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On the Periodic Solutions to the Kirchhoff-Bernstein Nonlinear Wave Equation (**).

Abstract. – *We prove the existence of periodic solutions to a class of nonlinear wave equations which contains, as a particular case, a classical equation proposed by Kirchhoff and studied by Bernstein. The very elementary method used here gives, however, a concrete and significant description of the solutions. On applying our result to the linear case one gets all the periodic solutions to the problem under consideration.*

1. – Introduction.

In his *Vorlesungen über Mechanik* ([Ki], 1883), G. Kirchhoff proposed the equation

$$u_{tt} - \left(1 + \gamma^2 \int_{-\infty}^{+\infty} u_x^2\, dx\right) u_{xx} = 0\,, \qquad \gamma > 0\,, \tag{1}$$

to describe the transverse motion of an elastic stretched string. In 1939 S. Bernstein [Be] proved local solvability for regular initial data and global solvability for analytic data, for the Cauchy problem. After Bernstein's paper a large number of authors have studied the above equations. See, for instance, [AS], [Ca 1,2], [DS 1,2], [Di 1,2,3], [GH], [Li], [Na], [Ni], [Po]. However, the problem of the existence of global solutions for arbitrarily large infinitely differentiable initial data is still

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open. Here, we want to construct time-periodic solutions to the problem

$$(2)\qquad \begin{cases} u_{tt} - \left(\int\limits_{\Omega} |\nabla u|^2 dx \right) \Delta u = 0 & \text{in } Q\,, \\ Bu|_{\Sigma} = 0\,, \end{cases}$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$, and Γ is its boundary. Moreover, $Q \equiv \mathbb{R} \times \Omega$, $\Sigma \equiv \mathbb{R} \times \Gamma$, and $m \in C(\mathbb{R}_0^+ ; \mathbb{R})$. $\mathbb{R}_0^+$ denotes the set of nonegative reals and $C(X, Y)$ denotes the set of continuous functions on X with values in Y. We denote by $\mathbb{R}^+$ the set of the positive reals. In equation (2) $B u_{|\Sigma} = 0$ denotes a suitable, homogeneous, time independent, boundary condition. Examples are: The Dirichlet boundary condition $u_{|\Sigma} = 0$; The Neumann boundary condition $(\partial u / \partial \nu)_{|\Sigma} = 0$; The mixed Dirichlet-Neumann boundary condition $u_{|\Sigma_1} = 0$, $(\partial u / \partial \nu)_{|\Sigma_2} = 0$, where $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Sigma_i = \mathbb{R} \times \Gamma_i$, $i = 1, 2$; The space periodic case; The case $\Omega = \mathbb{R}_n$.

The solutions are constructed by following the elementary method of separation of variables plus suitable reflections. However, this construction will be not presented below since it is more convenient to check that the «solutions» obtained solve the equations rather than justify rigorously its construction, step by step.

We start by considering the following generalization of problem (2). Let A be a linear or nonlinear operator, whose domain $D(A)$ and range are sets of real functions $\Phi(x)$ defined on Ω. We assume that, for each $\phi \in D(A)$ and each $\mu \in \mathbb{R}$, $\mu\phi \in D(A)$ and $A(\mu\phi) = |\mu|^q (\operatorname{sign}\mu) A\phi$, for some $q \in \mathbb{R}^+$. In particular $A(-\phi) = -A\phi$. If ψ is a real function of the real variable t, we define

$$A(\psi\phi)(t, x) = |\psi(t)|^q (\operatorname{sign}\psi(t))(A\phi)(x)\,.$$

For instance, if A is the differential operator defined in equation (22) then $q = p - 1$ and $A(\Psi(t)\,\Phi(x)) = |\Psi(t)|^{p-2}\,\Psi(t)\,A\Phi(x)$.

In the sequel we denote by λ^2 and $\overline{\phi}$ a *fixed* positive eigenvalue and a *fixed* eigenfunction of the operator $-A$, i.e.,

$$(3)\qquad -A\overline{\phi} = \lambda^2 \overline{\phi}\,.$$

The argument below is developed for each *fixed* couple $(\lambda^2, \overline{\phi})$.

Next, let F_i, $i = 1, \ldots, k$, be real functionals defined on $D(A)$ satisfying

$$(4)\qquad F_1(\mu\phi) = |\mu|^{l_i} F_i(\phi)\,,$$

for each $\mu \in \mathbb{R}$ and each $\phi \in D(A)$. Here $l_i \in \mathbb{R}_0^+$, $i = 1, \dots, k$. For instance, if $F(\phi) = \int_\Omega |\nabla \Phi|^2 \, dx$ then $l = 2$. Hence, if Ψ is a real function of the real variable t, one has $F_1(\psi\overline{\phi})(t) = |\psi(t)|^{l_1} F_1(\overline{\phi})$. Finally, let $m \in C(\mathbb{R}_0^k; \mathbb{R})$ be given.

Consider the equation

$$(5) \qquad \begin{cases} u_{tt} - m(F_1[u], \dots, F_k[u]) A u = 0\,, \\ u(t) \in D(A)\,, \qquad \forall t \in \mathbb{R}\,. \end{cases}$$

«Boundary conditions», if some, are included in equation $(5)_2$. Define

$$(6) \qquad M(s) \equiv m(s^{l_1} F_1[\overline{\phi}], \dots, s^{l_k} F_k[\overline{\phi}])\,,$$

for each $s \in \mathbb{R}_0^+$. Note that $M(\cdot)$ depends on the particular eigenfunction $\overline{\phi}$. Next let a be a *fixed* positive real number such that

$$(7) \qquad \int_\varrho^a s^q M(s)\, ds > 0\,, \qquad \forall \varrho \in [0, a)\,,$$

and

$$(8) \qquad \tau \equiv \int_0^a \left(\int_\varrho^a s^q M(s)\, ds \right)^{-1/2} d\varrho < +\infty\,.$$

The assumption (7) is satisfied in «almost all» cases. The assumption (8) is satisfied whenever $\lim\limits_{\varrho \to a^-} M(\varrho)/(a-\varrho)^{1-\varepsilon} < +\infty$ for some $\varepsilon > 0$. Note that this holds if $M(a) \neq 0$. In particular, if $m(y) > 0$ for each $y = (y_1, \dots, y_k) \in (\mathbb{R}^+)^k$, then each $a \in \mathbb{R}^+$ satisfies (7) and (8).

Now, we show that to each fixed triplet $(\lambda^2, \overline{\phi}, a)$ it corresponds a periodic solution $u(t, x)$ to problem (5) with period given by (13). Define

$$(9) \qquad z(y) \equiv \int_0^y \left(\int_\varrho^a s^q M(s)\, ds \right)^{-1/2} d\varrho\,, \qquad \forall y \in [0, a]\,.$$

Clearly, $z \in C([0, a]) \cap C^2([0, a])$. Moreover,

$$(10) \qquad z'(y) = \left(\int_y^a s^q M(s)\, ds \right)^{-1/2}, \qquad \forall y \in [0, a)\,,$$

and $z'(a) = +\infty$. Let $y = g(z)$ $(z \in [0, \tau])$ be the inverse function to

$z(y)$. It readily follows that $g \in C^2([0,\tau])$ and that, for each $z \in [0,\tau]$,

$$\text{(11)} \qquad \begin{cases} g'(z) = \left(\int_{g(z)}^{a} s^q M(s)\, ds \right)^{1/2}, \\ g''(z) = -\frac{1}{2}(g(z))^q M(g(z)) . \end{cases}$$

Moreover, $g(0) = 0$, $g(\tau) = a$, $g'(z) > 0$ if $z \in [0, \tau)$, $g'(\tau) = 0$, $g''(0) = 0$, $g''(\tau) = -(1/2)\, a^q M(a)$. We extend the function g to the whole $\mathbb{R}$ as follows: first, define $g(\tau + z) = g(\tau - z)$, for each $z \in [\tau, 2\tau]$. Then set $g(z) = -g(-z)$ for each $z \in [-2\tau, 0]$. Finally, extend g to the whole $\mathbb{R}$ as a periodic function with period 4τ. Note that $g \in C^2(\mathbb{R})$.

The functions

$$\text{(12)} \qquad u(t,x) = \pm\, g(\sqrt{2}\,\lambda t)\,\overline{\phi}(x)$$

are periodic solutions to the problem (5) with period given by

$$\text{(13)} \qquad T = 4\tau/\sqrt{2}\,\lambda .$$

The verification is left to the reader. ■

EXAMPLES. Assume that Ω is an open, bounded, subset of $\mathbb{R}^n$ and consider the problem (2). Let λ_n^2 and $\phi_n(x)$ denote the (positive) eigenvalues and the corresponding eigenfunctions to the problem

$$\text{(14)} \qquad \begin{cases} -\Delta \phi_n = \lambda_n^2 \phi_n & \text{in } \Omega , \\ B\phi_{n|\Gamma} = 0 . \end{cases}$$

Assume, for convenience, that $\int_\Omega \phi_n \phi_m \, dx = \delta_{m,n}$. Here, $A = -\Delta$, $q = 1$, $k = 1$, $F[u] = \int_\Omega |\nabla u|^2 \, dx$, $l = 2$. Choose $\overline{\phi}_n(x) = \lambda_n^{-1} \phi_n(x)$. Assume that $m(s) \geq 0$ for each $s \in \mathbb{R}_0^+$, and that $m(b) > 0$ for some b.

The above construction shows that to each couple $(a, n) \in [b, +\infty) \times \mathbb{N}$ it correspond two time periodic solutions

$$\text{(15)} \qquad u_{a,n}(t,x) = \pm\, \lambda_n^{-1} g_a(\sqrt{2}\,\lambda_n t)\, \phi_n(x)$$

to problem (2), with period given by

$$\text{(16)} \qquad T_{a,n} \equiv \frac{4}{\sqrt{2}\,\lambda_n} \int_0^a \left(\int_\varrho^a s\, m(s^2)\, ds \right)^{-1/2} d\varrho .$$

The function $g_a(z)$ is defined as done above for $g(z)$ (see eq. (9)) by replacing $s^q M(s)$ by $s\, m(s^2)$.

It is worth noting that equation (15) furnishes *all* the periodic solutions to the linear wave equation. In fact, if we set $m(s) \equiv 1$, $\forall s \in \mathbb{R}_0^+$, the formulae (15) gives

$$u_{a,n}(t, x) = \pm a\lambda_n^{-1} \sin(\lambda_n t)\, \phi_n(x). \tag{17}$$

Note that $A = \pm\lambda_n^{-1} a$ is an arbitrary, non zero, real number. Variations of the parameter a give rise here to variations of the amplitude A but not to variations of the period $2\pi/\lambda_n$. On the contrary, for the Kirchhoff-Bernstein equation (i.e., $m(s) = 1 + \gamma^2 s$) the period

$$T_{a,n}^{(\gamma)} = 4\lambda_n^{-1} \int_0^1 (1 - s^2)^{-1/2} \left[1 + \frac{1}{2}\gamma^2 a^2 (1 + s^2)\right]^{-1/2} ds \tag{18}$$

depends on the parameter a. Note that (independently of γ) the period $T_{a,n}^{(\gamma)}$ runs from $2\pi/\lambda_n$ (which corresponds to the linear wave equation) to 0, as the parameter a runs from 0 to $+\infty$.

Another example is given by the problem

$$\begin{cases} u_{tt} - m\left(\int_\Omega u^2\, dx, \int_\Omega |\nabla u|^2\, dx\right) \Delta u = 0 & \text{in } Q, \\ Bu_{|\Sigma} = 0, \end{cases} \tag{19}$$

where $m \in C(\mathbb{R}_0^+ \times \mathbb{R}_0^+ ; \mathbb{R})$. In this case the above result holds if we replace $m(s^2)$ by $m(\lambda_n^{-2} s^2, s^2)$. Here, τ_a and $g_a(y)$ depend on n.

Finally, we consider an example in which the operator A is nonlinear. Let Ω be an open, bounded, subset of $\mathbb{R}^n$, $n \geq 1$, and let $p > 2n/(n+2)$. Consider the Dirichlet boundary value problem

$$\begin{cases} u_{tt} - m[u] \sum_{i=1}^n (|\nabla u|^{p-2} u_{x_i})_{x_i} = 0 & \text{in } Q, \\ u_{|\Sigma} = 0, \end{cases} \tag{20}$$

where

$$m[u] \equiv m\left(\int_\Omega |u|^{l_1} dx, \ldots, \int_\Omega |u|^{l_r} dx; \int_\Omega |\nabla u|^{l_{r+1}} dx, \ldots, \int_\Omega |\nabla u|^{l_k} dx\right). \tag{21}$$

We assume that $1 \leq l_j \leq p^* \equiv n\, p/(n-p)$ for $j = 1, \ldots, r$; and that $1 \leq l_j \leq p$ for $j = r+1, \ldots, k$. Above, $m(y)$ is a real continuous function over $(\mathbb{R}_0^+)^k$. Assume, for convenience, that $m(y)$ is positive if $y \in (\mathbb{R}^+)^k$.

Set

$$Au \equiv -\sum_{j=1}^{n} (|\nabla u|^{p-2} u_{x_j})_{x_j}, \tag{22}$$

and $D(A) = W_0^{1,p}(\Omega)$. Moreover, let

$$F_i[u] = \int_\Omega |u|^{l_i} dx \quad \text{if } i = 1, \dots, r,$$

$$F_i[u] = \int_\Omega |\nabla u|^{l_i} dx \quad \text{if } i = r+1, \dots, k.$$

Here, $q = p - 1$. The above general result applies. To each $a > 0$ and each couple $(\lambda^2, \overline{\phi})$ such that $-A\overline{\phi} = \lambda^2 \overline{\phi}$, $\overline{\phi} \in W_0^{1,p}(\Omega)$, it corresponds a couple of time-periodic solutions of problem (21), given by the above construction. Note that the eigenvalue problem $-A\overline{\phi} = \lambda^2 \overline{\phi}$ admits nontrivial solutions since the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is compact.

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