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Équations aux dérivées partielles/*Partial Differential Equations*

On some diffusion equations in semiconductor theory

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Abstract – We show that the solutions $p(t, x)$, $n(t, x)$ to the initial-boundary value problem (1), (2), (3) are globally bounded if the initial data are bounded. If the initial data belong merely to $L^2(\Omega)$ then p and n are continuous with values in $L^2(\Omega)$ and satisfy the global estimate (11), where C_0, c, ν does not depend on the particular initial data.

Sur certaines équations de diffusion en théorie des semiconducteurs

Résumé – *On démontre que les solutions* $p(t, x)$, $n(t, x)$ *du problème aux limites* (1), (2), (3) *sont globalement bornées si les données initiales sont bornées. Si celles-ci appartiennent seulement à* $L^2(\Omega)$, *alors p et n sont continues à valeurs dans* $L^2(\Omega)$ *et vérifient l'estimation globale* (11), *où les constantes* C_0, c, ν *ne dépendent pas des données initiales particulières.*

Version française abrégée – On considère le système d'équations aux dérivées partielles

$$(1)\qquad \begin{cases} \partial p/\partial t - \nabla\cdot(D_1 \nabla p + \mu_1 p \nabla u) = R(p, n), \\ \partial n/\partial t - \nabla\cdot(D_2 \nabla n - \mu_2 n \nabla u) = R(p, n), \\ -\nabla\cdot(a \nabla u) = f + p - n \qquad \text{dans} \quad \mathbb{R}_+ \times \Omega, \end{cases}$$

avec les conditions aux limites

$$(2)\qquad \begin{cases} p = \phi(x), \qquad n = \psi(x), \qquad u = U(x) \quad \text{dans} \quad \mathbb{R}_+ \times D, \\ \partial p/\partial \nu = \partial n/\partial \nu = \partial u/\partial \nu = 0 \qquad \text{dans} \quad \mathbb{R}_+ \times B, \end{cases}$$

et les conditions initiales

$$(3)\qquad p(0, x) = p_0(x), \qquad n(0, x) = n_0(x).$$

Ce système décrit le transport de lacunes et d'électrons dans un semiconducteur. Ω est un ouvert de $\mathbb{R}^N$, $N \geqq 2$, de frontière lipschitzienne Γ. On désigne par ν la normale extérieure. D est un sous-ensemble fermé de Γ et $B = \Gamma/D$. Par commodité on suppose ici que $R(p, n)$ est donnée par l'expression classique

$$(4)\qquad R(p, n) = (1 - pn)/(r_0 + r_1 p + r_2 n),$$

où r_0, r_1 et r_2 sont des constantes positives. Pour une discussion des équations ci-dessus *voir* [8]. On suppose que ϕ, ψ, U sont les traces sur D de fonctions (désignées par les mêmes symboles) qui appartiennent à $H^1(\Omega) \cap L^\infty(\Omega)$. De plus, ϕ et ψ sont non négatives. Par commodité on suppose ici que $f \in L^\infty(Q)$ où Q désigne le cylindre $]0, +\infty[\times\Omega$. $|||\cdot|||_\infty$ désigne la norme dans l'espace $L^\infty(Q)$. On pourrait aussi supposer sans introduire des difficultés supplémentaires que ϕ, ψ et U, bornées, dépendent aussi de t.

Sous les hypothèses ci-dessus il existe une solution (p, n) du problème (1), (2), (3), telle que $p \geqq 0$, $n \geqq 0$, $p - \phi$, $n - \psi \in L^2_{\text{loc}}(0, +\infty; V)$, p_t, $n_t \in L^2_{\text{loc}}(0, \infty; V')$ et p, $n \in L^\infty_{\text{loc}}(0, +\infty; L^\infty(\Omega))$. Ici V désigne l'espace $\{w \in H^1 : w = 0 \text{ sur } D\}$ et V' est son espace dual. On a aussi $u - U \in L^\infty(0, T; V)$. La solution est unique dans la classe ci-dessus.

Note présentée par Haïm Brezis.

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On pose

$$M = \max\{\|p_0\|_\infty, \|n_0\|_\infty, \|\phi\|_\infty, \|\psi\|_\infty, 1\}.$$

Dans [1] on démontre le résultat suivant:

THÉORÈME 1. – *Sous les hypothèses ci-dessus on a*

$$|||p|||_\infty, |||q|||_\infty \leqq C\,M\,(1 + |||f|||_\infty^{N+2})$$

avec une constante C *qui dépend seulement de* N, a, D_1, D_2, μ_1, μ_2 *et* $\{\Omega, D, B\}$. *Si* $N = 2$ *on doit remplacer* $(N+2)$ *par* 3.

Dans le cas stationnaire on a le résultat suivant:

THÉORÈME 2. – *Soient* ϕ, ψ, $U \in H^1(\Omega) \cap L^\infty(\Omega)$, *avec* $\phi \geqq 0$, $\psi \geqq 0$ *et soit* $f \in L^r(\Omega)$, *avec* $r \in]2N, +\infty]$. *Alors le problème stationnaire* (1), (2) *admet une solution* (p, n, u) *avec* $p - \phi$, $n - \psi$, $u - U \in V$ *et* $p \geqq 0$, $n \geqq 0$. *De plus,* p *et* n *sont bornées dans* Ω *et vérifient la majoration*

$$\|p\|_\infty, \|q\|_\infty \leqq C\,M\,(1 + \|f\|_p^{2(1+1/\chi)}),$$

où $\chi = (2/N) - (4/r)$ *si* $N \geqq 3$ *et* $\chi = (1/2) - (2/r)$ *si* $N = 2$. *Ici* $M = \max\{\|\phi_0\|_\infty, \|\psi_0\|_\infty, 1\}$.

Dans la suite on va supposer que $N \leqq 3$. Soient $g \in L^2(\Omega)$ et $U \in H^1(\Omega) \cap L^\infty(\Omega)$ et soit $u \in H^1(\Omega)$ la solution variationnelle du problème elliptique

$$-\Delta u = g \quad \text{dans } \Omega, \qquad u = U \quad \text{sur } D, \qquad \partial u/\partial \nu = 0 \quad \text{sur } B.$$

On suppose que $\{\Omega, D, B\}$ soit tel que: il existe un espace fonctionnel Y et un exposant réel q ($q > 2$ si $N = 2$, $q = 3$ si $N = 3$) tels que si $g \in L^2(\Omega)$ et $U \in Y$, alors $\nabla u \in L^q(\Omega)$ et

$$\|\nabla u\|_q \leqq c\,(\|g\| + \|U\|_Y).$$

On désigne par $\|\cdot\|$ la norme dans $L^2(\Omega)$. On remarque que le théorème 3 ci-dessous est valable pour N entier quelconque si la propriété ci-dessus est vérifiée avec $q = N$. Mais on voit aisément qu'elle est fausse si $N > 4$. En revanche, si $N \leqq 3$, cette propriété est valable dans presque toutes les situations qui apparaissent dans les applications.

Dans l'article [2] on démontre le résultat suivant [existence d'un attracteur global dans $L^2(\Omega)$]:

THÉORÈME 3. – *Supposons que les hypothèses ci-dessus soient vérifiées. Alors, si* p_0, $n_0 \in L^2(\Omega)$, *il existe une solution unique* (p, n) *du problème* (1)-(3) *dans les espaces fonctionnels indiqués avant le théorème* 1. *Cette solution est continue dans* $[0, +\infty[$ *à valeurs dans* $L^2(\Omega)$ *et vérifie l'estimation globale*

$$\|p(t)\|^2 + \|n(t)\|^2 \leqq C_0 + c\,e^{-\nu t}(\|p_0\|^2 + \|n_0\|^2)$$

pour chaque $t \geqq 0$. *Ici la constante* C_0 *dépend de* $\|\phi\|_\infty$, $\|\psi\|_\infty$, $|||f|||_\infty$, *mais pas de* p_0, n_0 *et* U. *Les constantes* c *et* ν *ne dépendent pas des données* ϕ, ψ, U, f, p_0, *et* n_0.

We study the following system of nonlinear partial differential equations describing the transport of holes and electrons in a semiconductor device

$$(1) \qquad \begin{cases} \partial p/\partial t - \nabla \cdot (D_1 \nabla p + \mu_1 p \nabla u) = R(p, n), \\ \partial n/\partial t - \nabla \cdot (D_2 \nabla n - \mu_2 n \nabla u) = R(p, n), \\ -\nabla \cdot (a \nabla u) = f + p - n \qquad \text{in } \mathbb{R}_+ \times \Omega, \end{cases}$$

with boundary conditions

$$(2)\qquad \begin{cases} p = \phi(x), \quad n = \psi(x), \quad u = U(x) & \text{in} \quad \mathbb{R}_+ \times D, \\ \partial p/\partial \nu = \partial n/\partial \nu = \partial u/\partial \nu = 0 & \text{on} \quad \mathbb{R}_+ \times B, \end{cases}$$

and initial conditions

$$(3)\qquad p(0, x) = p_0(x), \qquad n(0, x) = n_0(x).$$

Here, Ω is a bounded Lipschitzian domain in $\mathbb{R}^N$. We assume that the boundary Γ of Ω is the union of two disjoint sets D and B, where the closed set D has non vanishing $(N-1)$-dimensional measure. We denote by ν the unit outward normal to Γ. The unknowns u, p, and n denote the electrostatic potential, the free hole carrier concentration and the free electron carrier concentration. f is the net density of ionized impurities and $R(p, n)$ the recombination term. The solutions $p(x, t)$ and $n(x, t)$ are required to be nonegative. We do not assume that the hole and electron diffusion coefficients D_1 and D_2 and the hole and electron mobilities μ_1 and μ_2 are (necessarily) connected by the Einstein relations $D_i = (k\,\vartheta_0)\,\mu_i$, $i = 1, 2$, where k is the Boltzmann's constant and ϑ_0 the constant temperature. For convenience we assume that D_1, D_2, μ_1, μ_2 and a (the dielectric permitivity) are positive constants. This leads to equations (1). For a more detailed description of the model *see* [8] and references therein.

The recombination term $R(p, n)$ is a locally Lipschitz continuous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\lim_{p+n \to +\infty} \frac{R(p, n)^+}{p+n} = 0,$$

where $z^+ = \max\{z, 0\}$, moreover $R(p, 0) \geqq 0$ for each $p \geqq 0$ and $R(0, n) \geqq 0$ for each $n \geqq 0$. We point out that these assumptions hold for the Shockley-Read-Hall recombination term

$$(4)\qquad R(p, n) = \frac{1 - pn}{r_0 + r_1\,p + r_2\,n}$$

where r_0, r_1 and r_2 are positive constants.

Under suitable hypotheses Gajewski and Gröger [3] show that the solutions p, n of the above problem belong to $L^\infty(0, T; L^\infty(\Omega))$ for each fixed T. However, the $L^\infty(\Omega)$-norm of $p(t)$ and $n(t)$ may blow up, at most exponentially, as t goes to $+\infty$. For previous, related results, *see* references in [3]. A main open question, in order to approach the problem of the qualitative behaviour of solutions for large values of t, is to know whether the solutions are uniformly bounded in $Q =]0, +\infty[\times\Omega$. This is the main concern in reference [1], where we prove theorem 1 below (a partial result was obtained by Gröger [5]. This author exhibits a sufficient condition in order that the above result holds. For the Shockley-Read-Hall recombination term (4) Gröger's condition corresponds to a smallness assumption on f).

Before stating our main theorem we introduce some notations. We set $Q =]0, +\infty[\times\Omega$ and we denote by $\|\cdot\|_q$ the norm in $L^q(\Omega)$ and by $|||\cdot|||_\infty$ that in $L^\infty(Q)$. We set $\|\cdot\| = \|\cdot\|_2$. We denote by V the Hilbert space $V = \{w \in H^1(\Omega) : w = 0 \text{ on } D\}$ and by V' its dual space. We set

$$M = \max\{\|p_0\|_\infty, \|n_0\|_\infty, \|\phi\|_\infty, \|\psi\|_\infty, 1\}.$$

We assume that ϕ, ψ, U are the traces on D of functions (denoted by the same symbols) that belong to $H^1(\Omega) \cap L^\infty(\Omega)$. Moreover, ϕ and ψ are non-negative. For simplicity, we assume here that $f \in L^\infty(Q)$ and that $R(p, n)$ is given by (4). We point out that our proofs and results are valid if ϕ, ψ and U are time-dependent and bounded in $]0, +\infty[\times D$, moreover ϕ_t, $\psi_t \in L^2_{\text{loc}}(0, +\infty; V')$.

Under the above hypotheses problem (1)-(3) has a unique solution $p \geqq 0$, $n \geqq 0$, u, such that $p - \phi$ and $n - \psi$ belong to $L^2_{\text{loc}}(0, +\infty; V)$, p_t and n_t belong to $L^2_{\text{loc}}(0, +\infty; V')$ and $u - U \in L^\infty(0, +\infty; V)$. Our main result in reference [1] is the following:

THEOREM 1. – *Under the above assumptions the solution (p, n) of problem* (1)-(3) *is bounded in Q. More precisely*

$$|||p|||_\infty,\ |||q|||_\infty \leqq C\, M\, (1 + |||f|||_\infty^{N+2}) \tag{5}$$

where the constant C depends only on N, a, D_1, D_2, μ_1, μ_2 and on $\{\Omega, B, D\}$. If $N = 2$, one has to replace $N + 2$ by 3.

For the stationary problem one has the following result.

THEOREM 2. – *Let ϕ, ψ, $U \in H^1(\Omega) \cap L^\infty(\Omega)$, with $\phi \geqq 0$, $\psi \geqq 0$ and let $f \in L^r(\Omega)$, $r \in]2N, +\infty[$. Then, the stationary problem* (1)-(2) *has a solution (p, n, u) such that $p - \phi$, $n - \psi$, $u - U \in V$; moreover $p \geqq 0$, $n \geqq 0$. Furthermore, p and n are bounded in Ω and satisfy the estimate*

$$\|p\|_\infty,\ \|n\|_\infty \leqq C\, M\, (1 + \|f\|_r^{2(1+1/\chi)}), \tag{6}$$

where $\chi = (2/N) - (4/r)$ if $N \geqq 3$, $\chi = (1/2) - (2/r)$ if $N = 2$. Here, $M = \max\{\|\phi_0\|_\infty, \|\psi_0\|_\infty, 1\}$.

A second basic question in order to study the asymptotic behaviour of the set of solutions is that of the existence (or non existence) of a (significant) functional space X and of a bounded set $B_0 \subset X$ that attracts (uniformly) each bounded subset B of X. We prove in reference [2] that this property holds for $X = L^2(\Omega)$. In order to prove this result, the first step consists in showing the existence and the uniqueness of a solution

$$p,\ n \in C(0, +\infty; L^2(\Omega)) \tag{7}$$

corresponding to each (arbitrary) pair of initial data p_0, $n_0 \in L^2(\Omega)$.

We prove the existence of a weak solution (p, n) to our problem in the class $p - \phi$, $n - \psi \in L^2_{\text{loc}}(0, +\infty; V)$, p_t, $n_t \in L^2_{\text{loc}}(0, +\infty; V'_N)$, where V'_N is the dual space of $V_N = \{v \in V : \nabla v \in L^N(\Omega)\}$; if $N = 2$ replace N by q, $q > 2$. In order to prove the uniqueness of the solution and also that p_t, $n_t \in L^2_{\text{loc}}(0, +\infty; V')$ [hence that (7) holds] we assume the property described below. Consider the elliptic mixed boundary value problem

$$-\Delta u = g \quad \text{in } \Omega, \qquad u = U \quad \text{on } D, \qquad \partial u/\partial \nu = 0 \quad \text{on } B. \tag{8}$$

We assume that there is a functional space Y and a real q ($q > 2$ if $N = 2$; $q = N$ otherwise) such that if $g \in L^2(\Omega)$ and $U \in Y$ then the variational solution u of problem (8) satisfies

$$\|\nabla u\|_q \leqq c\,(\|g\| + \|U\|_Y). \tag{9}$$

Note that this is an assumption on $\{\Omega, B, D\}$. This assumption is never satisfied if $N > 4$ since $H^1(\Omega)$ is not contained in $L^N(\Omega)$. If $N = 2$ it holds if Ω is a bounded domain with a polygonal boundary (or a regular transformation of such a set). In this case $q > 2$ can be arbitrarily fixed; moreover $Y = W^{1-(1/q), q}(B)$. This follows essentially from results by Lorenzi [7]. Since it is sufficient to have (9) for some $q > 2$, it seems possible to use Gröger's results [6].

If $N = 3$ and if Ω is a bounded convex set with a polyhedral boundary (or a regular transformation of it) then the solution of problem (8) belongs to $H^{3/2}(\Omega)$. This follows from results of Grisvard [4], at least if $U = 0$. Note that $\nabla u \in L^3(\Omega)$ since $H^{3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)$. It is worth noting that in [4] the author considers only homogeneous boundary conditions, but this seems inessential there. We also note that $W^{1,3}$-regularity holds under weaker hypotheses on the angles between faces than that needed to get $H^{3/2}$-regularity. But we do not know about precise statements in the literature.

One has the following result.

THEOREM 3. – *Let the assumption* (9) *hold and let* ϕ, ψ, U, f *and* $R(p, n)$ *be as in theorem* 1, *with* $U \in Y$. *Then, to each pair of initial data* $(p_0, n_0) \in L^2(\Omega)$ *corresponds a unique solution* (p, n) *of problem* (1)-(3) *in the class* $p - \phi, n - \psi \in L^2_{\text{loc}}(0, +\infty; V)$; $p_t, n_t \in L^2_{\text{loc}}(0, +\infty; V')$. *Moreover,* $p, n \in C(0, +\infty; L^2(\Omega))$ *and there is a positive constant* C_0 *(which depends on the norms* $\|\phi\|_\infty$, $\|\psi\|_\infty$, $|||f|||_\infty$ *but not on* p_0, n_0 *and* U*) such that*

$$\|p(t)\|^2 + \|n(t)\|^2 \leqq C_0 + c\,e^{-\nu t}(\|p_0\|^2 + \|n_0\|^2), \tag{10}$$

for each $t \geqq 0$. *The positive constants* c *and* ν *are independent of the data* ϕ, ψ, U, f, p_0 *and* n_0.

The above result shows that the set

$$B_0 = \{(\bar{p}, \bar{n}) \in L^2(\Omega) : \bar{p} \geqq 0,\ \bar{n} \geqq 0.\ \|\bar{p}\|^2 + \|\bar{n}\|^2 \leqq C_0\}$$

is a global bounded attractor in the space $L^2(\Omega)$.

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