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On the existence of branches of time-periodic solutions to the nonlinear vibrating string equation*

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1 Introduction

We are interested in the construction of time-periodic solutions to the nonlinear wave equation

$$y_{tt} = (\hat{\sigma}(y_x))_x \,, \qquad 0 < x < 1 \,, \tag{1.1}$$

with boundary conditions

$$y(0,t) = y(1,t) = 0 \,, \qquad t \in \mathbf{R} \,, \tag{1.2}$$

where $\hat{\sigma}(\cdot)$ is an odd function. As remarked in [4], Keller and Ting [5] have shown that, if $\hat{\sigma}'(0) > 0$, there are no time periodic solutions to (1.1) (1.2) which branch smoothly from the solutions $y(x,t) = a\sin(\sqrt{\hat{\sigma}'(0)}n\pi(t+\phi))\sin n\pi x$ of the linear wave equation $y_{tt} = \hat{\sigma}'(0)y_{xx}$. In [4] Greenberg considers analytic perturbations to $\hat{\sigma}(\gamma) = \gamma^3$ of the form

$$\hat{\sigma}(\gamma) = \gamma^3\big(1 + \sum_{k=1}^{\infty} \sigma_k \gamma^{2k}\big) \,, \tag{1.3}$$

where the series converges in some neighborhood of $\gamma = 0$, and shows that there are time periodic solutions to (1.1) (1.2) which branch smoothly from small amplitude ($0 \le a \ll 1$), time periodic, standing wave solutions $y(x,t) = aA(t)U(x)$ to the equation $y_{tt} = ((y_x)^3)_x$ with boundary conditions (1.2). These solutions are of the form $y(x,t) = aA(t)U(x) + O(a^3)$, and the period of oscillation is the same as the period of $A(t)$; see [4].

Our aim is to extend Greenberg's result to the case in which

$$\hat{\sigma}(\gamma) = |\gamma|^{m-1}\gamma(1 + R(\gamma)) \tag{1.4}$$

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for arbitrary $m > 1$, since there are no specific physical reasons to consider only the case $m = 3$ (nevertheless, the proof presents some simpler features). It is also of interest to verify that m can take values arbitrarily close to the singular value $m = 1$. We also want to drop the analyticity assumption on R. We will assume that R is an even function of class $C^1(]-b,b[) \cap C^3(]-b,b[-\{0\})$ for some $b > 0$, such that

$$\begin{aligned} &|R(\gamma)| \le c|\gamma|^{1+\beta_0}, \quad |R'(\gamma)| \le c|\gamma|^{\beta_0}, \quad &&\text{if } |\gamma| < b, \\ &|R''(\gamma)| \le c|\gamma|^{\beta_0-1}, \quad |R'''(\gamma)| \le c|\gamma|^{\beta_0-2}, \quad &&\text{if } 0 < |\gamma| < b, \end{aligned} \tag{1.5}$$

where

$$\beta_0 = 2/(m-1) \text{ if } 1 < m \le 3; \qquad \beta_0 = (m-1)/2 \text{ if } m \ge 3. \tag{1.6}$$

Remark 1.1 *Note that $\beta_0 \ge 1$, moreover $\beta_0 = 1$ if and only if $m = 3$. Hence, in the case $m = 3$, our assumption is $R(\gamma) = O(\gamma^2)$; compare with (1.3). We have some evidence to see that the hypotheses on β_0 may be weakened. We conjecture that it is sufficient to assume that $\beta_0 > 1/(m-1)$ when $m \in (1,3]$ and that $\beta_0 > (m-2)/2$ when $m \ge 3$. For $m = 3$, both assumptions yield $\beta_0 > 1/2$.*

Our proof is strongly based on that of Greenberg in [4], to which we refer the reader. The general scheme of our proof follows that of this author. These facts are pointed out here *once and for all*. In the proof given in [4] there are, however, two points that need more stringent argumentation. See remarks 4.1 and 4.2 below. We are grateful to J.M.Greenberg for useful discussions on these points.

In order to study the problem (1.1), (1.2), one scales the variables by setting

$$y(x,t) = a^s w u(x, at/\nu), \tag{1.7}$$

where a and w are positive constants and $s = 2/(m-1)$. Then, equation (1.1) becames

$$\nu^{-2} u_{\tau\tau} = (wa^{2+s})^{-1} (\hat{\sigma}(a^s w u_x))_x \,. \tag{1.8}$$

By taking into account equation (1.4) one gets

$$u_{\tau\tau} = \nu^2 w^{m-1} (\sigma(u_x))_x \,, \tag{1.9}$$

where $\sigma(\gamma) = \sigma(\gamma; a)$ is given by

$$\sigma(\gamma) = |\gamma|^{m-1}\gamma\big[\,1 + R\big(a^{\frac{2}{m-1}} w\gamma\big)\,\big]\,. \tag{1.10}$$

The boundary conditions are

$$u(0,\tau) = u(1,\tau) = 0\,, \qquad \tau \in \mathbf{R}\,. \tag{1.11}$$

Let us now consider solutions of the problem (1.9) (1.10) in the rectangle $Q \equiv \{(x,\tau) \,:\, 0 < x < 1/2,\ 0 < \tau < \pi/2\}$ with boundary conditions

$$u(0,\tau) = u_x(1/2,\tau) = 0\,, \qquad 0 \le \tau \le \pi/2\,, \tag{1.12}$$

$$u(x,0) = u_\tau(x,\pi/2) = 0\,, \qquad 0 \le x \le 1/2\,. \tag{1.13}$$

Since $\sigma(\gamma)$ is an odd function it follows that, if u is a solution of (1.9) (1.12) (1.13) in Q, then the following extension of u (obtained by reflections, and denoted again by u) is a solution of (1.9) (1.11) on the whole plane $\mathbf{R}^2$. Set, for each $x \in [0, 1/2]$, $u(x,\tau) = u(x, \pi - \tau)$ if $\pi/2 \le \tau \le \pi$; $u(x,\tau) = -u(x,-\tau)$ if $-\pi \le \tau \le 0$. Finally, extend $u(x,\tau)$ to all real τ as a τ-periodic function of period 2π (for each $x \in [0, 1/2]$). Next, extend u to all x in $[-1, 1]$ by defining $u(x,\tau) = u(1-x,\tau)$ if $1/2 \le x \le 1$, $u(x,\tau) = -u(-x,\tau)$ if $-1 \le x \le 0$. Finally, extend u to all real x as a x-periodic function of period 2. Hence, our aim now becomes that of solving the problem (1.9) (1.12) (1.13) in Q, where σ is given by (1.10).

In the particular case in which the function R vanishes identically, the above problem, more precisely, the problem

$$\nu^{-2} u_{\tau\tau} = w_0^{m-1}((u_x)^m)_x\,, \qquad \text{in } Q\,, \tag{1.9$'$}$$

with boundary conditions (1.12), (1.13) admits the solution

$$u(x,\tau) = A_0(\tau) U_0(x) \tag{1.14}$$

where $A_0(\tau)$ and $U_0(x)$ are defined by

$$\int_0^{A_0(\tau)} \frac{da}{(1-a^{m+1})^{1/2}} = \frac{\nu\tau}{[(m+1)/2]^{1/2}}\,, \qquad 0 \le \tau \le \pi/2\,, \tag{1.15}$$

and

$$\int_0^{U_0(\tau)} \frac{du}{(1-u^2)^{1/(m+1)}} = \Big(\frac{m+1}{2m}\Big)^{\frac{1}{m+1}} w_0^{\frac{1-m}{m+1}} x\,, \quad 0 \le x \le 1/2. \tag{1.16}$$

By definition

$$w_0 = 2^{\frac{m+1}{1-m}} \Big(\frac{2m}{m+1}\Big)^{\frac{1}{1-m}} \Big(\int_0^1 \frac{du}{(1-u)^2)^{1/(m+1)}} \Big)^{\frac{m+1}{1-m}} \tag{1.17}$$

$$\nu = [2(m+1)]^{1/2} \pi^{-1} \int_0^1 \frac{da}{(1-a^{m+1})^{1/2}}\,. \tag{1.18}$$

Note that

$$A_0'' + \nu^2 A_0^m = 0\,; \qquad (A_0')^2 = \frac{\nu^2(1-A_0^{m+1})}{(m+1)/2}\,, \tag{1.19}$$

$$w_0^{m-1}\big[(U_0')^m\big]' + U_0 = 0\,; \quad w_0^{m-1}(U_0')^{m+1} = \frac{m+1}{2m}(1-U_0^2)\,. \tag{1.20}$$

The functions $A_0(\tau)$ and $U_0(x)$ can be extended to all of $\mathbf{R}$ by suitable reflections (similar to that done above for $u(x,\tau)$). After these extensions, the function A_0 is even and periodic of period 2π and U_0 is even of period 2.

The function $u(x,\tau)$ given by equation (1.14) is a standing wave solution of problem (1.9) on $\mathbf{R}^2$ and satisfies, in particular, the boundary conditions (1.11).

Note that $A_0(0) = 0$, $A_0(\pi/2) = 1$, $A_0' > 0$ in $[0, \pi/2[$, $U_0(0) = 0$, $U_0(1/2) = 1$, $U_0' > 0$ in $[0, 1/2[$. In particular, $0 \le u(x,\tau) \le 1$ and $u_x > 0$ in Q. In view of this last inequality, in equation $(1.9)'$ we write u_x^m instead of $|u_x|^{m-1}u_x$. We leave the verification of the above results to the reader. The solution (1.14) gives rise, by using (1.7), to a family of nontrivial periodic solutions

$$y_0(x,t) = a^{\frac{2}{m-1}} w_0 A_0(at/\nu) U_0(x)\,, \tag{1.21}$$

of problem

$$y_{tt} = \big(|y_x|^{m-1} y_x \big)_x\,, \qquad 0 < x < 1\,, \tag{1.22}$$

with boundary conditions (1.2). These functions are t-periodic of period $2\pi\nu/a$. In the sequel we prove the following result.

Theorem 1.1 *Let be $m > 1$. For each $a > 0$ the problem (1.22), (1.2) admits a time periodic, standing wave solution of the form (1.21), where ν is given by (1.18); the time period is $2\pi\nu/a$. Let $\hat\sigma$ be given by equation (1.4), where the even function R satisfies (1.5). Then, to each sufficiently small positive value of the amplitude a the problem (1.1), (1.2) admits a non-trivial time-periodic solution of the form*

$$y(x,t) = y_0(x,t) + O\big(a^{\frac{m+3}{m-1}}\big)\,. \tag{1.23}$$

The period of this solution is again $2\pi\nu/a$.

Remark 1.2 *The reader should note that the function in (1.14) is a solution of the problem $(1.9)'$ (1.12) (1.13) also when $0 < m < 1$. Moreover, this solution is more regular that those when $m > 1$. In fact $A_0(\tau)$ is always of class $C^2([0,\pi/2])$ but $U_0(x)$ is of class $C^2([0,1/2)]$ only if $0 < m < 1$. If $m > 1$, $U_0(x)$ belongs to the Hölder class $C^{1,1/m}([0,1/2])$.*

We did not check if the proofs are adaptable to the case in which $m < 1$. It would be interesting to consider this problem.

Remark 1.3 *Nontrivial periodic solutions, with fixed period ν, of the problem (1.1) (1.2) may be seek as critical points of the functional*

$$\int_0^\nu \int_0^1 \big[\, \frac{1}{2} y_t^2 - G(y_x) \,\big]\, dx\, dt\,, \qquad G'(s) \equiv \hat\sigma(s)\,.$$

However, this way gives much less information than the method followed here (it may be suitable for studying n-dimensional problems in general domains).

2 Auxiliary results

The proof of the existence of a branch of periodic solutions to problem (1.1) (1.2) is strongly based on an accurate study of the corresponding linear problem (2.28) (2.29). This study is the aim of this section.

Some of the results stated here, in the particular case $\vartheta = 1/2$ are due to Greenberg; see [4].

For convenience, we set

$$\vartheta = \frac{m-1}{m+1}. \tag{2.1}$$

Lemma 2.1 *The nontrivial solutions* $(\lambda^2, S_\lambda(\phi))$ *of the problem*

$$\begin{cases} (\cos^\vartheta \phi\, S'_\lambda(\phi))' + \lambda^2 \cos^\vartheta \phi\, S_\lambda(\phi) = 0\,, & 0 < \phi < \pi/2\,, \\ S'_\lambda(0) = \lim\limits_{\phi \to (\pi/2)-} \cos^\vartheta \phi\, S'_\lambda(\phi) = 0\,, \end{cases} \tag{2.2}$$

are given by

$$\lambda_m^2 = 4m^2 + 2\vartheta m\,, \qquad S_m(\phi) = \sum_{k=0}^{m} S_m^k \cos^{2k}\phi\,, \tag{2.3}$$

where $m = 0, 1, 2, \ldots$. *The coefficients* S_m^k *are defined inductively by*

$$S_m^k = \frac{-(k+1)(2k+\vartheta+1)}{(m-k)(2(m+k)+\vartheta)} S_m^{k+1}\,, \qquad k = 0, \ldots, m-1\,, \tag{2.4}$$

and S_m^m *is normalized so that*

$$\int_0^{\pi/2} S_m^2(\phi) \cos^\vartheta \phi\, d\phi = 1\,. \tag{2.5}$$

Furthermore, one has

$$\int_0^{\pi/2} S_m S_n \cos^\vartheta \phi\, d\phi = \delta_{m,n}\,; \qquad \int_0^{\pi/2} S'_m S'_n \cos^\vartheta \phi\, d\phi = \lambda_m^2 \delta_{m,n} \tag{2.6}$$

and

$$|S_m(\phi)| \le \frac{K_0}{\cos^\vartheta \phi}\,, \qquad |S'_m(\phi)| \le \frac{K_0 \lambda_m}{\cos^\vartheta \phi}\,, \tag{2.7}$$

where $K_0 = \left[(2/\pi)^{1-\vartheta}(2(1-\vartheta) + \vartheta\pi)\right]^{1/2}$.

The proofs of (2.3)-(2.6), left to the reader, follow by standard arguments. The operator $\frac{d}{d\phi}(\cos^\vartheta \phi \frac{d\cdot}{d\phi})$ is selfadjoint in $L^2(0, \pi/2; \cos^\vartheta \phi\, d\phi)$. Moreover, the eigenfunctions $S_m(\phi)$, $m = 0, 1, \ldots$, are a complete system in this Hilbert space.

The proofs of (2.7) and the proofs of the corresponding estimates (2.14) in lemma 2.2 below are, respectively, extensions of those of equations (56) and (62) in [4]. Since Greenberg proves (56) in this last reference, we will prove (2.14), leaving to the reader the (similar) proof of (2.7).

Finally, we note that $S'_m(\pi/2) = 0$, $m = 0, 1, 2, \ldots$ (compare to equation $(2.2)_2$). See the beginning of the proof of lemma 2.3.

Lemma 2.2 *The nontrivial solutions* $(\mu^2, T_\mu(\psi))$ *of the problem*

$$\begin{cases} (\sin^\vartheta \psi \, T'_\mu(\psi))' + \mu^2 \sin^\vartheta \psi \, T_\mu(\psi) = 0\,, \quad 0 < \psi < \pi/2\,, \\ T_\mu(0) = T_\mu(\pi/2) = 0\,, \end{cases} \tag{2.8}$$

are given by

$$\begin{cases} \mu_n^2 = (2n-1)^2 + 3\vartheta(2n-1) + 2\vartheta^2\,, \\ T_n(\psi) = \sin^\vartheta \psi \sum_{k=1}^{n} T_n^k \cos^{2k-1} \psi\,, \end{cases} \tag{2.9}$$

where $n = 1, 2, \ldots$. *The coefficients* T_n^k *are defined inductively by* $T_n^0 = 0$ *and*

$$(2k+1)(2k)T_n^{k+1} = a(n,k,\vartheta)T_n^k + b(n,k,\vartheta)T_n^{k-1}\,, \tag{2.10}$$

for $k = 1, \ldots, n-1$. *The coefficients* a, b *above are given by*

$$\begin{cases} a(n,k,\vartheta) \;=\; -(2n-1)^2 - 3(2n-1)\vartheta - 2\vartheta^2 - 2\vartheta + \\ \qquad\qquad\quad + (2k-1)(4k-3) + 6k\vartheta, \\ b(n,k,\vartheta) \;=\; (2n-1)^2 - (2k-3)^2 + 6\vartheta(n-k+1)\,. \end{cases} \tag{2.11}$$

The coefficients T_n^1 *are normalized so that*

$$\int_0^{\pi/2} T_n^2(\psi) \sin^\vartheta \psi \, d\psi = 1\,. \tag{2.12}$$

Furthermore, one has

$$\int_0^{\pi/2} T_n T_m \sin^\vartheta \psi \, d\psi = \delta_{m,n}\,; \qquad \int_0^{\pi/2} T'_n T'_m \sin^\vartheta \psi \, d\psi = \mu_n^2 \delta_{m,n} \tag{2.13}$$

and

$$|T_n(\psi)| \le \frac{K_0}{\sin^\vartheta \psi}\,, \qquad |T'_n(\psi)| \le \frac{K_0 \mu_n}{\sin^\vartheta \psi}\,, \tag{2.14}$$

where K_0 *is the same as that in lemma 2.1.*

Proof. The proofs of (2.9) (2.10) follow standard arguments which consist in plugging in equation (2.8) a function $T_\mu(\psi)$ of the form $(2.9)_2$ and then on equalizing the coefficients of each distinct power of $\cos\psi$ to zero.

It is worth noting that the coefficients $b(n,k,\vartheta)$ do not vanish (they are strictly positive). The coefficients T_n^k may be obtained as follows. We start by giving to T_n^1 an arbitrary value $\rho \neq 0$. Then, equation (2.10) allows us to determine the values of $T_n^2, \ldots, T_n^n$ in this same order. Since $T_n^1 \neq 0$, the function $T_n(\psi)$ does not vanishes identically. Since $T_n(\psi)$ is proportional to ρ we can get (2.12) by an appropriate choice of ρ. We also remark that we have $T_n^n \neq 0$. If not, let n_0 be the smallest n for which $T_n^n = 0$. Since $T_n^n \neq 0$ for $n = 1, \ldots, n_0 - 1$, it follows that

$T_{n_0}(\psi)$ is a linear combination of the $T_n(\psi)$, $n = 1, \dots, n_0 - 1$. But this contradicts $(2.13)_1$. The proofs of (2.13) are obvious.

We now prove (2.14). For convenience we write $\mu = \mu_n$, $T(\psi) = T_n(\psi)$, for a fixed index n. On multiplying both sides of equation (2.8) by $\sin^{\vartheta}\psi\, T'(\psi)$, integrating over $[\psi, \pi/2]$, and by straightforward calculations, one gets

$$T'(\pi/2)^2 = (T'(\psi)^2 + \mu^2 T(\psi)^2)\sin^{2\vartheta}\psi + 2\vartheta\mu^2 \int_{\psi}^{\pi/2} \sin^{2\vartheta-1} s \cos s\, T_n(s)^2\, ds. \tag{2.15}$$

On dividing both sides of this equation by $\sin^{\vartheta}\psi$, integrating over $[0, \pi/2]$, and using (2.13), one obtains

$$T_n'(\pi/2)^2 \int_{\psi}^{\pi/2} \sin^{-\vartheta}\psi\, d\psi = 2\mu^2 + 2\vartheta\mu^2 Y \tag{2.16}$$

where

$$Y = \int_0^{\pi/2} \sin^{-\vartheta}\psi \left(\int_{\psi}^{\pi/2} \sin^{2\vartheta-1} s \cos s\, T^2(s)\, ds \right) d\psi\,.$$

An integration by parts shows that

$$Y = \int_0^{\pi/2} \sin^{\vartheta}\psi\, T^2(\psi) \left(\frac{\cos^{\vartheta}\psi}{\sin^{1-\vartheta}\psi} \int_0^{\psi} \frac{ds}{\sin^{\vartheta} s} \right) d\psi\,.$$

Since $s \geq (2/\pi)s$, it follows that the integral of $\sin^{-\vartheta} s$ on $[0, \psi]$ is bounded by $(\pi/2)^{\vartheta}(1-\vartheta)^{-1}\psi^{1-\vartheta}$. Hence, by taking into account (2.12), it follows easily that $Y \leq \pi/2(1-\vartheta)$. On the other hand, the integral of $\sin^{-\vartheta}\psi$ on $[0, \pi/2]$ is greater than $(\pi/2)^{1-\vartheta}(1-\vartheta)^{-1}$. Hence, from (2.16) one gets

$$T_n'(\pi/2) \leq (2/\pi)^{1-\vartheta}(2(1-\vartheta) + \vartheta\pi)\mu_n^2\,. \tag{2.17}$$

Finally, from (2.15) and (2.17) we conclude that $\sin^{2\vartheta}\psi\,[T_n'(\psi)^2 + \mu_n^2 T_n(\psi)^2]$ is bounded by the right hand side of (2.17). This proves (2.14). □

Lemma 2.3 *There is a positive constant K_1, independent of m, such that*

$$S_m(\phi)^2 \leq K_1(1+m^2)^{\vartheta/2}\,, \qquad S_m'(\phi)^2 \leq K_1(1+m^2)^{1+\vartheta/2}\,, \tag{2.18}$$

for each $\phi \in [0, \pi/2]$.

Proof. We use here some sharp results on Gegenbauer (or ultraspherical) polynomials proved in [2, 3]. Following [2, 3], Gegenbauer polynomials are polynomials in the real variable x which, in our case, is replaced by $\sin\phi$ i.e. we set $x = \sin\phi$. We point out that here the polynomials are not normalized as in [2, 3]. More precisely,

$$S_m(\phi) = C(m)C_{2m}^{\vartheta/2}(\sin\phi)\,, \qquad 0 \leq \phi \leq \pi/2\,, \tag{2.19}$$

where $C_n^\lambda(x)$ denotes the Gegenbauer polynomials as defined in [2, 3]. See [2] 3.15.1 and [3] 10.9.

For the convenience of the reader we prove the identity (2.19). We take (2.19) as the definition of $S_m(\phi)$ and we prove that $S_m(\phi)$ solves (2.2) when $\lambda^2 = 4m^2 + 2\vartheta m$. For convenience, we write $y(x) \equiv C_{2m}^{\vartheta/2}(x)$. From [3] 10.9, Eq. (16), it follows that the function $y(x)$ is even. Hence, $S_m(\phi)$ is even and, in particular, $S'_m(0) = 0$. On the other hand, $S'_m(\pi/2) = y'(1)\cos(\pi/2) = 0$. Hence, the boundary conditions (2.2) are satisfied. Next, from [3] 10.9 Eq. (14) one shows that

$$(1-x^2)y''(x) - (1+\vartheta)xy'(x) + (4m^2 + 2\vartheta m)y(x) = 0\,, \tag{2.20}$$

for each $x \in \mathbf{R}$. It readily follows that $S_m(\phi)$, defined by equation (2.19), satisfies $(2.2)_1$ for $\lambda^2 = 4m^2 + 2\vartheta m$. Clearly, this is independent of the particular value of the constant $C(m)$. Next, we prove that

$$C(m) = \left[\frac{(2m)!\,(2m+\vartheta/2)}{\Gamma(2m+\vartheta)}\right]^{1/2} 2^{\vartheta/2}\pi^{-1/2}\Gamma(\vartheta/2)\,. \tag{2.21}$$

In [2, 3] (see [2] 3.15.1 Eq. (17)) the polynomials $C_{2m}^{\vartheta/2}(x)$ are normalized by the condition

$$\int_0^1 [C_{2m}^{\vartheta/2}(x)]^2(1-x)^{(\vartheta-1)/2}\,dx = \frac{\pi\Gamma(2m+\vartheta)}{2^\vartheta\,(2m)!\,(2m+\vartheta/2)\Gamma(\vartheta/2)^2}\,,$$

where we have used the fact that $C_{2m}^{\vartheta/2}$ is even. Setting $x = \sin\phi$ in this equation, using (2.19), and choosing $C(m)$ in such a way that (2.5) holds, we obtain (2.21). □

Next, we prove the estimate $(2.18)_1$. We start by remarking that (see [3] 10.18 Eq. (7))

$$\max_{0\le x\le 1} |C_{2m}^{\vartheta/2}(x)| = |C_{2m}^{\vartheta/2}(1)| = \frac{\Gamma(2m+\vartheta)}{\Gamma(\vartheta)\,(2m)!}\,. \tag{2.22}$$

From (2.22) (2.19) and (2.21) it follows that

$$\max_{0\le\phi\le\pi/2} S_m(\phi)^2 = \frac{2^\vartheta\Gamma(\vartheta/2)^2}{\pi\Gamma(\vartheta)^2}(2m + \frac{\vartheta}{2})\frac{\Gamma(2m+\vartheta)}{(2m)!}\,. \tag{2.23}$$

In order to prove (2.18), it is sufficient to show that

$$\lim_{m\to\infty} \frac{(2m+\vartheta/2)\Gamma(2m+\vartheta)}{(1+m^2)^{\vartheta/2}(2m)!} = 2^\vartheta\,.$$

The quantity under the limit sign can be written as

$$2^\vartheta \cdot \frac{2m+\vartheta/2}{2m} \cdot \frac{\Gamma(2m+\vartheta)}{\Gamma(2m)(2m)^\vartheta} \cdot \Big(\frac{m}{\sqrt{1+m^2}}\Big)^\vartheta\,.$$

since $\Gamma(2m) = (2m-1)!$. Our thesis follows because

$$\lim_{n\to\infty} \frac{\Gamma(\vartheta+n)}{\Gamma(\vartheta)n^{\vartheta}} = 1 .$$

See, for instance [6] chap. 7, § 83, Eq (44).

□

Finally, we prove (2.18) by showing that

$$\max_{0\le\phi\le\pi/2} S'_m(\phi)^2 \le (4m^2+2\vartheta m) \max_{0\le\phi\le\pi/2} S_m(\phi)^2 \tag{2.24}$$

Multiplication of both sides of the equation (2.2) by $\cos^{\vartheta}\phi S'_m(\phi)$ yields

$$[\cos^{2\vartheta}\phi({S'_m}^2 + \lambda^2 S_m^2)]' = \lambda^2 S_m^2 (\cos^{2\vartheta}\phi)' ,$$

where $\lambda^2 = 4m^2 + 2\vartheta m$. Integration over $(\phi, \pi/2)$ and using straightforward techniques show that

$$S'_m(\phi)^2 + \lambda^2 S_m(\phi)^2 \le \lambda^2 \max_{\phi\le s\le\pi/2} S_m(s)^2 .$$

This proves (2.26). Note that this estimate also implies that $S'_m(\pi/2) = 0$.

Lemma 2.4 *Let λ_m and μ_n be as in lemmas 2.1 and 2.2 respectively. Then, for each $m \ge 0$ and each $n \ge 1$ one has*

$$\lambda_m^2 - \mu_n^2 = 4(m+n+\vartheta-1/2)(m-n+(1-\vartheta)/2) . \tag{2.25}$$

In particular, by setting $\overline{\vartheta} = (1-\vartheta)/2$,

$$|\lambda_m^2 - \mu_n^2| \ge 4(m+n-1/2)|m-n+\overline{\vartheta}| . \tag{2.26}$$

We leave the proof to the reader.

Define the linear operator

$$LV \equiv (\cos^{\vartheta}\phi \sin^{\vartheta}\psi V_{\psi})_{\psi} - (\cos^{\vartheta}\phi \sin^{\vartheta}\psi V_{\phi})_{\phi} , \tag{2.27}$$

and set $\Lambda = (0,\pi/2)\times(0,\pi/2)$. In the remaining of this section we study the auxiliary problem

$$LV = \cos^{\vartheta}\phi \sin^{\vartheta}\psi F \qquad \text{in } \Lambda , \tag{2.28}$$

with boundary conditions

$$\begin{cases} V_{\phi}(0,\psi) = \lim\limits_{\phi\to(\pi/2)-} \cos^{\vartheta}\phi V_{\phi}(\phi,\psi) = 0 , & 0\le\psi\le\pi/2 \\ V(\phi,0) = V(\phi,\pi/2) = 0 , & 0\le\phi\le\pi/2 , \end{cases} \tag{2.29}$$

where

$$F(\phi,\psi) = \sum_{\substack{m\ge0\\ n\ge1}} F_{m,n} S_m(\phi) T_n(\psi) . \tag{2.30}$$

By using lemmas 2.1, and 2.4 it follows that the solution V of the problem (2.28), (2.29) is given by

$$V(\phi,\psi)=\sum_{\substack{m\geq 0\\ n\geq 1}} V_{m,n}S_m(\phi)T_n(\psi) \tag{2.31}$$

where, for each couple of integers $m\geq 0,\ n\geq 1$,

$$V_{m,n}=\frac{F_{m,n}}{\lambda_m^2-\mu_n^2}\,. \tag{2.32}$$

In particular

$$|V_{m,n}|\leq \frac{|F_{m,n}|}{4(m+n-1/2)|m-n+\overline{\vartheta}|}\,. \tag{2.33}$$

Let β and δ be nonnegative reals. We define the norms

$$\| V\|^2_{\beta,\delta}=\sum_{\substack{m\geq 0\\ n\geq 1}}(1+m^2)^\beta\, n^{2\delta}\, V^2_{m,n}\,. \tag{2.34}$$

From now on, we shall use notations

$$d\mu_\phi=\cos^\vartheta\phi\, d\phi\ ,\qquad d\mu_\psi=\sin^\vartheta\psi\, d\psi\ ,\qquad d\mu=d\mu_\phi\, d\mu_\psi\,.$$

For completness, we state some significant equivalences between the norms (2.34) and suitable integrals. One has

$$\begin{aligned}
&\| F\|^2_{0,0}=\int\!\!\int F^2\,d\mu\,; \qquad \| F\|^2_{1,0}\simeq \| F\|^2_{0,0}+\int\!\!\int F^2_\phi\,d\mu\,; \qquad (2.35)\\
&\| F\|^2_{0,1}\simeq \int F^2_\psi\,d\mu\,; \qquad \| V\|^2_{1,1}\simeq \| V\|^2_{0,0}+\int\!\!\int V^2_{\phi\psi}\,d\mu\,;\\
&\| V\|^2_{2,0}\simeq \| V\|^2_{0,0}+\int\!\!\int (V_\phi\cos^\vartheta\phi)^2_\phi\,\frac{\sin^\vartheta\psi}{\cos^\vartheta\phi}\,d\phi\,d\psi\,;\\
&\| V\|^2_{0,2}\simeq \int\!\!\int (V_\psi\sin^\vartheta\psi)^2_\psi\,\frac{\cos^\vartheta\phi}{\sin^t\psi}\,d\phi\,d\psi\,,
\end{aligned}$$

where the integrals are over Λ. The proofs follow by using (2.6), (2.13), and the explicit expressions of λ_m^2 and μ_n^2. By the way, note that the fourth and the fifth equations show that

$$\int\!\!\int [(V_\phi\cos^\vartheta\phi)^2_\phi+(V_\phi\cos^\vartheta\phi)^2_\psi]\,\frac{\sin^\vartheta\psi}{\cos^\vartheta\phi}\,d\phi\,d\psi\leq \| V\|^2_{2,0}+\| V\|^2_{1,1}\,.$$

In particular, if V is a solution of (2.28), (2.29) and $F=F_1+F_2$, $\| F_1\|_{1,0}<+\infty$, $\| F_2\|_{0,1}<+\infty$, then the above inequality holds. Note that $V_\phi\cos^\vartheta\phi$ is the function which appears in the boundary condition $(2.29)_1$.

We have the following result.

Theorem 2.1 *Suppose that* $\| F \|_{1,0} < \infty$. *Then, the solution* V *of the problem (2.28), (2.29) satisfies the estimates*

$$\| V \|^2_{1+\alpha,1-\alpha} \le c \| F \|^2_{1,0}, \qquad -1 \le \alpha \le 1. \tag{2.36}$$

Moreover,

$$|V_\psi| \le \frac{c}{\sin^\vartheta \psi} \| F \|_{1,0}, \qquad |V_\phi| \le \frac{c}{\cos^\vartheta \phi} \| F \|_{1,0} \tag{2.37}$$

for each $(\vartheta, \psi) \in \Lambda$. *Similarly, if* $\| F \|_{0,1} < +\infty$, *then*

$$\| V \|^2_{1+\alpha,1-\alpha} \le c \| F \|^2_{0,1}, \qquad -1 \le \alpha \le 1. \tag{2.38}$$

Moreover,

$$|V_\psi| \le \frac{c}{\sin^\vartheta \psi} \| F \|_{0,1}, \qquad |V_\phi| \le \frac{c}{\cos^\vartheta \phi} \| F \|_{0,1} \tag{2.39}$$

for each $(\vartheta, \psi) \in \Lambda$.

Proof. It is sufficient to prove (2.36) and (2.38) for $\alpha = 1$ and $\alpha = -1$, since

$$\| V \|^2_{1+\alpha,1-\alpha} \le \| V \|^2_{2,0} + \| V \|^2_{0,2}, \qquad -1 \le \alpha \le 1.$$

By using (2.33) one gets

$$16 \| V \|^2_{2,0} \le \sum_{\substack{m \ge 0 \\ n \ge 1}} \frac{(1+m^2)^2}{n^2(m+n-1/2)^2(m-n+\overline{\vartheta})^2} n^2 F^2_{m,n}.$$

Hence, in order to prove (2.38) for $\alpha = 1$, it is sufficient to show that in the last inequality the coefficient of $n^2 F^2_{m,n}$ is bounded by a constant c, independent of the pair (m,n), $m,n \ge 1$. This holds since $m - n - 1/2 \ge m/4$ and $m - n + \overline{\vartheta} \ge m/2$, if $m \ge 2n$; $n^2 \ge (1+m)^2/8$ and $|m - n + \overline{\vartheta}| \ge \overline{\vartheta}$, if $m \le 2n$.

Next, we prove (2.38) for $\alpha = -1$. In this case we are lead to prove that

$$\frac{n^2}{(m+n+\vartheta-1/2)^2(m-n+\overline{\vartheta})^2} \le c,$$

where c is independent of (m,n). This holds, since $|m - n + \overline{\vartheta}| \ge \overline{\vartheta}$.

The estimate (2.36) for $\alpha = \pm 1$ is proved in a similar way.

□

Next, we prove (2.39). The proof of (2.37) is similar, and is left to the reader. Write

$$\begin{aligned} V \;=\; & \sum_{m=1}^{\infty} [S_m(\phi) - S_m(0)] \sum_{n=1}^{\infty} V_{m,n} T_n(\psi) + \\ & + \sum_{m=0}^{\infty} S_m(0) \sum_{n=1}^{\infty} V_{m,n} T_n(\psi) = U + W. \end{aligned} \tag{2.40}$$

Since $U_\psi(0,\psi) \equiv 0$, it readily follows that

$$|U_\psi(\phi,\psi)| \leq \left(\int_0^{\pi/2} \frac{d\phi}{\cos^\vartheta \phi} \right)^{1/2} \left(\int_0^{\pi/2} U_{\phi\psi}^2 \, d\mu_\phi \right)^{1/2} = \\ = \left\{ \sum_{m=1}^{\infty} \lambda_m^2 \Big(\sum_{n=1}^{\infty} V_{m,n} \, T_n'(\psi) \Big)^2 \right\}^{1/2} .$$

Hence, by using in particular (2.14), one gets

$$|U_\psi| \leq \frac{c}{\sin^\vartheta \psi} \left\{ \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} n |F_{m,n}| \frac{m}{|\lambda_m^2 - \mu_n^2|} \right) \right\}^{1/2} .$$

As usual, the positive constant c may change from one to another equation. By Cauchy-Schwarz inequality,

$$|U_\psi| \leq \frac{c}{\sin^\vartheta \psi} \| F \|_{0,1} \left(\sup_{m \geq 1} \sum_{n=1}^{\infty} \frac{m^2}{(\lambda_m^2 - \mu_n^2)^2} \right)^{1/2} .$$

Moreover,

$$\sum_{n=1}^{\infty} \frac{m^2}{(\lambda_m^2 - \mu_n^2)^2} \leq \sum_{n=1}^{\infty} \frac{1}{m - n + \overline{\vartheta})^2} \leq \\ \leq \sum_{p=0}^{\infty} \frac{1}{(\overline{\vartheta} + p)^2} + \sum_{p=0}^{\infty} \frac{1}{((1-\overline{\vartheta}) + p)^2} . \tag{2.41}$$

Hence $|U_\psi|$ is bounded on Λ by the right hand side of $(2.39)_1$.

Next, we consider W_ψ. Since $|S_m(0)| \leq K_0$, it readily follows that

$$|W_\psi| \leq K_0 \sum_{m=0}^{\infty} \Big(\sum_{n=1}^{\infty} n^2 F_{m,n}^2 \Big)^{1/2} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\frac{T_n'(\psi)}{\lambda_m^2 - \mu_n^2} \Big)^2 \right]^{1/2} .$$

By Cauchy-Schwarz inequality, and by using (2.14), (2.26), one gets

$$|W_\psi| \leq c \frac{\| F \|_{0,1}}{\sin^\vartheta \psi} \left(\sum_{\substack{m \geq 0 \\ n \geq 1}} \frac{1}{(n+m-1/2)^2 (m-n+\overline{\vartheta})^2} \right)^{1/2} . \tag{2.42}$$

In order to accomplish the proof of $(2.39)_1$ we show that the double series on the right hand side of (2.42) converges to a finite sum. Since $8(n+m-1/2)^2 \geq (m+n)^2+1$ and $(m-n+\overline{\vartheta})^2 \geq c((m-n)^2+1)$, for some $c > 0$, our assertion follows since the sum of the above double series is bounded by $8/c$ times the quantity

$$\sum_{m,n \in \mathbf{Z}} \frac{1}{(m-n)^2+1} \, \frac{1}{(m+n)^2+1} .$$

The sum of this last double series is bounded by

$$\sum_{p,q\in\mathbf{Z}} \frac{1}{1+p^2}\,\frac{1}{1+q^2} \le \left(\frac{\pi}{2}\right)^2 .$$

□

Next, we prove $(2.39)_2$. We write

$$\begin{aligned} V &= \sum_{n=1}^{\infty}[T_n(\psi) - T_n(\pi/2)] \sum_{m=0}^{\infty} T_{m,n} S_m(\phi) + \\ &\quad + \sum_{n=1}^{\infty} T_n(\pi/2) \sum_{m=0}^{\infty} T_{m,n} S_m(\phi) \equiv U + W . \end{aligned}$$

Since $U_\phi(\phi,\pi/2) \equiv 0$, by arguing as was done above for U_ψ (clearly, the functions U and W are now distinct from those in equation (2.40)), it readily follows that

$$|U_\phi| \le \frac{c}{\cos^\vartheta \phi} \| F \|_{0,1} \left(\sup_{n\ge1} \sum_{m=1}^{\infty} \frac{m^2}{(\lambda_m^2 - \mu_n^2)^2} \right)^{1/2} .$$

An argument similar to that of (2.41) shows that the sum of the above series is bounded by a constant independent of n.

Finally,

$$\begin{aligned} |W_\phi| &\le \left(\sum_{n=1}^{\infty} \frac{T_n^2(\pi/2)}{n^2} \right)^{1/2} \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} n V_{m,n} S'_m(\phi) \right)^2 \right\}^{1/2} \le \\ &\le \frac{c}{\cos^\vartheta \phi} \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} n |F_{m,n}| \cdot \frac{m}{|\lambda_m^2 - \mu_n^2|} \right)^2 \right\}^{1/2} \end{aligned}$$

which, in turn, is bounded by

$$\frac{c}{\cos^\vartheta \phi} \sup_{n\ge1} \sum_{m=0}^{\infty} \frac{m^2}{(\lambda_m^2 - \mu_n^2)^2} \| F \|_{0,1} .$$

Theorem 2.2 *Let V be the solution of problem (2.28), (2.29), defined by (2.31), (2.32). If $\| F \|_{1,0} < +\infty$ then*

$$|V_\phi| \le \frac{c}{\sin^\vartheta \psi} \| F \|_{1,0} \qquad \textit{in } \Lambda. \tag{2.43}$$

If $\| F \|_{0,1} < +\infty$ then

$$|V_\phi| \le \frac{c}{\sin^\vartheta \psi} \| F \|_{0,1} \qquad \textit{in } \Lambda. \tag{2.44}$$

Proof. Here, we apply the sharp estimates (2.18). From (2.31), (2.33), $(2.18)_2$ and (2.14) one obtains the estimate

$$|V_\phi| \leq \frac{c}{\sin^\vartheta \psi} \sum_{\substack{m\geq 0\\ n\geq 1}} \frac{(1+m^2)^{1/2+\vartheta/4}|F_{m,n}|}{|m+n-1/2||m-n+\overline{\vartheta}|} \,. \tag{2.45}$$

It readily follows that

$$|V_\phi| \leq \frac{c}{\sin^\vartheta \psi} \| F \|_{1,0} \left(\sum_{\substack{m\geq 0\\ n\geq 1}} \frac{1}{|m+n-1/2|^{2-\vartheta}(m-n+\overline{\vartheta})^2} \right)^{1/2} .$$

Consequently $(2.43)_1$ follows, since

$$\sum_{\substack{m\geq 0\\ n\geq 1}} |m+n-1/2|^{2-\vartheta}|m-n+\overline{\vartheta}|^2 \leq \sum_{p,q\in \mathbf{Z}} |p-1/2|^{\vartheta-2}\,|q+\overline{\vartheta}|^{-2} \,.$$

On the other hand, from (2.45) one gets,

$$|V_\phi| \leq \frac{c}{\sin^\vartheta \psi} \| F \|_{0,1} \left(\sum_{\substack{m\geq 0\\ n\geq 1}} \frac{(1+m^2)^{1+\vartheta/2}}{(m+n-1/2)^2(m-n+\overline{\vartheta})^2 n^2} \right)^{1/2} .$$

Moreover, the sum of the double series is bounded by a constant times the quantity

$$\sum_{\substack{m\geq 2n\\ n\geq 1}} \frac{1}{(1+m^2)^{1-\vartheta/2}n^2} + \sum_{\substack{0\leq m\leq 2n\\ n\geq 1}} \frac{1}{|m+n-1/2|^{2-\vartheta}(m-n+\overline{\vartheta})^2} \,,$$

which, in turn, is finite. Hence $(2.43)_2$ holds.

□

Finally, we have the following result.

Theorem 2.3 *Let V be the solution of problem (2.28), (2.29) and assume that $\| F \|_{1,0} < +\infty$ or that $\| F \|_{0,1} < +\infty$. Then V is Hölder-continuous on Λ with exponent $1-\vartheta$. More precisely,*

$$|V(\phi,\psi) - V(\phi_0,\psi_0)| \leq c \| F \|_{1,0}(|\phi-\phi_0|^{1-\vartheta} + |\psi-\psi_0|^{1-\vartheta}) \,, \tag{2.46}$$

for each couple of pairs $(\phi,\psi), (\phi_0,\psi_0) \in \Lambda$.

Proof. Use

$$V(\phi,\psi) - V(\phi_0,\psi_0) = \int_{\phi_0}^{\phi} V_\phi(\xi,\psi_0)\,d\xi + \int_{\psi_0}^{\psi} V_\psi(\phi,\eta)\,d\eta$$

and the estimates (2.37) or (2.39). □

3 An equivalent problem

For convenience, we introduce specific symbols to denote positive constants that appear often in the sequel. Set

$$m_1 = (m-1)/2 \,, \qquad m_2 = \sqrt{2/(m+1)} \,, \qquad \overline{w} = \nu w^{m_1} \,.$$

The Riemann invariants for the equation (1.9) are defined by the equations

$$\begin{cases} \alpha = 1/2(u_\tau - \overline{w}q(u_x)) \,, \\ \beta = 1/2(u_\tau + \overline{w}q(u_x)) \,, \end{cases} \tag{3.1}$$

where

$$q(\gamma) = \int_0^\gamma \sqrt{\sigma'(s)}\, ds \,. \tag{3.2}$$

Note that

$$\beta - \alpha = \overline{w}q(u_x) \,. \tag{3.3}$$

If $u(x,\tau)$ is a solution of (1.9), then $\alpha(x,\tau), \beta(x,\tau)$ are solutions of the problem

$$\begin{cases} \alpha_\tau + \overline{w}q'(u_x)\alpha_x = 0 \,, \\ \beta_\tau - \overline{w}q'(u_x)\beta_x = 0 \,, \end{cases} \tag{3.4}$$

in Q. Moreover, if u satisfies the boundary conditions (1.12), (1.13) then α and β satisfy the corresponding boundary conditions

$$\begin{cases} (\beta+\alpha)(0,\tau) = (\beta-\alpha)(1/2,\tau) = 0 & 0 \le \tau \le \pi/2, \\ (\beta-\alpha)(x,0) = (\beta+\alpha)(x,\pi/2) = 0 & 0 \le x \le 1/2. \end{cases} \tag{3.5}$$

Conversely, assume that $q(0) = 0$ and that $q(\gamma)$ is invertible, and let $\alpha(x,\tau)$, $\beta(x,\tau)$ be solutions of

$$\begin{cases} \alpha_\tau + \overline{w}q'\big(q^{-1}\big(\dfrac{\beta-\alpha}{\overline{w}}\big)\big)\alpha_x = 0 \,, \\ \beta_\tau - \overline{w}q'\big(q^{-1}\big(\dfrac{\beta-\alpha}{\overline{w}}\big)\big)\beta_x = 0 \,, \end{cases} \tag{3.6}$$

in Q, and satisfy the boundary conditions (3.5). Define $u(x,\tau)$ by the equations

$$u_x = q^{-1}\big(\frac{\beta-\alpha}{\overline{w}}\big) \,, \qquad u_\tau = \alpha + \beta \,. \tag{3.7}$$

The solutions of problem (3.7) exist and are defined up to an additive constant. In fact (3.6) implies that the derivative of the right hand side of $(3.7)_1$ with respect to τ is equal to the derivative of the right hand side of $(3.7)_2$ with respect to x. Moreover, for a suitable additive constant the function $u(x,\tau)$ is a solution of (1.9) (1.12) (1.13).

Hence the problem (1.9) (1.12) (1.13) is equivalent to the problem (3.6) (3.5).

We shall prove the existence of solutions α, β of the problem (3.6), (3.5) of the particular form

$$\begin{cases} \alpha = 1/2m_2\nu\sin(\phi - \psi)\ , \\ \beta = 1/2m_2\nu\sin(\phi + \psi)\ , \end{cases} \tag{3.8}$$

where $\phi(x,\tau), \psi(x,\tau)$ are functions on Q, satisfying $0 \leq \phi(x,\tau) \leq \pi/2$, $0 \leq \psi(x,\tau) \leq \pi/2$; clearly

$$\beta - \alpha = m_2\nu\sin\psi\cos\phi\ , \qquad \beta + \alpha = m_2\nu\sin\phi\cos\psi\ . \tag{3.9}$$

In this way our problem is transformed to the problem of finding a pair of functions ϕ, ψ with values in $[0, \pi/2]$ and solution of the problem

$$\begin{cases} (\phi - \psi)_\tau + c(\phi,\psi)(\phi - \psi)_x = 0 \\ (\phi + \psi)_\tau - c(\phi,\psi)(\phi + \psi)_x = 0 \end{cases} \tag{3.10}$$

in Q, with boundary conditions

$$\begin{cases} \phi(0,\tau) = 0\ , \quad \phi(1/2,\tau) = \pi/2\ , \quad 0 \leq \tau \leq \pi/2\ , \\ \psi(x,0) = 0\ , \quad \psi(x,\pi/2) = \pi/2\ , \quad 0 \leq x \leq 1/2\ , \end{cases} \tag{3.11}$$

where, by definition,

$$c(\phi,\psi) = \overline{w}q'\big(q^{-1}\big(\frac{m_2\nu}{\overline{w}}\cos\phi\sin\psi\big)\big)\ . \tag{3.12}$$

Note that c is the coefficient of α_x and β_x in the equation (3.6) and is also equal to $\sqrt{\nu^2 w^{m-1}\sigma'(u_x)}$ (recall the equation (1.9)), provided that (3.1) and (3.8) hold. If the couple (ϕ,ψ) solves the above problem then the couple (α,β), defined by equations (3.8), solves (3.6), (3.5).

In order to deal the equations (3.10) it is necessary to study carefully the properties of the coefficient $c(\phi,\psi)$, taking into account of the specific properties of the function $\sigma(\gamma)$ defined by the equation (1.10).

First of all, taking into account (1.10), (1.5), one easily verifies that, given (in **R**) a bounded neighborhood of the origin, the function $q' \circ q^{-1}$ is well defined in this neighborhood provided that the positive parameter a belongs to a sufficiently small neighborhood of the origin (since, roughly speaking, $\sigma(\gamma)$ is a "small" perturbation of $|\gamma|^{m-1}\gamma$; note that ν is fixed and that w takes values in a neighborhood of w_0 as shown in the sequel. In fact, $w = w(a)$ and $w(a) \to w_0$ and $a \to 0$).

We have the following result:

Proposition 3.1 *Under the hypotheses (1.5) (1.6) we have*

$$\begin{aligned} c(\phi,\psi) &= \sqrt{m}\,\big(\frac{m+1}{2m}\big)^{\vartheta/2} w^\vartheta \nu \cos^\vartheta\phi\sin^\vartheta\psi \\ &\quad \big\{1 + g\big(\big(\frac{m+1}{2m}\big)^{1/2} a^{1/\vartheta} w\cos\phi\sin\psi\big)\big\} \end{aligned} \tag{3.13}$$

where $g \in C([0,b]) \cap C^2(]0,b[)$ *for some* $b > 0$, *satisfies the estimates*

$$|g(x)| \le c|x| \ , \quad |g'(x)| \le c \ , \quad |g''(x)| \le cx^{-1} \ . \tag{3.14}$$

Before proving this proposition we establish a result which is independent of the particular hypotheses (1.5) (1.6).

Proposition 3.2 *Let* $\hat{\sigma}(\gamma)$ *be defined by the equation (1.4), where* $R(\gamma)$ *is of class* $C^1(-b,b)$. *Let* $\sigma(\gamma)$ *be as in (1.10). Then*

$$c(\phi,\psi) = \frac{\sqrt{m}\nu}{a} f\big(\frac{a^{1/\vartheta}w}{m_2\sqrt{m}}\cos\phi\sin\psi\big) \tag{3.15}$$

where $f = \ell \circ F^{-1}$ *and*

$$\begin{cases} \ell(t) \equiv t^{\frac{m-1}{2}}\sqrt{1+k(t)} \ , \\ F(t) \equiv \dfrac{m+1}{2}\displaystyle\int_0^t \ell(s)\,ds \ , \\ k(t) \equiv R(t) + \dfrac{1}{m}tR'(t) \ . \end{cases} \tag{3.16}$$

Proof. We have

$$c(\phi,\psi) = \overline{w}\sqrt{\sigma'(\gamma)} \quad \text{if} \quad \gamma \equiv q^{-1}\big(\frac{m_2\nu}{\overline{w}}\cos\phi\sin\psi\big) \ . \tag{3.17}$$

On the other hand, one easily shows that

$$\sigma'(\gamma) = \frac{m}{a^2w^{m-1}}\ell^2\big(a^{\frac{2}{m-1}}w\gamma\big) \tag{3.18}$$

and (from (3.2)) that

$$q(\gamma) = \frac{2\sqrt{m}}{m+1}a^{-1/\vartheta}w^{\frac{m+1}{2}}F\big(a^{\frac{2}{m-1}}w\gamma\big) \ . \tag{3.19}$$

Since (see (3.13)) $q(\gamma) = (m_2\nu/\overline{w})\cos\phi\sin\psi$ it follows, by using (3.19), that

$$a^{\frac{2}{m-1}}w\gamma = F^{-1}\big(\frac{m+1}{2\sqrt{m}}m_2a^{1/\vartheta}w\cos\phi\sin\psi\big) \ . \tag{3.20}$$

This equation together with (3.17) and (3.18) yields (3.15).

□

The proof of the proposition 3.1 follows from (3.15) and from the following result.

Proposition 3.3 *Let* f, F *and* ℓ *be defined as in proposition 3.2 and assume that* R *satisfies (1.5),(1.6). Then*

$$f(x) = x^{\vartheta}(1+g(x)) \ , \qquad x \in [0,b[\ , \tag{3.21}$$

for some $b > 0$ *where* g *satisfies the hypothesis in proposition 3.1.*

The proof will be given in section 5.

Instead of looking directly for a solution $\phi = \phi(x,\tau)$, $\psi = \psi(x,\tau)$ of the problem (3.10) (3.11), one looks for a pair of functions $x = X(\phi,\psi)$, $\tau = T(\phi,\psi)$ which are solutions of a suitable boundary value problem, such that the map $(\phi,\psi) \to (x,\tau)$ is invertible (from Λ onto Q) and the inverse map $(x,\tau) \to (\phi,\psi)$ solves (3.10) (3.11).

Consider the following linear system in Λ

$$\begin{cases} X_\phi - c(\phi,\psi)T_\psi = 0\ , \\ X_\psi - c(\phi,\psi)T_\phi = 0\ , \end{cases} \tag{3.22}$$

with boundary conditions

$$X(0,\psi) = 0\ , \quad X(\pi/2,\psi) = 1/2\ , \quad 0 \le \psi \le \pi/2\ , \tag{3.23}$$

$$T(\phi,0) = 0\ , \quad X(\phi,\pi/2) = \pi/2\ , \quad 0 \le \phi \le \pi/2\ . \tag{3.24}$$

One has the following result:

Theorem 3.1 *If $0 < a \ll 1$ then there is a Hölder continuous map $\mathcal{I} : (\phi,\psi) \to (x,\tau)$ ($x = X(\phi,\psi)$, $\tau = T(\phi,\psi)$) from $\overline{\Lambda}$ onto $\overline{Q}$ such that (3.22) (3.23) (3.24) hold. $\mathcal{I}$ is locally Lipschitz continuous and has non- vanishing Jacobian on Λ. Moreover, $\mathcal{I}$ is a one to one map from Λ onto Q. The norms $\| V \|_{1+\alpha,1-\alpha}$, for each $\alpha \in [-1,1]$, and the quantity $\sup_{(\phi,\psi)\in\Lambda} \{\sin^\vartheta \psi |V_\psi| + (\cos^\vartheta \phi + \sin^\vartheta \psi)|V_\phi|\}$ are finite, for $V = T$ and for $V = X$.*

The proof of theorem 3.1 will be carried out in section 4. Assume, for the time being, that this theorem holds. Then, by the implicit function theorem we have $T_\psi = J\phi_x$, $X_\psi = -J\phi_\tau$, $T_\phi = -J\psi_x$, $X_\phi = J\psi_\tau$. Hence, it follows from (3.22) that the equations (3.10) are satisfied. Moreover, the boundary conditions (3.11) are satisfied as a consequence of (3.22) (3.23) together with the surgectivity and the continuity on $\overline{\Lambda}$ of the map $(\phi,\psi) \to (x,\tau)$. Hence the existence of a family of non trivial periodic solutions of the problem (1.1) (1.2) (1.4) (branching from the nontrivial periodic solution of this same problem when $\hat{\sigma}(\gamma) = |\gamma|^{m-1}\gamma$) is proved if we can prove the theorem 3.1.

4 Proof of theorem 3.1

The strategy of the proof is the following. By eliminating X from (3.22) we easily get the equation

$$(c(\phi,\psi)T_\psi)_\psi - (c(\phi,\psi)T_\phi)_\phi = 0 \qquad \text{in } \Lambda \tag{4.1}$$

with the boundary conditions

$$\begin{cases} T(\phi,0) = 0\ ,\ T(\phi,\pi/2) = \pi/2\ , & 0 \le \phi \le \pi/2\ , \\ T_\phi(0,\psi) = \lim_{\phi \to (\pi/2)-} \cos^\vartheta \phi T_\phi = 0\ , & 0 \le \psi \le \pi/2\ . \end{cases} \tag{4.2}$$

This problem admits the trivial (constant) solution $T = \pi/2$. Hence, a good strategy is to look for solution T which are perturbations ($0 < a \ll 1$) of the particular solution T_0 obtained by setting $R \equiv 0$ in the equation (1.10). Once the solution T is known we define

$$X(\phi, \psi) = \int_0^\phi c(s,\psi)\, T_\psi(s,\psi)\, ds\ , \qquad 0 \le \phi \le \pi/2\ . \tag{4.3}$$

The pair (X, T) satisfies (3.22) (3.23) (3.24) except, eventually, for the boundary condition $X(\pi/2, \psi) = \pi/2$. This condition will be fulfilled by making a suitable choice for the free parameter w.

Proof of theorem 3.1 Define $T^0 = T^0(\psi)$ by setting

$$T^0_\psi = (k_0 \sin^\vartheta \psi)^{-1}\ , \qquad T^0(0) = 0 \tag{4.4}$$

and by choosing k_0 in such a way that $T^0(\pi/2) = \pi/2$. It readily follows that T^0 is a solution of problem (4.1) (4.2) when $c(\phi, \psi) = k \cos^\vartheta \phi \sin^\vartheta \psi$, where k is any constant. The constant k_0 is given by

$$k_0 \equiv \frac{2}{\pi}\int_0^{\pi/2} \frac{d\alpha}{\sin^\vartheta \alpha} = \sqrt{\frac{m+1}{2}}\,\nu\ . \tag{4.5}$$

The second relation in (4.5) follows easily by the change of variables $\sin\alpha = y^{1/(1-\vartheta)}$.

Next, we look for a solution of problem (4.1) (4.2) of the form

$$T(\phi,\psi) = T^0(\psi) + a^{1/\vartheta} T^1(\phi,\psi)\ , \tag{4.6}$$

i.e.; we look for a solution $T^1(\phi, \psi)$ of the non-homogeneous linear equation

$$\begin{aligned}(\cos^\vartheta \phi \sin^\vartheta \psi T^1_\psi)_\psi - (\cos^\vartheta \phi \sin^\vartheta \psi T^1_\phi)_\phi &= \\ = \cos^\vartheta \phi \sin^\vartheta \psi\, [F_0 + F_1(T^1_\psi) + F_2(T^1_\phi)]\ ,\end{aligned} \tag{4.7}$$

where

$$F_0 = \frac{-g_\psi}{k_0 a^{1/\vartheta}(1+g)\sin^\vartheta \psi},\quad F_1(A) = \frac{-g_\psi A}{1+g},\quad F_2(A) = \frac{g_\phi A}{1+g}\ , \tag{4.8}$$

which satisfy the homogeneous boundary conditions

$$\begin{cases} T^1(\phi,0) = T^1(\phi,\pi/2) = 0\ , & 0 \le \phi \le \pi/2\ , \\ T^1_\phi(0,\psi) = \lim\limits_{\phi\to(\pi/2)-} \cos^\vartheta \phi\, T^1_\phi = 0\ , & 0 \le \psi \le \pi/2\ . \end{cases} \tag{4.9}$$

In equations (4.8), $g = g\big(\sqrt{\frac{m+1}{2m}}\, a^{1/\vartheta} w \cos\phi \sin\psi\big)$; see (3.13). The above problem will be solved by a fixed point argument. In the sequel we denote by $c, c_0, c_1, \ldots,$

positive constants which may depend on m but not on a or on w. The same symbol can be used to denote distinct constants. In the sequel it is always assume that

$$wa^{1/\vartheta} \leq c_0 \,, \tag{4.10}$$

for some suitable constant c_0, which can be made smaller from equation to equation, if necessary. From (3.14) and (4.10) it follows that, for sufficiently small values of c_0 one has $1+g \geq 1/2$ on Λ and also

$$\| \frac{g_\phi}{1+g} \|_\infty + \| \frac{g_\psi}{1+g} \|_\infty + \| (\frac{g_\phi}{1+g})_\psi \|_\infty + \| (\frac{g_\psi}{1+g})_\phi \|_\infty \leq c a^{1/\vartheta} w \tag{4.11}$$

where $\| \ \|_\infty$ is the usual $L^\infty(\Lambda)$ norm. Since (see (2.35))

$$\| fA \|_{1,0}^2 \leq c\,(\| f \|_\infty + \| f_\phi \|_\infty)^2 \, \| A \|_{1,0}^2 \,,$$

and similarly for $\| \ \|_{0,1}$, it readily follows that

$$\begin{cases} \| F_1(A) \|_{1,0}^2 \leq c a^{2/\vartheta} w^2 \| A \|_{1,0}^2 \,, \\ \| F_2(A) \|_{1,0}^2 \leq c a^{2/\vartheta} w^2 \| A \|_{0,1}^2 \,, \\ \| F_0 \|_{1,0}^2 \leq c w^2 \,; \end{cases} \tag{4.12}$$

note that $F_0 = F_1(\sin^\vartheta \psi / k_0 A^{1/\vartheta})$.

Next, let $\| \widetilde{T} \|_{1,1} < +\infty$ and consider the problem

$$LT^1 = F_0 + F_1(\widetilde{T}_\psi) + F_2(\widetilde{T}_\phi) \tag{4.13}$$

with boundary conditions (4.9). By theorem 2.1 together with (4.12) one gets

$$\| T^1 \|_{1+\alpha,1-\alpha}^2 \leq c_1 w^2 (1 + a^{2/\vartheta} \| \widetilde{T} \|_{1,1}^2) \,. \tag{4.14}$$

Note that $\| \widetilde{T}_\phi \|_{0,1} \leq c \| \widetilde{T} \|_{1,1}$ and that $\| \widetilde{T}_\psi \|_{1,0} \leq c(\| \widetilde{T}_\psi \|_{0,0} + \| \widetilde{T} \|_{1,1}) \leq c(\| \widetilde{T} \|_{0,1} + \| \widetilde{T} \|_{1,1}) \leq c \| \widetilde{T} \|_{1,1}$; see (2.35).

Set

$$\mathbf{K} = \{ T \ : \ \| T \|_{1,1}^2 \leq 2 c_1 w^2 \}$$

and define a map S by setting $S(\widetilde{T}) = T^1$, where T_1 is the solution of (4.13) (4.9). Assume, in the equation (4.10), that $c_0 \leq 1/\sqrt{2c_1}$. Then by (4.14),

$$\| T^1 \|_{1+\alpha,1-\alpha}^2 \leq 2 c_1 w^2 \,, \qquad \forall \, \alpha \in [-1,1] \,. \tag{4.15}$$

In particular $S(\mathbf{K}) \subset \mathbf{K}$. If $T^1 = S(\widetilde{T})$, $T_0^1 = S(\widetilde{T}_0)$, then

$$L(T^1 - T_0^1) = \cos^\vartheta \phi \sin^\vartheta \psi [F_1((\widetilde{T} - \widetilde{T}_0)_\psi) + F_2((\widetilde{T} - \widetilde{T}_0)_\phi)].$$

Since $c_1 w^2 a^{2/\vartheta} \leq 1/2$ one has

$$\| T^1 - T_0^1 \|_{1,1}^2 \leq \frac{1}{2} \| \widetilde{T} - \widetilde{T}_0 \|_{1,1}^2 .$$

This shows that the map S has a (unique) fixed point $T^1 = \widetilde{T}$ in $\mathbf{K}$ which is the desired solution of problem (4.7) (4.9). Moreover, again by theorem 2.1, one has

$$|T_\phi^1| \leq \frac{cw}{\cos^\vartheta \phi} , \quad |T_\psi^1| \leq \frac{cw}{\sin^\vartheta \psi} , \quad \text{on } \Lambda . \tag{4.16}$$

Theorem 2.2 shows that

$$|T_\phi^1| \leq \frac{cw}{\sin^\vartheta \psi} \qquad \text{on } \Lambda . \tag{4.17}$$

□

Remark 4.1 *The estimates (2.36) (2.37) by themselves do not seem to be sufficient to prove the existence of the above fixed point. In fact,* $\| \widetilde{T} \|_{2,0} < +\infty$ *does not imply that* $\| \widetilde{T}_\phi \|_{1,0} < +\infty$ *(hence, does not imply that* $\| F_2(\widetilde{T}_\phi) \|_{1,0} < +\infty$*); consider, for instance, the function* $\widetilde{T}(\phi) = \int_0^\phi (1/\cos^\vartheta s)\, ds$. *It seems necessary to use (2.38) (2.39), as was done above.*

□

Let now T be defined by the equation (4.6). From (4.4) and $(4.16)_2$ it follows that

$$T_\psi \geq \frac{c}{\sin^\vartheta \psi} \qquad \text{on } \Lambda , \tag{4.18}$$

if in the equation (4.10) c_0 is sufficiently small. Next define $X(\phi, \psi)$ by the equation (4.3). It readily follows that the pair (X, T) satisfies the equation (3.22) and the boundary conditions $(3.23)_1$ and (3.24). Now we choose $w = w(a)$ in such a way that $(3.23)_2$ is satisfied. From $(3.22)_2$, (3.13) and $(4.2)_4$ it follows that $X_\psi(\pi/2, \psi) \equiv 0$. Hence

$$X(\pi/2, \psi) = \int_0^{\pi/2} c(\phi, \psi)\, T_\psi(\phi, \psi)\, d\phi \equiv \overline{x} \tag{4.19}$$

is independent of ψ. From (4.6) and $(4.16)_2$ it readily follows that

$$\overline{x} = \frac{\sqrt{m}}{k_0} \left(\frac{m+1}{2m} \right)^{\vartheta/2} \nu w^\vartheta \int_0^{\pi/2} \cos^\vartheta \phi \, d\phi + c\nu\varepsilon(w, a)$$

where $\varepsilon(w, a) = (1 + wa^{1/\vartheta}) O(w^{1+\vartheta} a^{1/\vartheta})$. Moreover $\varepsilon(w, a)$ is a continuous function of w (and of a) since $g \in C^1$. By taking into account that (setting $u = \cos \phi$)

$$\int_0^{\pi/2} \cos^\vartheta \phi \, d\phi = \int_0^1 \frac{du}{(1 - u^2)^{1/(m+1)}}$$

it readily follows that

$$\overline{x} = \frac{1}{2}\,(\frac{w}{w_0})^{\vartheta} + c\nu\varepsilon(w,a)\ .$$

Let now $(w_1/w_0)^{\vartheta} = 1/8$, $(w_2/w_0)^{\vartheta} = 3/8$ and let $a > 0$ be such that $c\nu\varepsilon(w,a) < 1/4$, for each $w \in [w_1, w_2]$. Since $\overline{x}(w_1) < 1/2$ and $\overline{x}(w_2) > 1/2$, there is $w \in]w_1, w_2[$ such that $\overline{x} = 1/2$. Hence, it follows from (4.19) that to each positive value of the parameter a, $0 < a \ll 1$, there corresponds a value of the parameter w such that $(3.23)_2$ holds.

Finally we prove that the map $(\phi,\psi) \to T(\phi,\psi)$ is one to one from $\overline{\Lambda}$ onto $\overline{Q}$.

Remark 4.2 *From $(3.22)_1$ (3.13) and (4.18) it follows that*

$$X_\phi \geq c\nu w^{\vartheta} \cos^{\vartheta}\phi\ , \qquad 0 \leq \phi < \pi/2\ . \tag{4.20}$$

This estimate, together with (4.18) and the boundary conditions (3.23) (3.24), is not sufficient to prove that the map $(\phi,\psi) \to (X,T)$ is one to one. Moreover, the Jacobian $J = X_\phi T_\psi - X_\psi T_\phi$ may change sign in Λ for arbitrarily small values of the positive parameter a. In fact, (4.18) and (4.20) show that

$$X_\phi T_\psi \geq c\nu w^{\vartheta}\frac{\cos^{\vartheta}\phi}{\sin^{\vartheta}\psi} \qquad \text{on } \Lambda\ . \tag{4.21}$$

On the other hand (4.6) $(4.16)_1$ and (3.13) show that $0 \leq X_\psi T_\phi \leq c(\phi,\psi) T_\phi^2 \leq c w^{2+\vartheta} a^{2/\vartheta} \sin^{\vartheta}\psi (\cos^{\vartheta}\phi)^{-1}$. Hence

$$J \geq c w^{\vartheta}\,(\frac{\cos^{\vartheta}\phi}{\sin^{\vartheta}\psi} - c(w a^{1/\vartheta})^2\,\frac{\sin^{\vartheta}\psi}{\cos^{\vartheta}\phi})\ . \tag{4.22}$$

When $a = 0$ one has $J > 0$ on Λ. However this result does not follow from (4.22) if $a > 0$. This shows that the estimates (4.16) by themselves are not sufficient to prove that $J > 0$ on Λ. This will be done using also the estimate (4.17), a consequence of theorem 2.2.

□

By using (4.17) instead of $(4.16)_1$ one gets

$$0 \leq X_\psi T_\phi \leq c w^{2+\vartheta} a^{2/\vartheta}\frac{\cos^{\vartheta}\phi}{\sin^{\vartheta}\psi}\ .$$

Hence,

$$J \geq c w^{\vartheta}\,(1 - c(w a^{1/\vartheta})^2)\,\frac{\cos^{\vartheta}\phi}{\sin^{\vartheta}\psi} > 0 \quad \text{on } \Lambda\ . \tag{4.23}$$

The equation (4.23) shows that the map $\mathcal{I} : (\phi,\psi) \to (X,T)$ is locally invertible on Λ. This map is also (Hölder-)continuous on $\overline{\Lambda}$. This last property together to

(3.23) (3.24) shows that $\mathcal{I}$ maps the boundary $\partial\Lambda$ onto the boundary ∂Q in the "natural" way, i.e.,

$$\mathcal{I}(\{(0,\psi) : 0 \leq \psi \leq \pi/2\}) = \{(0,T) : 0 \leq T \leq 1/2\} ,$$

and so on. It follows that $\mathcal{I}$ maps Λ onto Q. Since the inverse images of compact subsets of Q are compact subsets of Λ, and since Λ and Q are simply connected it follows that $\mathcal{I}$ is globally invertible on Λ (by Caccioppoli's theorem; see [1], Chapter 3, Theorems 1.7 and 1.8).

5 Proof of proposition 3.3

Clearly $k \in C^1([0,b]) \cap C^2(]0,b[)$, moreover

$$|k(t)| \leq ct^{\beta_0+1} , \quad |k'(t)| \leq ct^{\beta_0} , \quad |k''(t)| \leq ct^{\beta_0-1} . \tag{5.1}$$

Denote by $s = s(x)$ the inverse of the function $x = F(s)$ and write $s(x)$ in the form

$$s(x) = x^{\frac{2}{m+1}}(1 + h(x)) . \tag{5.2}$$

Note that $h(x) \equiv 0$ when $R(t) \equiv 0$. One has

$$F(s) = s^{\frac{m+1}{2}} + O(s^{\frac{m+3}{2}+\beta_0}) .$$

By setting $s = s(x)$ in this last equation and by straightforward manipulations we prove that

$$h(x) = O(x^{\frac{2\beta_0+2}{m+1}}) . \tag{5.3}$$

Next, by using (5.1) (5.2) and (5.3) we prove that

$$f(x) = \ell(s(x)) = x^{\vartheta}\left[1 + O(x^{\frac{(\beta_0+1)(m-1)}{m+1}}) + O(x^{\frac{2(\beta_0+1)}{m+1}})\right] .$$

Since g is defined by the equation $f(x) = x^{\vartheta}(1 + g(x))$ $(3.14)_1$ follows easily.

□

Next we differentiate the relation $\ell(s(x)) = x^{\vartheta}(1 + g(x))$ with respect to x and we replace there $\ell'(s(x))$ by the expression obtained from differentiation of (3.16) with respect to t (for $t = s(x)$) and replace $s'(x)$ by $1/F'(s(x))$. This gives (use also (5.2))

$$\frac{m+1}{m-1}xg'(x) + g(x) = -\frac{h(x)}{1+h(x)} + \frac{k'(s(x))\,x^{\frac{2}{m+1}}}{(m-1)[1+k(s(x))]} . \tag{5.4}$$

Hence

$$xg'(x) = O(g(x)) + O(x^{\frac{2(1+\beta_0)}{m+1}}) . \tag{5.5}$$

It readily follows $(3.14)_2$.

□

Next, we differentiate (5.4) with respect to x. Suitable manipulations show that

$$\frac{m+1}{m-1} x g''(x) + \frac{2m}{m-1} g'(x) = O(h'(x)) + O\big(x^{\frac{2(\beta_0+1)}{m+1}-1}\big) \ . \qquad (5.6)$$

On the other hand, differentiation of $x = F(s(x))$ yields

$$1 = \frac{m+1}{2} s(x)^{\frac{m-1}{2}} \sqrt{1+k(s(x))}\big[x^{\frac{2}{m+1}}(1+h(x))\big]' \ .$$

It readily follows that

$$\frac{m+1}{2} x^{\frac{2}{m+1}} h'(x) = x^{\frac{1-m}{m+1}} \left[\frac{1}{(1+h(x))\sqrt{1+k(s(x))}} - 1 + h(x) \right] . \qquad (5.7)$$

The expression within the square bracket has the form

$$\frac{1-(1+h(x))(1+(1/2)k(s(x)) + O(k(s(x))^2)}{(1+h(x))\sqrt{1+k(s(x))}} + h(x)$$

which, in turn, has the form $O(h(x)) + O(k(s(x))) = O(x^{2(\beta_0+1)/(m+1)})$.

Hence, from (5.7)

$$h'(x) = O\big(x^{\frac{2\beta_0 - m+1}{m+1}}\big) \ .$$

Consequently, we get from (5.6) that

$$g''(x) = O(g'(x)/x) + O\big(x^{\frac{2(\beta_0+1)}{m+1}-2}\big) \ .$$

This yields $(3.14)_3$. □

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