\|\nabla U\|^2 + \frac{2c_0}{a}\|f+p-n\|^2. \tag{2.6}$$

This estimate, together with

$$|(\phi\nabla u, \nabla(p-\phi))| \leq c\|\phi\|_\infty^2\|\nabla u\|^2 + (D_1/4\mu_1)\|\nabla(p-\phi)\|^2$$

allows us to deduce from (2.5) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|p-\phi\|^2 + \frac{D_1}{4}\|\nabla(p-\phi)\|^2 + \frac{\mu_1}{2a}\int(f+p-n)(p-\phi)^2\,dx \\ \leq c\|\phi\|_\infty^2(\|\nabla U\|^2 + \|f+p-n\|^2) + c\|\nabla\phi\|^2 + \int R(p,n)(p-\phi)\,dx.\end{aligned} \tag{2.7}$$

A similar estimate holds for $n-\psi$. By multiplying this last estimate by μ_1, the estimate (2.7) by μ_2 and by adding, side by side, the two estimates one finds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\mu_2\|p-\phi\|^2 + \mu_1\|n-\psi\|^2) + \frac{\rho\mu_0^2}{4}\|\nabla(p-\phi, n-\psi)\|^2 \\ &\quad + \frac{\mu_0^2}{2a}\int(f+p-n)[(p-\phi)^2 - (n-\psi)^2]\,dx \\ &\leq c(\|\phi\|_\infty^2 + \|\psi\|_\infty^2)(\|\nabla U\|^2 + \|f+p-n\|^2) + c\|\nabla(\phi,\psi)\|^2 \\ &\quad + \int R(p,n)[\mu_2(p-\phi) + \mu_1(n-\psi)]\,dx.\end{aligned} \tag{2.8}$$

Next, we prove that

$$(f+p+n)[(p-\phi)^2 - (n-\psi)^2] \geq -[f^2 + 3(\phi+\psi)^2]|(p-\phi)+(n-\psi)|. \tag{2.9}$$

Denote by A the left-hand side of (2.9). One has

$$A = (f+\phi-\psi)[(p-\phi)^2 - (n-\psi)^2] + [(p-\phi)-(n-\psi)]^2[(p-\phi)+(n-\psi)].$$

Moreover

$$\begin{aligned}|(f+\phi-\psi)[(p-\phi)^2 - (n-\psi)^2]| &\leq \frac{1}{4}|f+\phi-\psi|^2\,|(p-\phi)+(n-\psi)| \\ &\quad + [(p-\phi)-(n-\psi)]^2|(p-\phi)+(n-\psi)|.\end{aligned}$$

Assume that $p+n \geq \phi+\psi$. Then $A \geq -(1/4)|f+\phi-\psi|^2 \cdot ((p-\phi)+(n-\psi))$, hence (2.9) holds. If $p+n \leq \phi+\psi$ then $|A| = |f+p-n|\,|(p-\phi)-(n-\psi)|\cdot|(p-\phi)+(n-\psi)|$. Since this last quantity is less than or equal to $(|f|+\phi+\psi)2(\phi+\psi)|(p-\phi)+(n-\psi)|$, (2.9) holds.

Finally, one easily verifies that

$$R(p,n)[\mu_2(p-\phi)+\mu_1(n-\psi)] \leq c(1+\phi^2+\psi^2), \tag{2.10}$$

by considering, separately, the case in which the expression between square brackets is positive and the case in which it is negative. From (2.8), (2.9), (2.10) one gets, by doing straightforward calculations, and by using (2.3),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\mu_2\|p-\phi\|^2+\mu_1\|n-\psi\|^2)+\frac{\rho\mu_0^2}{8}\|\nabla(p-\phi,n-\psi)\|^2 \leq \quad (2.11)\\ c[\|f\|_\infty^4(1+\|\phi\|_\infty^2+\|\psi\|_\infty^2)+\|\phi\|_\infty^4+\|\psi\|_\infty^4+\|\nabla\phi\|^2+\|\nabla\psi\|^2+\|\nabla U\|^4+1]\\ +ce^{-\nu t}(\|\phi\|_\infty^2+\|\psi\|_\infty^2)(\|p_0\|^2+\|u_0\|^2).\end{aligned}$$

By integration over $[0, T]$ one gets, in particular,

$$\begin{aligned}\int_0^T \|\nabla(p-\phi,n-\psi)\|^2\,dt \leq c(1+\|\phi\|^2+\|\psi\|^2)(1+\|p_0\|^2+\|n_0\|^2)\\ +cT[(1+\|\phi\|_\infty^2+\|\psi\|_\infty^2)\|f\|_\infty^4+\|\phi\|_\infty^4+\|\psi\|_\infty^4+\|\nabla U\|^4 \quad (2.12)\\ +\|\nabla\phi\|^2+\|\nabla\psi\|^2+1]. \qquad \square\end{aligned}$$

Now, recall that we have been denoting by (p_0, n_0) the initial data $(\tilde{p}_0, \tilde{n}_0)$ given by (1.4) and by (p, n, u) the solution of problem (1.1)–(1.3) with initial data $(\tilde{p}_0, \tilde{n}_0)$. By Theorem 1.1, since $(\tilde{p}_0, \tilde{n}_0)$ belongs to $L^\infty_+(\Omega)$, the solution $\tilde{p}, \tilde{n}, \tilde{u}$ satisfies, for each fixed k, $\tilde{p}-\phi, \tilde{n}-\psi \in L^2_{\mathrm{loc}}(0,+\infty; V)$, $\tilde{p}', \tilde{n}' \in L^2_{\mathrm{loc}}(0,+\infty; V')$, $\tilde{p}, \tilde{n} \in L^\infty(0,+\infty; L^\infty)$, $\tilde{u}-U \in L^\infty(0,+\infty; V)$. For convenience, $w' = w_t$.

Next, we pass to the limit as $k \to +\infty$. For convenience, we denote by C positive constants that may depend on the same parameters on which the constants of type c may depend plus an eventual dependence (increasing) on the norms $\|p_0\|$, $\|n_0\|$, $\|f\|_\infty$, $\|\phi\|_\infty$ and $\|\psi\|_\infty$ but not on k and on T. If dependence on T is allowed, we write $C(T)$. Hence,

$$\begin{cases} \|(\tilde{p},\tilde{n})\|_{2,\infty} \leq C\,, & \|\nabla(\tilde{p},\tilde{n})\|_{2,2;T} \leq C(T), \\ \|(\tilde{p}-\phi,\tilde{n}-\psi)\|_{L^2(0,T;V)} \leq C(T)\,, & \|\tilde{u}-U\|_{L^\infty(0,T;V)} \leq C(T). \end{cases} \tag{2.13}$$

On the other hand, for each $v \in V \supset V_N$, one has

$$\langle\tilde{p}', v\rangle = \int (D_1\nabla\tilde{p}+\mu_1\tilde{p}\nabla\tilde{u})\cdot\nabla v\,dx + \int R(\tilde{p},\tilde{n})v\,dx. \tag{2.14}$$

For convenience we assume that $N \geq 3$, and leave some details to the reader when $N = 2$. From $(2.13)_{1,2}$ it follows, by interpolation between $L^\infty(o, T; L^2)$ and $L^2(0, T; L^{2^*})$, that

$$\|(\tilde{p},\tilde{n})\|_{\lambda,\vartheta;T} \leq C(T), \tag{2.15}$$

if

$$\frac{2}{\vartheta}+\frac{N}{\lambda}\geq\frac{N}{2}\ ,\quad 2<\vartheta<+\infty. \tag{2.16}$$

If $N=2$ it must be $(2/\vartheta)+(N/\lambda)>N/2$.

For completeness we note that the assumption

$$|R(p,n)|\leq c(1+|p|^{\alpha}+|n|^{\alpha}), \tag{2.17}$$

for some positive constants c and α, where $\alpha<2+2/N$, is sufficient here in order to prove that

$$\|R(\tilde{p},\tilde{n})\|_{L^2(0,T;V_N')}\leq C(T). \tag{2.18}$$

In fact, under the above assumption, one has

$$\Big|\int R(\tilde{p},\tilde{n})v\,dx\Big|\leq c\left(1+\|\tilde{p}\|^{\alpha}_{2+\frac{2}{N}}+\|\tilde{n}\|^{\alpha}_{2+\frac{2}{N}}\right)\|v\|_{V_N}$$

since $V_N\subset L^p$ for each finite p, hence for $p=2(N+1)/[2(N+1)-\alpha N]$. Hence, by using (2.16), (2.18) follows.

If we assume the stronger condition $\alpha=1+4/N(\alpha<1+4/N$ if $N=2)$ then

$$\|R(\tilde{p},\tilde{n})\|_{L^2(0,T;V')}\leq C(T). \tag{2.19}$$

In fact

$$\Big|\int R(\tilde{p},\tilde{n})v\,dx\Big|\leq c\|v\|_V\int_{\Omega}\left(1+|\tilde{p}|^{\hat{2}\alpha}+|\tilde{n}|^{\hat{2}\alpha}\right)^{\alpha/\hat{2}\alpha}dx,$$

where $\hat{2}=2N/(N+2)$. Since the exponents $\vartheta=2\alpha$ and $\lambda=\hat{2}\alpha$ satisfy (2.16), the integral on the right-hand side of the above inequality belongs to $L^2(0,T)$, for $t\in[0,T]$. This yields (2.19). Note that the recombination term defined in (1.9) satisfies the above assumption for $\alpha=1$. □

Next we study the first term on the right hand side of (2.14). From $(2.13)_2$ it follows that

$$\Big|\int_0^T\Big(\int\nabla\tilde{p}\cdot\nabla v\,dx\Big)\,dt\Big|\leq C(T)\|v\|_{L^2(0,T;V)}. \tag{2.20}$$

On the other hand (we left the case $N=2$ to the reader) one has $|(\tilde{p}\nabla u,\nabla v)|\leq\|\tilde{p}\|_{2^*}\|\nabla\tilde{u}\|\|\nabla v\|_N$. Hence

$$\Big|\int_0^T\Big(\int\tilde{p}\nabla\tilde{u}\cdot\nabla v\,dx\Big)\,dt\Big|\leq C(T)\|v\|_{L^2(0,T;V_N)}. \tag{2.21}$$

We get a stronger result under the assumptions of Theorem 1.3. In fact, if (1.16) holds and if U belongs to Y, then

$$\Big|\int_0^T\Big(\int\tilde{p}\nabla\tilde{u}\cdot\nabla v\,dx\Big)\,dt\Big|\leq C(T)\|v\|_{L^2(0,T;V)}. \tag{2.22}$$

In fact, $|(\tilde{p}\nabla\tilde{u}, \nabla v)| \le C\|\tilde{p}\|_{2^*}\|\nabla\tilde{u}\|_N\|\nabla v\|$. Since $\|\nabla\tilde{u}\|_N \le c(\|f\|_\infty + \|\tilde{p}\| + \|\tilde{n}\| + \|U\|_y) \le C(T)$, (2.22) follows easily, by using $(2.13)_2$.

The above arguments apply as well to $\tilde{n}$. Hence, from (2.14), (2.20), (2.21), and (2.18), and from similar equations concerning $\tilde{n}$, one shows that

$$\|\partial_t(\tilde{p}, \tilde{n})\|_{L^2(0,T;V_N')} \le C(T). \tag{2.23}$$

Moreover, if $\alpha \le 1 + 4/N$("<," if $N = 2$), if (1.16) holds and if $U \in Y$, then we can use (2.22) and (2.19) in order to prove that

$$\|\partial_t(\tilde{p}, \tilde{n})\|_{L^2(0,T;V')} \le C(T). \tag{2.24}$$

Passage to the limit as $k \to +\infty$. Next, we use the uniform estimates (2.13), (2.21), (2.22) or (2.13), (2.23), (2.24) in order to pass to the limit, as $k \to +\infty$, by using compactness theorems. We start by using Theorem 5.1, Chapter 1, page 17 in reference [13]. We set in this theorem $p_0 = p_1 = 2$, $B_0 = V$, $B_1 = V_N'$ for proving Theorem 1.2, $B_1 = V'$ for proving Theorem 1.3, and $B = L^{q_0}(\Omega)$, $q_0 < 2^*(q_0 < +\infty$, if $N = 2)$. The theorem guarantees the compactness of the sequences $\{(\tilde{p}, \tilde{n})\}$ in $L^2(0, T; L^{q_0})$, as $k \to +\infty$, for each fixed $T > 0$. Let us denote by $\{(\tilde{p}, \tilde{n})\}$ a subsequence which is convergent in the above topology to some limit, denoted by (p, m). Convergence in $L^2(0, T; L^2)$ together with the boundedness in $L^2(0, T; H^1)$, hence in $L^2(0, T; L^{2^*})$, yields (by interpolation) convergence in $L^2(0, T; H^s)$, for each $s < 1$, and in $L^2(0, T; L^q)$, for each $q < 2^*$. We also can assume that $\tilde{p}$ and $\tilde{n}$ converge to p and n almost everywhere in Q_T.

On the other hand, (2.13) shows that

$$(\tilde{p} - \phi, \tilde{n} - \psi) \rightharpoonup (p - \phi, n - \psi) \tag{2.25}$$

weakly in $L^2(0, T; V)$; moreover, (2.23) or (2.24) show that

$$(\partial_t\tilde{p}, \partial_t\tilde{n}) \rightharpoonup (\partial_t p, \partial_t n) \tag{2.26}$$

weakly in $L^2(0, T; V_N')$ or in $L^2(0, T; V')$, depending on the assumptions (those of Theorem 1.2 or those of Theorem 1.3). Finally, $\tilde{u} - U \rightharpoonup u - U$ weakly in $L^2(0, T; V)$ (weakly-$*$ in $L^\infty(0, T; V)$).

Next, consider the sequence of intervals of time $[0, m]$, $m \in \mathbb{N}$. Choose first a subsequence $(\tilde{p}, \tilde{n}, \tilde{u})$ that converges, as explained above, with respect to the interval $[0, 1]$. Denote this subsequence by $(p_{k_1}^{(1)}, n_{k_1}^{(1)}, u_{k_1}^{(1)})$, where k_1 is a strictly increasing sequence of positive integers. Then pick up a subsequence $(p_{k_2}^{(2)}, n_{k_2}^{(2)}, u_{k_2}^{(2)})$ of the previous sequence, that converges with respect to the interval $[0, 2]$. And so on. The diagonal sequence $(p_{k_m}^{(m)}, n_{k_m}^{(m)}, u_{k_m}^{(m)})$, $m = 1, 2, \ldots$, is convergent (with respect to the above topologies) on each fixed interval $[0, T]$, to a well-defined limit (p, n, u) in $[0, +\infty)$. It is now a standard exercise to verify that the limit (p, n, u) is the desired solution (in Q_∞) of our problem. In particular, this limit satisfies the global estimate (2.3) and the estimate (2.12).

Nevertheless, let us make some remarks on the way to passing to the limit as $k \to +\infty$. We denote by $(\tilde{p}, \tilde{n}, \tilde{u})$ the subsequence constructed above. Just for convenience, we set here $D_1 = D_2 = \mu_1 = \mu_2 = 1$. Let v be a "testing-function" belonging to $L^2(0, T; V_{N+\epsilon})$, $\epsilon > 0$. One has

$$-\langle \nabla \cdot (\nabla \tilde{p} + \tilde{p}\nabla \tilde{u}), v\rangle = \int_0^T (\nabla \tilde{p} + \tilde{p}\nabla \tilde{u}, \nabla v)\, dt.$$

Clearly,

$$\left|\int_0^T ((\tilde{p} - p)\nabla \tilde{u})\, dt\right| \le \|\tilde{p} - p\|_{q,2;T} \|\nabla \tilde{u}\|_{2,\infty;T} \|\nabla v\|_{N+\epsilon,2;T},$$

where $1/q = (1/2) - 1/(N + \epsilon)$. Hence the right-hand side of the above inequality goes to zero as $k \to +\infty$ (k runs over a subsequence of the positive integers).

Since $(\tilde{p}\nabla \tilde{u}, \nabla v) = ((\tilde{p} - p)\nabla \tilde{u}, \nabla v) + (p\nabla \tilde{u}, \nabla v)$, it readily follows that

$$\lim_{k \to +\infty} \int_0^T (\nabla \tilde{p} + \tilde{p}\nabla \tilde{u}, v)\, dt = \int_0^T (\nabla p + p\nabla u, v)\, dt. \tag{2.27}$$

By using (2.25) or (2.26), (2.27), and the almost-everywhere convergence of $R(\tilde{p}, \tilde{n})$ to $R(p, m)$ together with (2.18) or (2.19), one shows that the limit (p, n, u) satisfies the equation $(1.1)_1$ in $L^2(0, T; V'_{N+\epsilon})$, for each fixed $T > 0$. Since in the equation $(1.1)_1$ each term belongs to $L^2(0, T; V'_N)$, the equation holds in this space. Under the hypothesis of Theorem 1.3 each term belongs to $L^2(0, T; V')$, hence the equation holds in this space. Similar results hold for the equation $(1.1)_2$ (passing to the limit in equation $(1.1)_3$ is obvious).

Next, $(1.2)_1$ is satisfied in the sense that $p - \phi$ and $n - \psi$ belong to $L^2_{\text{loc}}(0, +\infty; V)$, and similarly for $(1.3)_1$. The boundary conditions of the Neumann type are satisfied in the usual weak (variational) sense, since the text functions v belong to $L^2(0, T; V_N)$ or to $L^2(0, T; V)$, hence are "free" on B.

Finally, by setting in the lemma at the end of page 5024 in reference [1], $A_0 = H^1$, $A_1 = V'_N$ (or V'), $p_0 = p_1 = 2$, $m = 1$, $j = 0$, one shows that $\tilde{p}$ converges to p in $C(0, T; V'_N)$, as $k \to +\infty$. Since $\tilde{p}(0) = \min\{p_0, k\}$ it follows that $\tilde{p}(0)$ converges to p_0 in $L^2(\Omega)$, hence in V'_N. It follows that $p(0) = p_0$. Similarly, we show that $n(0) = n_0$. Note that p and n belong to $C(0, +\infty; V'_N)$ and that, under the assumptions of Theorem 1.3 they belong to $C(0, +\infty; V'_N)$ and that, under the assumptions of Theorem 1.3 they belong to $C(0, +\infty; L^2(\Omega))$. See [14, Chapter 1, Section 3, Theorem 3.1 and Section 2 Proposition 2.1].

Uniqueness. Here, we prove Theorem 1.4. Without loss of generality, we set $D_1 = D_2 = \mu_1 = \mu_2 = 1$. By taking the difference between the equation $(1.1)_1$ written for a solution (p, n, u) and the same equation written for a solution (q, m, v) one gets

$$\frac{\partial (p - q)}{\partial t} - \nabla \cdot [\nabla (p - q) + (p - q)\nabla u + q\nabla (u - v)] = R(p, n) - R(q, m),$$

which is satisfied in the sense of $L^2(0, T; V')$ due, in particular, to the fact that $\nabla u \in L^\infty(0, T; L^N)$. We consider below the case $N \neq 2$ and leave to the reader the small modifications to be done if $N = 2$. In general, the modification consists in replacing N by $N_1 (N_1 > 2)$ and 2^* by a suitable, sufficiently large, real number.

Since $p - q \in L^2(0, T; V)$ one gets from the above equation

$$\frac{1}{2}\frac{d}{dt}\|p-q\|^2 + \|\nabla(p-q)\|^2 + \int (p-q)\nabla u \cdot \nabla(p-q)\,dx + \int q\nabla(u-v)\cdot\nabla(p-q)\,dx = \int [R(p,n)-R(q,m)](p-q)\,dx. \tag{2.28}$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\|n-m\|^2 + \|\nabla(n-m)\|^2 - \int (n-m)\nabla u \cdot \nabla(n-m)\,dx - \int m\nabla(u-v)\cdot\nabla(n-m)\,dx = \int [R(p,n)-R(q,m)](n-m)\,dx. \tag{2.29}$$

Moreover,

$$a\int \nabla(u-v)\cdot\nabla w\,dx = \int [(p-q)-(n-m)]w\ dx, \tag{2.30}$$

for each $w \in V$. On the other hand

$$\int (p-q)\nabla u\cdot\nabla(p-q)\,dx = \frac{1}{2a}\int (f+p-n)(p-q)^2\,dx.$$

Note that $p - q \in L^2(0, T; L^{2^*})$, $\nabla u \in L^\infty(0, T; L^N)$ and $\nabla(p-q) \in L^2(0, T; L^2)$.

Since $V \hookrightarrow L^{2^*}$ one has

$$\Big|\int_\Omega (p-n)(p-q)^2\,dx\Big| \le \|p-n\|_N\|p-q\|_{2^*}\|p-q\| \le C_\epsilon\|p-n\|_N^2\|p-q\|^2 + \epsilon\|\nabla(p-q)\|^2, \tag{2.31}$$

for an arbitrary $\epsilon > 0$. Moreover $\|p-n\|_N^2 \in L^1(0, T)$ since $N \le 4$. Similarly,

$$\Big|\int f(p-q)^2\,dx\Big| \le C_\epsilon\|f\|_N^2\|p-q\|^2 + \epsilon\|\nabla(p-q)\|^2; \tag{2.32}$$

moreover, $\|f\|_N^2 \in L^1(0, T)$.

Next, by using (2.28), and by taking into account that $\nabla(u - v) \in L^\infty(0, T; L^N)$ (by the assumption (1.18)), one has

$$\int \nabla(u-v)\cdot\nabla[q(p-q)]\,dx = \frac{1}{a}\int [(p-q)-(n-m)]q(p-q)\,dx.$$

Hence,

$$\int q\nabla(u-v)\cdot\nabla(p-q)\,dx = -\int \nabla q\cdot\nabla(u-v)(p-q)\,dx$$
$$+\frac{1}{a}\int q[(p-q)-(n-m)](p-q)\,dx = I_1 + I_2. \tag{2.33}$$

Since $|I_1| \le \|\nabla q\|_{2^*}\|\nabla(u-v)\|_N\|p-q\|$ it readily follows, by the assumption (1.18), that

$$|I_1| \le c\|\nabla q\|_{2^*}\left(\|p-q\|^2 + \|n-m\|^2\right), \tag{2.34}$$

where $\|\nabla q\|_{2^*} \in L^2(0,T)$. On the other hand

$$|I_2| \le \|q\|_N\|p-q\|_{2^*}\left(\|p-q\| + \|n-m\|\right).$$

Hence

$$|I_2| \le C_\epsilon\|q\|_N^2\left(\|p-q\|^2 + \|n-m\|^2\right) + \epsilon\|\nabla(p-q)\|^2;$$

moreover, $\|q\|_N^2 \in L^1(0,T)$. From this last estimate, and from (2.33) and (2.34), one gets

$$\left|\int q\nabla(u-v)\cdot\nabla(p-q)\,dx\right| \le h(t)\left(\|p-q\|^2 + \|n-m\|^2\right) + \epsilon\|\nabla(p-q)\|^2, \tag{2.35}$$

where $h(t) \in L^1(0,T)$. Finally, (1.19) shows that the absolute value of the integral on the right-hand side of (2.28) is bounded by

$$c\left(1 + \|(p,q,n,m)\|_{\beta N}\right)^\beta\left(\|p-q\|_{2^*} + \|n-m\|_{2^*}\right)\left(\|p-q\| + \|n-m\|\right),$$

hence by

$$C_\epsilon(1 + \|(p,q,n,m)\|_{\beta N})^{2\beta}(\|p-q\|^2 + \|n-m\|^2) + \epsilon(\|\nabla(p-q)\|^2 + \|\nabla(n-m)\|^2).$$

Since $\beta = 4/N$ (when $N \ne 2$), this shows that

$$\left|\int [R(p,n) - R(q,m)](p-q)\,dx\right| \le h(t)\left(\|p-q\|^2 + \|n-m\|^2\right)$$
$$+\epsilon\left(\|\nabla(p-q)\|^2 + \|\nabla(n-m)\|^2\right),$$

where $h(t) \in L^1(0,T)$.

From (2.28), together with (2.31), (2.32), (2.35), (2.36), and also from (2.29), together with estimates (for $n-m$, etc.) similar to (2.31), (2.32), (2.35), (2.36), one gets

$$\frac{1}{2}\frac{d}{dt}\left(\|(p-q)\|^2 + \|(n-m)\|^2\right) \le h(t)\left(\|(p-q)\|^2 + \|(n-m)\|^2\right),$$

where $h(t) \in L^1(0, T)$. This proves (1.21). □

Finally if f belongs to $L^\infty(0, T; L^2)$ the right-hand side of $(1.1)_3$ belongs to the same space. The assumption (1.16) shows that the conclusion of Proposition 1.5 holds.

REFERENCES

[1] J.P. Aubin, *Un théorème de compacité*, C.R. Acad. Sc. Paris, t. 256 (1963), 5042–5044.

[2] H. Beirão da Veiga, *Remarks on the flow of holes and electrons in crystalline semiconductors*, "Navier-Stokes equations and related non-linear problems," ed. Adélia Sequeira, Plenum Press (1995), to appear.

[3] H. Beirão da Veiga, *On some diffusion equations in semiconductor theory*, C.R. Acad. Sc. Paris, t. 320, série I (1995).

[4] R. Dautray and J.L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Volumes 2 and 5, Springer-Verlag, Berlin, 1992.

[5] K. Gröger, *On the boundedness of solutions to the basic equations in semiconductors theory*, Math. Nachr., 129 (1986), 167–174.

[6] K. Gröger, *A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann., 283 (1989), 679–687.

[7] H. Gajewski and K. Gröger, *On the basic equations for carrier transport in semiconductors*, J. Math. Anal. Appl., 113 (1986), 12–35.

[8] H. Gajewski and K. Gröger, *Semiconductor equations for variable mobilities based on Boltzmann statistics for Fermi-Dirac statistics*, Math. Nachr., 140 (1989), 7–36.

[9] P. Grisvard, "Singularities in Boundary Value Problems," Research Notes in Appl. Math., 22, Masson and Springer-Verlag, 1992.

[10] A. Lorenzi, *A mixed boundary value problem for the Laplace equation in an angle* (1^{st}*·part*), Rend. Sem. Mat. Univ. Padova, 54 (1975), 147–183.

[11] O.A. Ladyzhenskaya, "The Boundary Value Problems of Mathematical Physics," Springer-Verlag, New York, 1985 (original Russian edition: Nauka 1973).

[12] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Amer. Math. Soc., Providence, 1968 (original russian edition: Moscow 1967).

[13] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[14] J.L. Lions and E. Manages, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.

[15] P.A. Markowich, C.A. Ringhofer and C. Schmeiser, "Semiconductor Equations," Springer-Verlag, Wien, 1990.

[16] M.S. Mock, *An initial value problem for semiconductor device theory*, SIAM J. Math. Anal., 5 (1974), 597–612.

[17] M.S. Mock, *Asymptotic behaviour of solutions of transport equations for semiconductor devices*, J. Math. Anal. Appl., 49 (1975), 215–225.

[18] J.L. Moll, "Physics of Semiconductors," McGraw-Hill, New York, 1964.

[19] E. Shamir, *Regularization of mixed second-order elliptic problems*, Israel J. of Math., 6 (1968), 150–168.

[20] W. Van Roosbroeck, *Theory of the flow of electrons and holes in germanium and other semiconductors*, Bell. Sys. Tech. J., 29 (1950), 560–607.