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Data Dependence in the Mathematical Theory of Compressible Inviscid Fluids

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1. Introduction

The aim of this paper is to describe a method that allows us to prove strong continuous dependence of solutions on the data for a large class of nonlinear partial differential equations. This problem is closely connected with that of the dependence of solutions of linear differential equations on the coefficients of the operators, and our method also allows us to get sharp results on that problem. It is worth noting that these problems can usually be solved easily in the elliptic and parabolic cases but are still unsolved for a large class of hyperbolic initial-boundary value problems.

We are particularly interested in giving a proof of the strong continuous dependence of solutions on the data for the motion of compressible inviscid fluids in domains with boundary. Since this problem is important in itself, we shall study it in detail instead of stating more general theorems, easily obtained by adapting the method followed here. In order to avoid further technicalities we consider a fluid filling the half space $\mathbb{R}^3_+$, and we study our problem in the space $H^k(\mathbb{R}^3_+)$ for the particular value $k = 3$ (see Theorem 3.2 below). The proof can easily be adapted to the case $k \geqq 3$, however, and also to open regular sets.

A main goal of the general theory of evolutionary partial differential equations is to extend to this field various results which are valid for ordinary differential equations. The main result of this last theory, namely, the theorem of existence, uniqueness, and continuous dependence on the data, gives rise to the notion of a well-posed problem in Hadamard's classical sense. On studying partial differential equations, the finite-dimensional space is replaced by a Hilbert space H and the ordinary differential equation by an equation $u' + A(u) = 0$, where A is an unbounded operator in H. After proving existence and uniqueness of the solution $u(t)$ for arbitrary initial data $u_0 \in H$, it remains to show that

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0 \quad \text{if} \quad \lim_{n\to\infty} \|u_0^n - u_0\| = 0,$$

where $\|\cdot\|$ denotes the norm in the space H. Convergence of $u_n(t)$ to $u(t)$ with respect to weaker norms can usually be obtained easily but is unacceptable as an ultimate result. In particular, weak results have no geometrical significance in terms of trajectories in the Hilbert space H.

The problem of continuous dependence of the solution on the initial data is particularly significant for the fundamental equations of motion of compressible inviscid fluids. Here it is necessary to distinguish between the Cauchy problem and the mixed problem, and between the incompressible and the compressible case.

Cauchy problem. The continuous dependence of the solution on the data was proved by Kato [K], both for incompressible and compressible fluids. For the incompressible case, see also Kato & Ponce [KP] and references therein.

Mixed problem, incompressible case. For incompressible fluids the compatibility conditions reduce simply to the initial velocity being tangent to the boundary. In this respect the problem is close to the Cauchy problem and, in fact, is still approachable by Kato's perturbation theory, as shown in reference [BV3].

The continuous dependence of solutions on the data was first proved by Ebin & Marsden [EM] by using techniques of Riemannian geometry on infinite-dimensional manifolds; see also [E2]. The corresponding result for non-homogeneous fluids was proved by Marsden [M]. Completely analytical proofs were given by Kato & Lai [KL] and (with a completely different method) in [BV3], for all $p \in (1, \infty)$. For non-homogeneous fluids an analytical proof was given in [BV4].

Mixed problem, compressible case. This is the most difficult situation to handle since the equations and the boundary conditions are particularly delicate. As far as I know, the result of this paper is the first one to appear in the literature. For that reason, it seems appropriate to review the main existence theorems*. The first such result was proved by Ebin [E1], assuming that the initial velocity is subsonic and the initial density is close to constant. Existence for arbitrarily large initial data was first proved in [BV1] and, independently, by Agemi [A]. In reference [BV1] a central role is played by the operators curl and divergence. These also play a central role in Schochet's paper [S1], where the author proves the existence of the solution in the general case $p = p(\rho, s)$ and also studies the incompressible limit. These results were extended to a class of first-order hyperbolic systems in [S2]. Reference [BV1] treats fluids in bounded domains. Reference [BV5], by following similar ideas, treats the case $\Omega = \mathbb{R}^3_+$. While it is not necessary to assume that the velocity field $v(t, \cdot)$ is square-integrable over $\mathbb{R}^3_+$ (see [BV5]), this assumption will nevertheless be made here for convenience and simplicity.

Finally, we call the reader's attention to the references [BV7] and [BV8] where the results proved below are extended to spaces $H^k(\Omega)$, $k \geq 3$, and to non-barotropic fluids. Recently, I also obtained similar results for fully nonlinear hyperbolic mixed problems.

* For the existence theorem for the Cauchy problem see [K, KM1, KM2].

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* For the existence theorem for the Cauchy problem see [K, KM1, KM2].

2. Notation

Here $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}^+$ the set of positive reals. We set $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}^+$ and $\Gamma = \mathbb{R}^2 \times \{0\}$, and let $\nu = (0, 0, 1)$ be the unit normal vector to Γ. We denote by z a generic point in $\mathbb{R}^3$. It is convenient to use distinct notations for the tangential directions z_1, z_2 and for the normal direction z_3. For that reason we write $z = (y, x)$ where $y = (y_1, y_2) = (z_1, z_2)$ and $x = z_3$.

Functions will be defined for the most part on $\mathbb{R}^3_+$. For that reason the symbol $\mathbb{R}^3_+$ will be dropped from the usual notation. For instance, L^2 denotes $L^2(\mathbb{R}^3_+)$, and so on. We define H^1_0 as the closure of C^∞_0 in H^1 and we set $H^k_0 = H^k \cap H^1_0$ for $k \in \mathbb{N}$. Note that H^k_0 is not the closure of C^∞_0 in H^k. We also use the abbreviated notation $\int F$ to denote integrals over $\mathbb{R}^3_+$.

We set $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial z_i$, $i = 1, 2, 3$. The symbol ∂_y denotes either of the derivatives ∂_1 or ∂_2. As usual, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index. We also set $(v \cdot \nabla)\, w = \Sigma_{i=1}^3 v_i(\partial_i w)$.

The norm in L^p, $p \in [1, +\infty]$, is denoted by $|\cdot|_p$. The norm in L^2 is however denoted in general by $\|\cdot\|$. Furthermore,

$$\|D^k u\|^2 = \sum_{|\alpha| = k} \|\partial^\alpha u\|^2, \quad \|u\|_k^2 = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|^2, \quad \|D^l u\|_k^2 = \sum_{l \leq |\alpha| \leq k+l} \|\partial^\alpha u\|^2.$$

For $T > 0$, we define $Q_T = [0, T] \times \mathbb{R}^3_+$ and $\Sigma_T = [0, T] \times \Gamma$. For brevity we set

$$C_T(H^k) = C([0, T]; H^k), \qquad L^2_T(H^k) = L^2(]0, T[; H^k),$$

and so on. The canonical norms in the above two spaces are denoted respectively by $\|\cdot\|_{k,T}$ and $|||\cdot|||_{k,T}$. The norm in $C_T(\bar{Q})$ is denoted by $|\cdot|_{\infty,T}$. These notations and others in the sequel will be used both for scalar and for vector fields.

Given an arbitrary function $f(t, z)$ we denote by $f(t)$, for each fixed t, the function $f(t, \cdot)$.

If X and Y are Banach spaces, $\mathscr{L}(X, Y)$ denotes the Banach space of bounded linear operators from X into Y. Moreover, $\mathscr{L}(X) = \mathscr{L}(X, X)$.

Finally, for estimating norms of products of functions we shall use the well-known embedding theorems of Sobolev without explicit mention. In particular, we recall that

$$H^2 \hookrightarrow L^\infty \quad \text{and} \quad H^1 \hookrightarrow L^6 \cap L^2 \hookrightarrow L^4.$$

3. Main Results

The initial-boundary problem for barotropic motion of a compressible inviscid fluid obeys the following equations (see for instance [Se, Sections C.I and E.I, II] and [Sd, IV, Section 1]),

$$
(3.1)\qquad
\begin{aligned}
\rho[\partial_t v + (v\cdot\nabla)\,v] + \nabla p(\rho) &= 0 \quad \text{in } Q_T,\\
\partial_t \rho + \nabla\cdot(\rho v) &= 0 \quad \text{in } Q_T,\\
v\cdot\nu &= 0 \quad \text{on } \Sigma_T,\\
v(0) = v_0,\qquad \rho(0) &= \rho_0,
\end{aligned}
$$

where v is the velocity field, ρ the density and p the pressure. The function $p:\mathbb{R}^+\to\mathbb{R}$ is given and assumed to be of class C^4 with $p'(s)>0$ for all $s\in\mathbb{R}^+$. We denote by σ a *fixed* positive constant, the value of the density at infinity. We set $H_\sigma=\{\rho:\rho-\sigma\in H^3\}$ and we define for each $f\in H^3_\sigma$,

$$m(f) = \inf_{z\in\mathbb{R}^3_+} f(z).$$

In equation (3.1) we assume that the initial data satisfy the following assumptions:

$$(3.2)\qquad v_0\in H^3,\quad v_0\cdot\nu=0\quad \text{on } \Gamma,\quad \rho_0\in H^3_\sigma,\quad m(\rho_0)>0,$$

together with the compatibility conditions

$$(3.3)\qquad \partial_x\rho_0 = 0,\quad \partial_x[\nabla\cdot(\rho_0 v_0)] = 0\quad \text{on } \Gamma.$$

For the following result see [BV5, Theorem 1.1] and also [BV1].

Theorem 3.1. *Under the above hypotheses, there are two positive constants T and c (which depend only on the quantities $\|v_0\|_3$, $\|\rho_0-\sigma\|_3$, and $m(\rho_0)$), such that there exists a unique solution (v,ρ) of problem* (3.1) *in the class of (v,ρ) satisfying v, $\rho-\sigma\in C_T(H^3)$. Moreover,*

$$(3.4)\qquad \sum_{j=0}^{3} (\|\partial_t^j v\|_{3-j,T} + \|\partial_t^j(\rho-\sigma)\|_{3-j,T}) \leqq c.$$

The constants c and T also depend on the value of the constant σ and on the particular function $p(\cdot)$, but this dependence is not taken into account since σ and $p(\cdot)$ are fixed once and for all. It is worth noting that an upper bound for c and a lower bound for T depend only on upper bounds for the norms $\|v_0\|_3$ and $\|\rho_0-\sigma\|_3$ and on a lower bound for $m(\rho_0)$. Finally the estimate (3.4) is equivalent to the estimate for $j=0$, since the estimates for $j=1, 2, 3$ then follow from the equations.

Now we state the main result of the paper.

Theorem 3.2. *Let v_0 and ρ_0 satisfy the assumptions* (3.2) *and* (3.3) *and let (v,ρ) be a solution in Q_{T_0} of problem* (3.1) *for some $T_0>0$.* Let (v_0^n,ρ_0^n), $n\in\mathbb{N}$, *be a sequence of initial data satisfying the hypotheses* (3.2) and (3.3). *Denote by (v_n,ρ_n) the solution of problem* (3.1) *with initial data (v_0^n,ρ_0^n) and assume that*

$$(3.5)\qquad \lim_{n\to\infty} (\|v_0^n - v_0\|_3 + \|\rho_0^n-\rho_0\|_3) = 0.$$

and E.I, II] and [Sd, IV, Section 1]),

$$
(3.1)\qquad
\begin{aligned}
\rho[\partial_t v + (v\cdot\nabla)v] + \nabla p(\rho) &= 0 \quad \text{in } Q_T,\\
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$$(3.4)\qquad \sum_{j=0}^{3}(\|\partial_t^j v\|_{3-j,T}+\|\partial_t^j(\rho-\sigma)\|_{3-j,T})\leq c.$$

The constants c and T also depend on the value of the constant σ and on the particular function $p(\cdot)$, but this dependence is not taken into account since σ and $p(\cdot)$ are fixed once and for all. It is worth noting that an upper bound for c and a lower bound for T depend only on upper bounds for the norms $\|v_0\|_3$ and $\|\rho_0-\sigma\|_3$ and on a lower bound for $m(\rho_0)$. Finally the estimate (3.4) is equivalent to the estimate for $j=0$, since the estimates for $j=1,2,3$ then follow from the equations.

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$$(3.5)\qquad \lim_{n\to\infty}(\|v_0^n-v_0\|_3+\|\rho_0^n-\rho_0\|_3)=0.$$

Then for sufficiently large values of n, the solutions (v_n,ρ_n) exist in Q_{T_0} and

$$(3.6)\qquad \lim_{n\to\infty}\sum_{j=0}^{3}(\|\partial_t^j(v_n-v)\|_{3-j,T_0}+\|\partial_t^j(\rho_n-\rho)\|_{2-j,T_0})=0.$$

It is convenient to make the change of variables $g=\log(\rho/\sigma)$ and to introduce the function $h(s)\equiv p'(\sigma e^s)$, for $s\in\mathbb{R}$. Clearly $h\in C^3(\mathbb{R};\mathbb{R}^+)$. The equations (3.1) are then equivalent to

$$
(3.7)\qquad
\begin{aligned}
\partial_t v + (v\cdot\nabla)v + h(g)\nabla g &= 0 \quad \text{in } Q_T,\\
\partial_t g + v\cdot\nabla g + \nabla\cdot v &= 0 \quad \text{in } Q_T,\\
v\cdot\nu &= 0 \quad \text{on } \Sigma_T,\\
v(0)=v_0,\quad g(0) &= g_0,
\end{aligned}
$$

where, by definition, $g_0(z)=\log(\rho_0(z)/\sigma)$. The assumptions (3.2) and (3.3) become

$$(3.8)\qquad v_0\in H^3,\quad v_0\cdot\nu=0 \quad \text{on } \Gamma,\quad g_0\in H^3,$$

and

$$(3.9)\qquad \partial_x g_0=0,\quad \partial_x[v_0\cdot\nabla g_0+\nabla\cdot v_0]=0 \quad \text{on } \Gamma.$$

It is easy to verify that Theorem 3.1 is equivalent to the following result (see [BV5, Theorem 1.2 and corollary]).

Theorem 3.3. *Let v_0 and g_0 satisfy the assumptions* (3.8) *and* (3.9). *There exist positive constants c and T, universal with respect to bounded sets of initial data, such that there is a unique solution (v,g) of problem* (3.7) *in Q_T with*

$$(3.10)\qquad \sum_{j=0}^{3}(\|\partial_t^j v\|_{3-j,T}+\|\partial_t^j g\|_{3-j,T})\leq c.$$

Remark. We say that T and a positive constant are *universal* with respect to bounded sets of initial data if a positive lower bound for T and an upper bound for c depend only on upper bounds for the norms $\|v_0\|_3$ and $\|g_0\|_3$.

Now let (v_0^n,g_0^n) be a sequence of initial data satisfying the conditions

$$(3.8)^n\qquad v_0^n\in H^3,\quad v_0^n\cdot\nu=0 \quad \text{on } \Gamma,\quad g_0^n\in H^3,$$

$$(3.9)^n\qquad \partial_x g_0^n=0,\quad \partial_x[v_0\cdot\nabla g_0^n+\nabla\cdot v_0^n]=0 \quad \text{on } \Gamma.$$

The next result, which is equivalent to Theorem 3.2, will be proved in the following sections.

Theorem 3.4. *Let (v_0,g_0) and (v_0^n,g_0^n), $n\in\mathbb{N}$, satisfy the assumptions* (3.8), (3.9), *and* $(3.8)^n$, $(3.9)^n$, *and let*

$$(3.11)\qquad \lim_{n\to\infty}(\|v_0^n-v_0\|_3+\|g_0^n-g_0\|_3)=0.$$

Moreover, let $(v, g) \in C_{T_0}(H^3)$ *be a solution of* (3.7) *in* Q_{T_0} *for some* $T_0 > 0$. *Then for sufficiently large values of* n *the solutions* (v_n, g_n) *of the problems*

$$(3.7)^n \qquad \begin{aligned} \partial_t v_n + (v_n \cdot \nabla) v_n + h(g_n) \nabla g_n = 0 \quad &\text{in } Q_T, \\ \partial_t g_n + v_n \cdot \nabla g_n + \nabla \cdot v_n = 0 \quad &\text{in } Q_T, \\ v_n \cdot \nu = 0, \quad &\text{on } \Sigma_T, \\ v_n(0) = v_0^n, \quad g_n(0) = g_0^n & \end{aligned}$$

exist for $T = T_0$. *Moreover*

$$(3.12) \qquad \lim_{n\to\infty} \sum_{j=0}^{3} (\| \partial_t^j (v_n - v) \|_{3-j,T} + \| \partial_t^j (g_n - g) \|_{3-j,T}) = 0.$$

Note that by Theorem 3.3 there are positive universal constants c and T such that the solutions (v, ρ) and (v_n, ρ_n) of problems (3.7) and $(3.7)^n$ exist on Q_T and respectively satisfy (3.10) and

$$(3.10)^n \qquad \sum_{j=0}^{3} (\| \partial_t^j v_n \|_{3-j,T} + \| \partial_t^j g_n \|_{3-j,T}) \leqq c.$$

Theorem 3.4 will be proved for the above choice of T (more precisely, for some universal value of T). Then a bootstrap argument easily shows that the result holds on $[0, T_0]$, since by a continuation argument the solution (v, ρ) exists in $[0, T_0 + \delta]$ for some $\delta > 0$.

Generic positive universal constants (see the definition above) are denoted by the symbol c. Thus the value of c or of c_1, c_2, etc., may change from relation to relation. Note that $\| \partial_t^j v_n \|_{3-j,T}$ and $\| \partial_t^j g_n \|_{3-j,T}$ can be replaced in the "right-hand sides" of various estimates by constants c.

We denote by h_0 the positive constant $h_0 = h(0)$ and for convenience, we set

$$(3.13) \qquad \begin{aligned} \bar{l}(t, x) &= h[g(t, x)], \qquad \bar{l}_n(t, x) = h[g_n(t, x)], \\ l(t, x) &= \bar{l}(t, x) - h_0, \qquad l_n(t, x) = \bar{l}_n(t, x) - h_0. \end{aligned}$$

Since $|g_n(t, z)| \leqq c$ for all $(t, z) \in Q_T$ and for all $n \in \mathbb{N}$, and since $h(\cdot)$ is a positive C^3 function, it readily follows that

$$(3.14) \qquad c^{-1} \leqq \bar{l}_n(t, z) \leqq c \quad \text{for all } (t, z) \in \bar{Q}_T$$

and that

$$(3.15) \qquad \sum_{j=0}^{3} \| \partial_t^j l_n \|_{3-j,T} \leqq c.$$

4. Weak Continuous Dependence on the Data

Lemma 4.1. *Let the hypothesis of Theorem 3.4 hold. Then*

$$(4.1) \qquad \lim_{n\to\infty} \sum_{j=0}^{2} (\| \partial_t^j (v - v_n) \|_{2-j,T} + \| \partial_t^j (g - g_n) \|_{2-j,T}) = 0,$$

$$(4.2) \qquad \lim_{n\to\infty} \sum_{j=0}^{2} \| \partial_t^j (l - l_n) \|_{2-j,T} = 0.$$

Moreover, let $(v, g) \in C_{T_0}(H^3)$ *be a solution of* (3.7) *in* Q_{T_0} *for some* $T_0 > 0$. *Then for sufficiently large values of n the solutions* (v_n, g_n) *of the problems*

$$(3.7)^n \qquad \begin{aligned} &\partial_t v_n + (v_n \cdot \nabla) v_n + h(g_n) \nabla g_n = 0 \quad \text{in } Q_T, \\ &\partial_t g_n + v_n \cdot \nabla g_n + \nabla \cdot v_n = 0 \quad \text{in } Q_T, \\ &v_n \cdot \nu = 0, \quad \text{on } \Sigma_T, \\ &v_n(0) = v_0^n, \quad g_n(0) = g_0^n \end{aligned}$$

exist for $T = T_0$. *Moreover*

$$(3.12) \qquad \lim_{n\to\infty} \sum_{j=0}^{3} (\|\partial_t^j(v_n - v)\|_{3-j,T} + \|\partial_t^j(g_n - g)\|_{3-j,T}) = 0.$$

Note that by Theorem 3.3 there are positive universal constants c and T such that the solutions (v, ρ) and (v_n, ρ_n) of problems (3.7) and $(3.7)^n$ exist on Q_T and respectively satisfy (3.10) and

$$(3.10)^n \qquad \sum_{j=0}^{3} (\|\partial_t^j v_n\|_{3-j,T} + \|\partial_t^j g_n\|_{3-j,T}) \leqq c.$$

Theorem 3.4 will be proved for the above choice of T (more precisely, for some universal value of T). Then a bootstrap argument easily shows that the result holds on $[0, T_0]$, since by a continuation argument the solution (v, ρ) exists in $[0, T_0 + \delta]$ for some $\delta > 0$.

Generic positive universal constants (see the definition above) are denoted by the symbol c. Thus the value of c or of c_1, c_2, etc., may change from relation to relation. Note that $\|\partial_t^j v_n\|_{3-j,T}$ and $\|\partial_t^j g_n\|_{3-j,T}$ can be replaced in the "right-hand sides" of various estimates by constants c.

We denote by h_0 the positive constant $h_0 = h(0)$ and for convenience, we set

$$(3.13) \qquad \begin{aligned} \bar{l}(t, x) &= h[g(t, x)], & \bar{l}_n(t, x) &= h[g_n(t, x)], \\ l(t, x) &= \bar{l}(t, x) - h_0, & l_n(t, x) &= \bar{l}_n(t, x) - h_0. \end{aligned}$$

Since $|g_n(t, z)| \leqq c$ for all $(t, z) \in Q_T$ and for all $n \in \mathbb{N}$, and since $h(\cdot)$ is a positive C^3 function, it readily follows that

$$(3.14) \qquad c^{-1} \leqq \bar{l}_n(t, z) \leqq c \quad \text{for all } (t, z) \in \bar{Q}_T$$

and that

$$(3.15) \qquad \sum_{j=0}^{3} \|\partial_t^j l_n\|_{3-j,T} \leqq c.$$

4. Weak Continuous Dependence on the Data

Lemma 4.1. *Let the hypothesis of Theorem 3.4 hold. Then*

$$(4.1) \qquad \lim_{n\to\infty} \sum_{j=0}^{2} (\|\partial_t^j(v - v_n)\|_{2-j,T} + \|\partial_t^j(g - g_n)\|_{2-j,T}) = 0,$$

$$(4.2) \qquad \lim_{n\to\infty} \sum_{j=0}^{2} \|\partial_t^j(l - l_n)\|_{2-j,T} = 0.$$

Proof. We start by showing that

$$(4.3) \qquad \lim_{n\to\infty} (\|v - v_n\|_{0,T} + \|g - g_n\|_{0,T}) = 0.$$

We form the difference between the equation $(3.7)_1^n$ and equation $(3.7)_1$, multiply each side of the equation obtained by $v_n - v$ and then integrate the result over $\mathbb{R}_+^3$. The same operations can be carried out for equations $(3.7)_2^n$ and $(3.7)_2$, with the multiplier now being $\bar{l}(g - g_n)$. After this process we add the two equations thus obtained. Straightforward calculations then yield the estimate

$$\frac{d}{dt} \int |v - v_0|^2 + \bar{l}|g - g_n|^2 \leqq c \int |v - v_n|^2 + |g - g_n|^2$$

for each $t \in [0, T]$. Since $\bar{l} \geqq c > 0$ on Q_T and since

$$\int |v - v_n|^2 + \bar{l}|g - g_n|^2 \leqq c(\|v_0 - v_0^n\|^2 + \|g_0 - g_0^n\|^2)$$

for $t = 0$, it readily follows that

$$\|v - v_n\|_{0,T}^2 + \|g - g_n\|_{0,T}^2 \leqq c e^{cT}(\|v_0 - v_0^n\|^2 + \|g_0 - g_0^n\|^2).$$

This gives (4.3). Since $\|\cdot\|_2 \leqq c\|\cdot\|_3^{2/3}\|\cdot\|_0^{1/3}$, it next follows that $\|v - v_n\|_{2,T}^2 + \|g - g_n\|_{2,T}^2$ tends to zero as n tends to infinity. By using the equations (3.7) and $(3.7)_n$ to express the derivatives with respect to t, we then obtain condition (4.1). Straightforward calculations finally yield (4.2); note that $|h^{(k)}(g) - h^{(k)}(g_n)| \leqq c|g - g_n|$ for $k = 0, 1, 2$.

5. An Equivalent Formulation of the Main Equations

Here we introduce the system of equations

$$(5.1) \qquad \begin{aligned} &\partial_t \zeta + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla) v + (\nabla \cdot v)\zeta = 0, \\ &(\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (h(g)\nabla g) = \Sigma(\partial_i v_j)(\partial_j v_i), \\ &-\nabla \cdot v = \partial_t g + v \cdot \nabla g, \\ &\nabla \times v = \zeta \quad \text{in } Q_T; \\ &v \cdot \nu = 0, \quad \partial_\nu g = 0 \quad \text{on } \Sigma_T; \\ &\zeta(0) = \zeta_0, \quad g(0) = g_0, \quad \partial_t g(0) = g_1, \quad \nabla \cdot v(0) = \nabla \cdot v_0 \end{aligned}$$

where v_0 and g_0 satisfy (3.8) and (3.9), and where, moreover,

$$(5.2) \qquad \zeta_0 = \nabla \times v_0, \quad g_1 = -(v_0 \cdot \nabla g_0 + \nabla \cdot v_0).$$

From equation (3.9) it follows that

$$(5.3) \qquad \partial_\nu g_0 = \partial_\nu g_1 = 0 \quad \text{on } \Gamma,$$

which are just the compatibility conditions for the system (5.1). One easily checks (see Appendix) that a pair (v, g) is a solution of problem (3.7) if and only if (ζ, v, g) is a solution of (5.1). Hence the two systems are equivalent. Clearly, a pair (v_n, g_n) is a solution of problem $(3.7)^n$ if and only if (ζ_n, v_n, g_n) is a solution of

$$\partial_t \zeta_n + (v_n \cdot \nabla) \zeta_n - (\zeta_n \cdot \nabla) v_n + (\nabla \cdot v_n) \zeta_n = 0,$$

$$(\partial_t + v_n \cdot \nabla)^2 g_n - \nabla \cdot \big(h(g_n) \nabla g_n\big) = \Sigma(\partial_i v_{n,j})(\partial_j v_{n,i}),$$

$$-\nabla \cdot v_n = \partial_t g_n + v_n \cdot \nabla g_n,$$

$$\nabla \times v_n = \zeta_n \quad \text{in } Q_T;$$

$$v_n \cdot \nu = 0, \qquad \partial_x g_n = 0 \quad \text{on } \Sigma_T;$$

$$\zeta_n(0) = \zeta_0^n, \quad g_n(0) = g_0^n, \quad \partial_t g_n(0) = g_1^n, \quad \nabla \cdot v_n(0) = \nabla \cdot v_0^n,$$

where

$$(5.2)^n \qquad \zeta_0^n = \nabla \times v_0^n, \quad g_1^n = -(v_0^n \cdot \nabla g_0^n + \nabla \cdot v_0^n).$$

Clearly,

$$(5.3)^n \qquad \partial_x g_0^n = \partial_x g_1^n = 0 \quad \text{or } \Gamma.$$

Note that by (3.11)

$$(5.4) \qquad \lim_{n\to\infty} (\| \zeta_0^n - \zeta_0 \|_2^2 + \| g_0^n - g_0 \|_3^2 + \| g_1^n - g_1 \|_2^2) = 0.$$

Remark. By working directly on the system (3.7) one obtains L^2 and H^1 energy estimates for the solution (v, g); see [BV2]. Higher-order interior estimates can be obtained as well, since derivatives satisfy equations similar to that satisfied by (v, g). Tangential higher-order estimates hold up to the boundary since the condition $v_3 = 0$ on Σ_T yields corresponding boundary conditions for tangential derivatives. This last argument fails for normal derivatives. A classical alternative device is to express, near the boundary, the normal derivatives $\partial_x v_1, \partial_x v_2, \partial_x v_3, \partial_x g$ in terms of the other first-order derivatives. For system (3.7) this device fails since the 4×4 matrix that should allow us to solve (algebraically) this system for the above four normal derivatives has rank 2 on Σ_T. These obstacles can be overcome by using the system (5.1). In fact, (i) The second-order equation $(5.1)_2$ can be solved algebraically (near the boundary) for $\partial_x^2 g$, since the corresponding coefficient in that equation does not vanish near Σ_T. (ii) For equation $(5.1)_1$ there is no substantial distinction between boundary and interior estimates, since neither boundary nor compatibility conditions are prescribed for this equation.

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$$\partial_t \zeta_n + (v_n \cdot \nabla)\zeta_n - (\zeta_n \cdot \nabla) v_n + (\nabla \cdot v_n)\zeta_n = 0,$$

$$(\partial_t + v_n \cdot \nabla)^2 g_n - \nabla \cdot (h(g_n)\nabla g_n) = \Sigma(\partial_i v_{n,j})(\partial_j v_{n,i}),$$

$$-\nabla \cdot v_n = \partial_t g_n + v_n \cdot \nabla g_n,$$

$$\nabla \times v_n = \zeta_n \quad \text{in } Q_T;$$

$$v_n \cdot \nu = 0, \quad \partial_x g_n = 0 \quad \text{on } \Sigma_T;$$

$$\zeta_n(0) = \zeta_0^n, \quad g_n(0) = g_0^n, \quad \partial_t g_n(0) = g_1^n, \quad \nabla \cdot v_n(0) = \nabla \cdot v_0^n,$$

where

$$\zeta_0^n = \nabla \times v_0^n, \quad g_1^n = -(v_0^n \cdot \nabla g_0^n + \nabla \cdot v_0^n). \tag{5.2^n}$$

Clearly,

$$\partial_x g_0^n = \partial_x g_1^n = 0 \quad \text{or } \Gamma. \tag{5.3^n}$$

Note that by (3.11)

$$\lim_{n\to\infty} (\|\zeta_0^n - \zeta_0\|_2^2 + \|g_0^n - g_0\|_3^2 + \|g_1^n - g_1\|_2^2) = 0. \tag{5.4}$$

Remark. By working directly on the system (3.7) one obtains L^2 and H^1 energy estimates for the solution (v, g); see [BV2]. Higher-order interior estimates can be obtained as well, since derivatives satisfy equations similar to that satisfied by (v, g). Tangential higher-order estimates hold up to the boundary since the condition $v_3 = 0$ on Σ_T yields corresponding boundary conditions for tangential derivatives. This last argument fails for normal derivatives. A classical alternative device is to express, near the boundary, the normal derivatives $\partial_x v_1$, $\partial_x v_2$, $\partial_x v_3$, $\partial_x g$ in terms of the other first-order derivatives. For system (3.7) this device fails since the 4×4 matrix that should allow us to solve (algebraically) this system for the above four normal derivatives has rank 2 on Σ_T. These obstacles can be overcome by using the system (5.1). In fact, (i) The second-order equation $(5.1)_2$ can be solved algebraically (near the boundary) for $\partial_x^2 g$, since the corresponding coefficient in that equation does not vanish near Σ_T. (ii) For equation $(5.1)_1$ there is no substantial distinction between boundary and interior estimates, since neither boundary nor compatibility conditions are prescribed for this equation.

6. Some Auxiliary Results

The aim of this section is to prove Theorem 6.3. We start by recalling some results for the mixed problem

$$\begin{aligned} &(\partial_t + v\cdot\nabla)^2 \Psi - \nabla\cdot(\bar{l}\nabla\Psi) = F \quad \text{in } Q_T,\\ &\partial_x \Psi = 0 \quad \text{on } \Sigma_T,\\ &\Psi(0) = \Psi_0, \quad \partial_t\Psi(0) = \Psi_1, \end{aligned} \tag{6.1}$$

where $\|\partial_t^j l\|_{3-j,T} \leqq c$, $\|\partial_t^j v\|_{3-j,T} \leqq c$ for $j = 0, 1, 2$; $v\cdot\nu = 0$ on Σ_T; and

$$c^{-1} \leqq \bar{l}(t, x) \equiv l(t, x) + h_0.$$

Note that the pairs (v, l) and (v_n, l_n), $n \in \mathbb{N}$, defined in the previous section satisfy these estimates uniformly with respect to n.

We assume that

$$\Psi_0 \in H^2, \quad \Psi_1 \in H^1, \quad \partial_x\Psi_0 = 0 \text{ on } \Gamma, \tag{6.2}$$

$$F \in L_T^2(H^1). \tag{6.3}$$

The following results were proved in [BV5, Theorems 3.1 and 3.2].

Theorem 6.1. *Under the hypotheses* (6.2) *and* (6.3), *there exists a sufficiently small "universal" constant T such that*

$$\sum_{j=0}^{1} \|\partial_t^j \Psi\|_{2-j,T}^2 \leqq c(\|\Psi_0\|_2^2 + \|\Psi_1\|_1^2 + |||F|||_{1,T}^2). \tag{6.4}$$

Theorem 6.2. *If* (6.2) *and* (6.3) *are replaced by the stronger hypotheses*

$$\Psi_0 \in H^3, \quad \Psi_1 \in H^2; \quad \partial_x\Psi_0 = \partial_x\Psi_1 = 0 \quad \text{on } \Gamma, \tag{6.5}$$

$$F \in L_T^2(H^2), \quad \partial_t F \in L_T^2(H^1), \tag{6.6}$$

then

$$\sum_{j=0}^{2} \|\partial_t^j \Psi\|_{3-j,T}^2 \leqq c(\|\Psi_0\|_3^2 + \|\Psi_1\|_2^2 + \|F(0)\|_1^2 + |||F|||_{2,T}^2 + |||\partial_t F|||_{1,T}^2). \tag{6.7}$$

For convenience we assume from now on that $T \leqq 1$. Consider the systems

$$\begin{aligned} &(\partial_t + v\cdot\nabla)^2\phi - \nabla\cdot(\bar{l}\nabla\phi) = f \quad \text{in } Q_T,\\ &\partial_x\phi = 0 \quad \text{on } \Sigma_T,\\ &\phi(0) = \phi_0, \quad (\partial_t\phi)(0) = \phi_1, \end{aligned} \tag{6.8}$$

and

$$(6.8)^n \qquad \begin{aligned} (\partial_t + v_n \cdot \nabla)^2 \phi_n - \nabla \cdot (\bar{l}_n \cdot \nabla \phi_n) &= f_n \quad \text{in } Q_T, \\ \partial_x \phi_n &= 0 \quad \text{on } \Sigma_T, \\ \phi_n(0) = \phi_0^n, \quad \partial_t \phi_n(0) &= \phi_1^n \end{aligned}$$

(for $n \in \mathbb{N}$), where $\phi_0, \phi_0^n \in H^2$, $\phi_1, \phi_1^n \in H^1$, $\partial_x \phi_0 = \partial_x \phi_0^n = 0$ on Γ, and $f, f_n \in L_T^2(H^1)$. Suppose that, moreover, $\|\phi_0^n\|_2 \leqq c$, $\|\phi_1^n\|_1 \leqq c$, $|||f_n|||_{1,T} \leqq c$, $|||\partial_t f_n|||_{0,T} \leqq c$, uniformly with respect to n. Here, $\bar{l} = h(g)$, $\bar{l}_n = h(g_n)$, and the couples (v, g), (v_n, g_n) are the solutions of (3.7) and $(3.7)^n$, respectively. Actually, we shall use only the properties (4.1), (3.10) and $(3.10)^n$ (for $j = 0, 1, 2$), and the tangency of the vectors v and v_n to the boundary. The existence of the solutions ϕ and ϕ_n of problems (6.8) and $(6.8)^n$ is guaranteed by Theorem 6.1. Finally, we assume that

$$(6.9) \qquad \lim_{n\to\infty} (\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2) = 0.$$

The remainder of this section is devoted to proving Theorem 6.3, a result which establishes the strong continuous dependence of the solution of the hyperbolic equation (6.8) on the coefficients.

Theorem 6.3. *Under the above hypotheses, to each $\varepsilon > 0$ there corresponds a positive constant $C(\varepsilon)$ such that*

(6.10)

$$\begin{aligned} \sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c\varepsilon &+ c(\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2) \\ &+ c|||f_n - f|||_{1,T}^2 \\ &+ C^2(\varepsilon)\left(\sum_{j=0}^{1} |||\partial_t^j(v_n - v)|||_{2-j,T}^2 + |||l_n - l|||_{2,T}^2\right). \end{aligned}$$

Corollary 6.4. *Under the above hypotheses, to each $\varepsilon > 0$ there corresponds an integer $N = N(\varepsilon)$ such that*

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c\varepsilon + c|||f_n - f|||_{1,T}^2 \quad \text{for } n \geqq N(\varepsilon).$$

Proof. Let $\varepsilon > 0$ be given and fix $\phi_0^\varepsilon \in H^3$, $\phi_1^\varepsilon \in H^2$, $f_\varepsilon \in L_T^2(H^2)$ such that $\partial_t f_\varepsilon \in L_T^2(H^1)$, $\partial_x \phi_0^\varepsilon = \partial_x \phi_1^\varepsilon = 0$ on Γ, and also

$$\|\phi_0^\varepsilon - \phi_0\|_2^2 + \|\phi_1^\varepsilon - \phi_1\|_1^2 + |||f_\varepsilon - f|||_{1,T}^2 < \varepsilon.$$

Consider the solution ϕ_ε of the problem

$$(6.11) \qquad \begin{aligned} (\partial_t + v \cdot \nabla)^2 \phi_\varepsilon - \nabla \cdot (\bar{l} \nabla \phi_\varepsilon) &= f_\varepsilon \quad \text{in } Q_T, \\ \partial_x \phi_\varepsilon = 0 \quad \text{on } \Sigma_T, \quad \phi_\varepsilon(0) = \phi_0^\varepsilon, &\quad \partial_t \phi_\varepsilon(0) = \phi_1^\varepsilon. \end{aligned}$$

and

$$(\partial_t + v_n \cdot \nabla)^2 \phi_n - \nabla \cdot (\bar{l}_n \cdot \nabla \phi_n) = f_n \quad \text{in } Q_T,$$

$$(6.8)^n \qquad \partial_x \phi_n = 0 \quad \text{on } \Sigma_T,$$

$$\phi_n(0) = \phi_0^n, \quad \partial_t \phi_n(0) = \phi_1^n$$

(for $n \in \mathbb{N}$), where $\phi_0, \phi_0^n \in H^2$, $\phi_1, \phi_1^n \in H^1$, $\partial_x \phi_0 = \partial_x \phi_0^n = 0$ on Γ, and $f, f_n \in L_T^2(H^1)$. Suppose that, moreover, $\|\phi_0^n\|_2 \leqq c$, $\|\phi_1^n\|_1 \leqq c$, $|||f_n|||_{1,T} \leqq c$, $|||\partial_t f_n|||_{0,T} \leqq c$, uniformly with respect to n. Here, $\bar{l} = h(g)$, $\bar{l}_n = h(g_n)$, and the couples (v, g), (v_n, g_n) are the solutions of (3.7) and $(3.7)^n$, respectively. Actually, we shall use only the properties (4.1), (3.10) and $(3.10)^n$ (for $j = 0, 1, 2$), and the tangency of the vectors v and v_n to the boundary. The existence of the solutions ϕ and ϕ_n of problems (6.8) and $(6.8)^n$ is guaranteed by Theorem 6.1. Finally, we assume that

$$(6.9) \qquad \lim_{n\to\infty} (\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2) = 0.$$

The remainder of this section is devoted to proving Theorem 6.3, a result which establishes the strong continuous dependence of the solution of the hyperbolic equation (6.8) on the coefficients.

Theorem 6.3. *Under the above hypotheses, to each $\varepsilon > 0$ there corresponds a positive constant $C(\varepsilon)$ such that*

(6.10)

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c\varepsilon + c(\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2) + c|||f_n - f|||_{1,T}^2 + C^2(\varepsilon)\left(\sum_{j=0}^{1} |||\partial_t^j(v_n - v)|||_{2-j,T}^2 + |||l_n - l|||_{2,T}^2\right).$$

Corollary 6.4. *Under the above hypotheses, to each $\varepsilon > 0$ there corresponds an integer $N = N(\varepsilon)$ such that*

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c\varepsilon + c|||f_n - f|||_{1,T}^2 \quad \text{for } n \geqq N(\varepsilon).$$

Proof. Let $\varepsilon > 0$ be given and fix $\phi_0^\varepsilon \in H^3$, $\phi_1^\varepsilon \in H^2$, $f_\varepsilon \in L_T^2(H^2)$ such that $\partial_t f_\varepsilon \in L_T^2(H^1)$, $\partial_x \phi_0^\varepsilon = \partial_x \phi_1^\varepsilon = 0$ on Γ, and also

$$\|\phi_0^\varepsilon - \phi_0\|_2^2 + \|\phi_1^\varepsilon - \phi_1\|_1^2 + |||f_\varepsilon - f|||_{1,T}^2 < \varepsilon.$$

Consider the solution ϕ_ε of the problem

$$(\partial_t + v \cdot \nabla)^2 \phi_\varepsilon - \nabla \cdot (\bar{l} \nabla \phi_\varepsilon) = f_\varepsilon \quad \text{in } Q_T,$$

$$(6.11) \qquad \partial_x \phi_\varepsilon = 0 \quad \text{on } \Sigma_T, \quad \phi_\varepsilon(0) = \phi_0^\varepsilon, \quad \partial_t \phi_\varepsilon(0) = \phi_1^\varepsilon.$$

By applying Theorem 6.2 to this system we get

$$(6.12) \qquad \sum_{j=0}^{2} \|\partial_t^j \phi_\varepsilon\|_{3-j,T}^2 \leqq C^2(\varepsilon),$$

where

$$C^2(\varepsilon) \equiv c(\|\phi_0^\varepsilon\|_3^2 + \|\phi_1^\varepsilon\|_2^2 + \|f_\varepsilon(0)\|_1^2 + |||f_\varepsilon|||_{2,T}^2 + |||\partial_t f_\varepsilon|||_{1,T}^2).$$

Next we estimate $\phi_n - \phi_\varepsilon$. By taking the difference between equations $(6.8)^n$ and $(6.11)_\varepsilon$ we get

$$(6.13)_n \qquad (\partial_t + v_n \cdot \nabla)^2 (\phi_n - \phi_\varepsilon) - \nabla \cdot [\bar{l}_n \nabla (\phi_n - \phi_\varepsilon)] = F_n$$

where

$$F_n = -(\partial_t + v_n \cdot \nabla)[(v_n - v) \cdot \nabla \phi_\varepsilon] - (v_n - v) \cdot \nabla(\partial_t \phi_\varepsilon + v \cdot \nabla \phi_\varepsilon) + \nabla \cdot [(l_n - l)\nabla \phi_\varepsilon] + f_n - f_\varepsilon.$$

Moreover,

$$\partial_x(\phi_n - \phi_\varepsilon) = 0 \text{ on } \Sigma_T, \quad (\phi_n - \phi_\varepsilon)(0) = \phi_0^n - \phi_0^\varepsilon, \quad \partial_t(\phi_n - \phi_\varepsilon)(0) = \phi_1^n - \phi_1^\varepsilon.$$

Now we apply Theorem 6.1 to the solution $\Psi = \phi_n - \phi_\varepsilon$ of problem $(6.13)_n$. Let us estimate F_n. For each $t \in [0, T]$ one has

$$\|F_n\|_1 \leqq \|(\partial_t + v_n \cdot \nabla)(v_n - v)\|_1 \|\nabla \phi_\varepsilon\|_2 + \|v_n - v\|_2 \|(\partial_t + v_n \cdot \nabla)\nabla \phi_\varepsilon\|_1 + \|v_n - v\|_2^2 \|\partial_t \phi_\varepsilon + v \cdot \nabla \phi_\varepsilon\|_2 + \|l_n - l\|_2 \|\phi_\varepsilon\|_3 + \|f_n - f_\varepsilon\|_1.$$

By using (6.2) we readily see that

$$\|F_n\|_1 \leqq C(\varepsilon)\left(\sum_{j=0}^{1} \|\partial_t^j(v_n - v)\|_{2-j,T} + \|l_n - l\|_2\right) + \|f_n - f_\varepsilon\|_1.$$

(Here we denote products of c and $C(\varepsilon)$ by the same symbol $C(\varepsilon)$.) Hence,

(6.14)

$$|||F_n|||_{1,T}^2 \leqq C^2(\varepsilon)\int_0^T \left(\sum_{j=0}^{1} \|\partial_t^j(v_n - v)\|_{2-j}^2 + \|l_n - l\|_2^2\right) dt + \int_0^T \|f_n - f_\varepsilon\|_1^2 \, dt.$$

Consequently, by applying (6.4) to $\Psi = \phi_n - \phi_\varepsilon$, we get

$(6.15)_n^\varepsilon$

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi_\varepsilon)\|_{2-j,T}^2 \leqq c(\|\phi_0^n - \phi_0^\varepsilon\|_2^2 + \|\phi_1^n - \phi_1^\varepsilon\|_1^2 + |||f_n - f_\varepsilon|||_{1,T}^2) + C^2(\varepsilon)\left(\sum_{j=0}^{1} |||\partial_t^j(v_n - v)|||_{2-j,T}^2 + |||l_n - l|||_{2,T}^2\right).$$

Instead of estimating $\phi_n - \phi_\varepsilon$ by taking the difference between equations $(6.8)_n$ and (6.11), we estimate $\phi - \phi_\varepsilon$ by taking the difference between equations (6.8) and (6.11). This yields

$$(6.15)^{\varepsilon} \quad \sum_{j=0}^{1} \|\partial_t^j(\phi - \phi_\varepsilon)\|_{2-j,T}^2 \leqq c(\|\phi_0 - \phi_0^\varepsilon\|_2^2 + \|\phi_1 - \phi_1^\varepsilon\|_1^2 + |||f - f_\varepsilon|||_{1,T}^2).$$

From $(6.15)_n^\varepsilon$ and $(6.15)^\varepsilon$ we get

(6.16)

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c(\|\phi_0 - \phi_0^\varepsilon\|_2^2 + \|\phi_1 - \phi_1^\varepsilon\|_1^2 + |||f - f_\varepsilon|||_{1,T}^2)$$
$$+ c(\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2 + |||f_n - f|||_{1,T}^2)$$
$$+ C^2(\varepsilon)\left(\sum_{j=0}^{1} |||\partial_t^j(v_n - v)|||_{2-j,T}^2 + |||l_n - l|||_{2,T}^2\right).$$

This proves Theorem 6.3. Corollary 6.4 follows immediately from (6.10), in view of (6.9) and Lemma 4.1. □

Theorem 6.6. *Let $n \in \mathbb{N}$. Assume that v and v_n satisfy* (3.10) *and* $(3.10)^n$ *respectively, that $v \cdot \nu = v_n \cdot \nu = 0$ on Σ_T, that f, f_n are uniformly bounded in $L_T^2(H^2)$, and that ζ_0 and ζ_0^n are defined as in* (5.2) *and* $(5.2)^n$. *Let ζ and ζ_n be the solutions of the systems*

$$(6.17) \qquad \partial_t\zeta + (v \cdot \nabla)\zeta = f \quad \textit{in } Q_T, \qquad \zeta(0) = \zeta_0,$$

$$(6.17)^n \qquad \partial_t\zeta_n + (v_n \cdot \nabla)\zeta_n = f_n \quad \textit{in } Q_T, \qquad \zeta_n(0) = \zeta_0^n,$$

respectively. Then, to each $\varepsilon > 0$ there corresponds a positive constant $C(\varepsilon)$ such that

(6.18)

$$\|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + c\{\|\zeta_0 - \zeta_0^n\|_2^2 + |||f - f_n|||_{2,T}^2 + C^2(\varepsilon)|||v - v_n|||_{2,T}^2\}.$$

Corollary 6.7. *Assume that the hypotheses of Theorem 6.6 hold, that $v - v_n$ satisfies* (4.1), *and that $\|\zeta_0 - \zeta_0^n\|_2 \to \infty$ as $n \to \infty$. Then, to each $\varepsilon > 0$ there corresponds an integer $N(\varepsilon)$ such that*

$$(6.19) \qquad \|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + c|||f - f_n|||_{2,T}^2 \quad \textit{for } n \geqq N(\varepsilon).$$

Proof of Theorem 6.6. From [BV5, Lemma (5.1)], if ξ is the solution of the problem

$$(6.20) \qquad \partial_t\xi + (w \cdot \nabla)\xi = F \text{ in } Q_T, \qquad \xi(0) = \xi_0,$$

where $w \in C_T(H^3)$, $w \cdot \nu = 0$ on Σ_T, $F \in L_T^2(H^k)$, $\xi_0 \in H^k$ (for $k = 0, 1, 2$, or 3), then

$$\|\xi\|_{k,T} \leqq \left(\|\xi_0\|_k + \int_0^T \|F(t)\|_k\, dt\right) \exp(c\|w\|_{3,T}\, T).$$

tions (6.8) and (6.11). This yields

$$(6.15)^{\varepsilon} \quad \sum_{j=0}^{1} \|\partial_t^j(\phi - \phi_\varepsilon)\|_{2-j,T}^2 \leqq c(\|\phi_0 - \phi_0^\varepsilon\|_2^2 + \|\phi_1 - \phi_1^\varepsilon\|_1^2 + |||f - f_\varepsilon|||_{1,T}^2).$$

From $(6.15)_n^\varepsilon$ and $(6.15)^\varepsilon$ we get

(6.16)

$$\sum_{j=0}^{1} \|\partial_t^j(\phi_n - \phi)\|_{2-j,T}^2 \leqq c(\|\phi_0 - \phi_0^\varepsilon\|_2^2 + \|\phi_1 - \phi_1^\varepsilon\|_1^2 + |||f - f_\varepsilon|||_{1,T}^2)$$
$$+ c(\|\phi_0^n - \phi_0\|_2^2 + \|\phi_1^n - \phi_1\|_1^2 + |||f_n - f|||_{1,T}^2)$$
$$+ C^2(\varepsilon)\left(\sum_{j=0}^{1} |||\partial_t^j(v_n - v)|||_{2-j,T}^2 + |||l_n - l|||_{2,T}^2\right).$$

This proves Theorem 6.3. Corollary 6.4 follows immediately from (6.10), in view of (6.9) and Lemma 4.1. □

Theorem 6.6. *Let $n \in \mathbb{N}$. Assume that v and v_n satisfy* (3.10) *and* $(3.10)^n$ *respectively, that $v \cdot \nu = v_n \cdot \nu = 0$ on Σ_T, that f, f_n are uniformly bounded in $L_T^2(H^2)$, and that ζ_0 and ζ_0^n are defined as in* (5.2) *and* $(5.2)^n$. *Let ζ and ζ_n be the solutions of the systems*

$$(6.17) \qquad \partial_t \zeta + (v \cdot \nabla)\zeta = f \quad \text{in } Q_T, \qquad \zeta(0) = \zeta_0,$$

$$(6.17)^n \qquad \partial_t \zeta_n + (v_n \cdot \nabla)\zeta_n = f_n \quad \text{in } Q_T, \qquad \zeta_n(0) = \zeta_0^n,$$

respectively. Then, to each $\varepsilon > 0$ there corresponds a positive constant $C(\varepsilon)$ such that

(6.18)

$$\|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + c\{\|\zeta_0 - \zeta_0^n\|_2^2 + |||f - f_n|||_{2,T}^2 + C^2(\varepsilon)|||v - v_n|||_{2,T}^2\}.$$

Corollary 6.7. *Assume that the hypotheses of Theorem 6.6 hold, that $v - v_n$ satisfies* (4.1), *and that $\|\zeta_0 - \zeta_0^n\|_2 \to \infty$ as $n \to \infty$. Then, to each $\varepsilon > 0$ there corresponds an integer $N(\varepsilon)$ such that*

$$(6.19) \qquad \|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + c|||f - f_n|||_{2,T}^2 \quad \textit{for } n \geqq N(\varepsilon).$$

Proof of Theorem 6.6. From [BV5, Lemma (5.1)], if ξ is the solution of the problem

$$(6.20) \qquad \partial_t \xi + (w \cdot \nabla)\xi = F \text{ in } Q_T, \qquad \xi(0) = \xi_0,$$

where $w \in C_T(H^3)$, $w \cdot \nu = 0$ on Σ_T, $F \in L_T^2(H^k)$, $\xi_0 \in H^k$ (for $k = 0, 1, 2$, or 3), then

$$\|\xi\|_{k,T} \leqq \left(\|\xi_0\|_k + \int_0^T \|F(t)\|_k\, dt\right) \exp(c\|w\|_{3,T}\, T).$$

Hence, if $T \leqq 1$ and $\|w\|_{3,T} \leqq c$, one has

$$(6.21) \qquad \|\xi\|_{k,T}^2 \leqq c(\|\xi_0\|_k^2 + |||F|||_{k,T}^2).$$

Consider the problem

$$(6.17)_\varepsilon \qquad \partial_t \zeta_\varepsilon + (v \cdot \nabla)\zeta_\varepsilon = f_\varepsilon \quad \text{in } Q_T, \qquad \zeta_\varepsilon(0) = \zeta_0^\varepsilon,$$

where $\zeta_0^\varepsilon \in H^3$, $f_\varepsilon \in L_T^2(H^3)$, $\|\zeta_0^\varepsilon - \zeta_0\|_2^2 < \varepsilon$, $|||f_\varepsilon - f|||_{2,T}^2 < \varepsilon$. The estimate (6.21) applied to the solution ζ_ε of problem $(6.17)_\varepsilon$ shows that

$$(6.22) \qquad \|\zeta_\varepsilon\|_{3,T} \leqq C(\varepsilon),$$

where $C(\varepsilon) = c(\|\zeta_0^\varepsilon\|_3 + |||f_\varepsilon|||_{3,T})$.

By forming the difference between the equations $(6.17)^n$ and $(6.17)_\varepsilon$ we obtain

$$\partial_t(\zeta_n - \zeta_\varepsilon) + (v_n \cdot \nabla)(\zeta_n - \zeta_\varepsilon) = -[(v_n - v)\cdot\nabla]\zeta_\varepsilon + f_n - f_\varepsilon.$$

Then by applying the estimate (6.21) for $k = 2$ to $\zeta_n - \zeta_\varepsilon$ and by using (6.22) one gets

$$(6.23)^\varepsilon \quad \|\zeta_\varepsilon - \zeta_n\|_{2,T}^2 \leqq c\{\|\zeta_0^n - \zeta_0^\varepsilon\|_2^2 + |||f_n - f_\varepsilon|||_{2,T}^2 + C^2(\varepsilon)|||v_n - v|||_{2,T}^2\}.$$

Hence,

$$(6.24) \quad \|\zeta_n - \zeta_\varepsilon\|_{2,T}^2 \leqq c\{\|\zeta_0 - \zeta_0^n\|_2^2 + |||f - f_n|||_{2,T}^2 + C^2(\varepsilon)|||v - v_n|||_{2,T}^2\}$$
$$+ c\{\|\zeta_0^\varepsilon - \zeta_0\|_2^2 + |||f_\varepsilon - f|||_{2,T}^2\}.$$

Moreover,

$$\|\zeta_\varepsilon - \zeta\|_{2,T}^2 \leqq c\{\|\zeta_0 - \zeta_0^\varepsilon\|_2^2 + |||f - f_\varepsilon|||_{2,T}^2\}.$$

Hence $\|\zeta - \zeta_n\|_{2,T}^2$ is bounded by the right-hand side of (6.18). □

Remark. Alternatively, Theorem 6.6 can also be proved by using the method of reference [KL] or by using the representation formulae (5.8), $(5.8)^n$ together with Theorem 6.4 in [BV3].

7. Proof of Theorem 3.4

By applying Corollary 6.7 to the solutions ζ and ζ_n of equations $(5.1)_1$ and $(5.1)_1^n$ we see that

$$\|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + |||(\zeta\cdot\nabla)v - (\zeta_n\cdot\nabla)v_n - (\nabla\cdot v)\zeta + (\nabla\cdot v_n)\zeta_n|||_{2,T}^2$$

for $n \geqq N(\varepsilon)$. Hence

$$\|\zeta - \zeta_n\|_{2,T}^2 \leqq c\varepsilon + c|||v - v_n|||_{3,T}^2.$$

On the other hand,

$$\|v - v_n\|_{3,T}^2 \leqq c(\|\zeta - \zeta_n\|_{2,T}^2 + \|\nabla\cdot(v - v_n)\|_{2,T}^2).$$

Consequently,

$$\|v - v_n\|_{3,T}^2 \leqq c\varepsilon + c\|\nabla\cdot(v - v_n)\|_{2,T}^2 + c_1|||v - v_n|||_{3,T}^2.$$

Assume from now on that $T < (2c_1)^{-1}$. By taking into account equations $(3.7)_2$ and $(3.7)_2^n$, we readily obtain

$$\|v - v_n\|_{3,T}^2 \leqq c\varepsilon + c \sum_{j=0}^{1} \|\partial_t^j (g - g_n)\|_{3-j,T}^2 + c\|v_n - v\|_{2,T}^2.$$

Since $\|v_n - v\|_{2,T} \to 0$ as $n \to \infty$, we get the following result.

Theorem 7.1. *To each $\varepsilon > 0$ there corresponds an integer $N(\varepsilon)$ such that*

$$\|v - v_n\|_{3,T}^2 \leqq c\varepsilon + c \sum_{j=0}^{1} \|\partial_t^j (g - g_n)\|_{3-j,T}^2 \quad \text{for } n \geqq N(\varepsilon). \tag{7.1}$$

Now we return to equations (5.1) and $(5.1)^n$, namely

$$\begin{aligned} &(\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (\bar{l} \nabla g) = \Sigma (\partial_i v_j)(\partial_j v_i) \quad \text{in } Q_T, \\ &\partial_x g = 0 \text{ on } \Sigma_T, \quad g(0) = g_0, \quad \partial_t g(0) = g_1, \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &(\partial_t + v_n \cdot \nabla)^2 g_n - \nabla \cdot (\bar{l}_n \nabla g_n) = \Sigma (\partial_i v_{n,j})(\partial_j v_{n,i}) \quad \text{in } Q_T, \\ &\partial_x g_n = 0 \text{ on } \Sigma_T, \quad g_n(0) = g_0^n, \quad \partial_t g_n(0) = g_1^n, \end{aligned} \tag{7.2^n}$$

where the initial data satisfy equation (5.4). For each tangential variable y, we apply the differentiation operator ∂_y to the above equations. Putting $\partial_y g = g_y$, we obtain from (7.2) that

$$\begin{aligned} &(\partial_t + v \cdot \nabla)^2 g_y - \nabla \cdot (\bar{l} \nabla g_y) = f \quad \text{in } Q_T, \\ &\partial_x (g_y) = 0 \text{ on } \Sigma_T, \quad g_y(0) = \partial_y g_0, \quad \partial_t g_y(0) = \partial_y g_1, \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} f = &-\partial_t (\partial_y v \cdot \nabla g) - (\partial_y v) \cdot \nabla (\partial_t g + v \cdot \nabla g) - v \cdot \nabla (\partial_y v \cdot \nabla g) \\ &+ 2\Sigma (\partial_i \partial_y v_j)(\partial_j v_i) + \nabla \cdot [h'(g)(\partial_y g) \nabla g]. \end{aligned} \tag{7.4}$$

From equation $(7.2)^n$ we get

$$\begin{aligned} &(\partial_t + v_n \cdot \nabla)^2 g_{n,y} - \nabla \cdot (\bar{l}_n \nabla g_{n,y}) = f_n \quad \text{on } Q_T, \\ &\partial_x (g_{n,y}) = 0 \text{ on } \Sigma_T, \quad g_{n,y}(0) = \partial_y g_0^n, \quad \partial_t g_{n,y}(0) = \partial_y g_{1,n}, \end{aligned} \tag{7.3^n}$$

where

$(7.4)^n$ f_n is obtained by replacing v and g in the right-hand side of (7.4) by v_n and g_n.

Straightforward calculations show that $\|f_n\|_{1,T} \leqq c$ and $\|\partial_t f_n\|_{0,T} \leqq c$.

Moreover,

$$\|f(t) - f_n(t)\|_1^2 \leqq c \sum_{j=0}^{1} (\|\partial_t^j (v(t) - v_n(t))\|_{3-j}^2 + \|\partial_t^j (g(t) - g_n(t))\|_{3-j}^2). \tag{7.5}$$

Assume from now on that $T < (2c_1)^{-1}$. By taking into account equations $(3.7)_2$ and $(3.7)_2^n$, we readily obtain

$$\|v - v_n\|_{3,T}^2 \leqq c\varepsilon + c\sum_{j=0}^{1} \|\partial_t^j(g - g_n)\|_{3-j,T}^2 + c\|v_n - v\|_{2,T}^2.$$

Since $\|v_n - v\|_{2,T} \to 0$ as $n \to \infty$, we get the following result.

Theorem 7.1. *To each $\varepsilon > 0$ there corresponds an integer $N(\varepsilon)$ such that*

$$(7.1)\qquad \|v - v_n\|_{3,T}^2 \leqq c\varepsilon + c\sum_{j=0}^{1} \|\partial_t^j(g - g_n)\|_{3-j,T}^2 \quad \text{for } n \geqq N(\varepsilon).$$

Now we return to equations (5.1) and $(5.1)^n$, namely

$$(7.2)\qquad \begin{aligned} &(\partial_t + v\cdot\nabla)^2 g - \nabla\cdot(\bar{l}\nabla g) = \Sigma(\partial_i v_j)(\partial_j v_i) \quad \text{in } Q_T,\\ &\partial_x g = 0 \text{ on } \Sigma_T, \quad g(0) = g_0, \quad \partial_t g(0) = g_1, \end{aligned}$$

and

$$(7.2)^n\qquad \begin{aligned} &(\partial_t + v_n\cdot\nabla)^2 g_n - \nabla\cdot(\bar{l}_n\nabla g_n) = \Sigma(\partial_i v_{n,j})(\partial_j v_{n,i}) \quad \text{in } Q_T,\\ &\partial_x g_n = 0 \text{ on } \Sigma_T, \quad g_n(0) = g_0^n, \quad \partial_t g_n(0) = g_1^n, \end{aligned}$$

where the initial data satisfy equation (5.4). For each tangential variable y, we apply the differentiation operator ∂_y to the above equations. Putting $\partial_y g = g_y$, we obtain from (7.2) that

$$(7.3)\qquad \begin{aligned} &(\partial_t + v\cdot\nabla)^2 g_y - \nabla\cdot(\bar{l}\nabla g_y) = f \quad \text{in } Q_T,\\ &\partial_x(g_y) = 0 \text{ on } \Sigma_T, \quad g_y(0) = \partial_y g_0, \quad \partial_t g_y(0) = \partial_y g_1, \end{aligned}$$

where

$$(7.4)\qquad \begin{aligned} f = &-\partial_t(\partial_y v\cdot\nabla g) - (\partial_y v)\cdot\nabla(\partial_t g + v\cdot\nabla g) - v\cdot\nabla(\partial_y v\cdot\nabla g)\\ &+2\Sigma(\partial_i\partial_y v_j)(\partial_j v_i) + \nabla\cdot[h'(g)(\partial_y g)\nabla g]. \end{aligned}$$

From equation $(7.2)^n$ we get

$$(7.3)^n\qquad \begin{aligned} &(\partial_t + v_n\cdot\nabla)^2 g_{n,y} - \nabla\cdot(\bar{l}_n\nabla g_{n,y}) = f_n \quad \text{on } Q_T,\\ &\partial_x(g_{n,y}) = 0 \text{ on } \Sigma_T, \quad g_{n,y}(0) = \partial_y g_0^n, \quad \partial_t g_{n,y}(0) = \partial_y g_{1,n}, \end{aligned}$$

where

$(7.4)^n$ f_n is obtained by replacing v and g in the right-hand side of (7.4) by v_n and g_n.

Straightforward calculations show that $\|f_n\|_{1,T} \leqq c$ and $\|\partial_t f_n\|_{0,T} \leqq c$.

Moreover,

$$(7.5)\qquad \|f(t) - f_n(t)\|_1^2 \leqq c\sum_{j=0}^{1} \left(\|\partial_t^j(v(t) - v_n(t))\|_{3-j}^2 + \|\partial_t^j(g(t) - g_n(t))\|_{3-j}^2\right).$$

Finally, by applying Corollary 6.4 to the solutions $\phi = g_y$ and $\phi_n = g_{n,y}$ of equations (7.3) and $(7.3)^n$, we obtain

$$(7.6)\qquad \begin{aligned} &\|g - g_n\|_{2,T}^2 + \sum_{i=1}^{2} \|\partial_i(g - g_n)\|_{2,T}^2\\ &\leqq c\varepsilon + c\sum_{j=0}^{1} \left(|||\partial_t^j(v - v_n)|||_{3-j,T}^2 + |||\partial_t^j(g - g_n)|||_{3-j,T}^2\right). \end{aligned}$$

Here the first term on the left-hand side of (7.6) comes from an application of Corollary 6.4 to the solutions $\phi = g$ and $\phi_n = g_n$ of equations (7.2) and $(7.2)^n$, respectively.

Next, by using differentiation with respect to t instead of with respect to y, we get equations similar to (7.3), $(7.3)^n$, (7.4) and $(7.4)^n$, with y of course replaced everywhere by t. Here, the initial conditions for $g_t = \partial_t g$ are $g_t(0) = g_1$ and $\partial_t g_t(0) = g_2$. An explicit expression for g_2 in terms of v_0, g_0, g_1 is obtained from equations $(3.7)_1$ and $(3.7)_2$. Similarly, one gets initial conditions $g_{n,t}(0) = g_1^n$ and $\partial_t g_{n,t}(0) = g_2^n$, and an explicit expression for g_2^n. Note that $\|g_2 - g_2^n\|_1 \to 0$ as $n \to \infty$. As above, we show that

$$(7.7)\qquad \|g - g_n\|_{2,T}^2 + \|\partial_t(g - g_n)\|_{2,T}^2 + \|\partial_t^2(g - g_n)\|_{1,T}^2$$

is bounded by the right-hand side of (7.6) plus an additional term $c\|\partial_t^2(v - v_n)\|_{1,T}^2$. By adding this last estimate to (7.6), we obtain, for $n \geqq N(\varepsilon)$,

$$(7.8)\qquad \begin{aligned} &\|g - g_n\|_{2,T}^2 + \sum_{i=1}^{2} \|\partial_i(g - g_n)\|_{2,T}^2 + \sum_{j=1}^{2} \|\partial_t^j(g - g_n)\|_{3-j,T}^2\\ &\leqq c\varepsilon + c\sum_{j=0}^{2} |||\partial_t^j(v - v_n)|||_{3-j,T}^2 + c\sum_{j=0}^{1} |||\partial_t^j(g - g_n)|||_{3-j,T}^2. \end{aligned}$$

Finally, we want to estimate $\|\partial_x^2(g - g_n)\|_{0,T}^2$ (see [BV5] for similar calculations). We start by deriving expressions for $\partial_x^2 g$ and $\partial_x^2 g_n$ near the boundary Γ.

Consider the coefficient of $\partial_x^2 g_n$ in equation $(7.2)_1^n$, namely $-(\bar{l}_n - v_{3,n}^2)$. Since $\|v_n\|_{3,T} \leqq c$, it follows that v_n is Lipschitz continuous on $\mathbb{R}_+^3$, uniformly with respect to n and t. Moreover, $v_{3,n} = 0$ on Σ_T. Consequently $|v_{3,n}(t, y, x)| \leqq c_2 x$ on Q_T. On the other hand $\bar{l}_n(t, z) \geqq c_0^{-1}$ on Q_T. Hence there are constants $c_3 = 2c_0c_2$ and $c = 2c_0$ such that

$$\bar{l}_n - v_{3,n}^2 \geqq c^{-1} \text{ on } E = (0, T)\times S \quad \text{for } n \in \mathbb{N},$$

where S is the strip $S = \{(y, x) : 0 < x < c_3^{-1}\}$. Consequently, equation $(7.2)_1^n$ yields an expression for $\partial_x^2 g_n$ on E. By taking the derivative of this expression with respect to x, we then obtain a relation for $\partial_x^3 g_n$ on E. Similarly, $(7.2)_1$

yields on the set E the relation

$$\partial_{x}^{2} g = (\bar{l} - v_3^2)^{-1} \Big\{ \partial_{t}^{2} g + \partial_t (v \cdot \nabla g) + v \cdot (\nabla\, \partial_t g) + \sum_{i=1}^{2} v_i \partial_i (v \cdot \nabla g) + v_3 \partial_x \Big(\sum_{i=1}^{2} v_i \partial_i g \Big) + v_3 (\partial_x v_3)(\partial_x g) - \bar{l}(\partial_1^2 g + \partial_2^2 g) - \nabla \bar{l} \cdot \nabla g - \Sigma (\partial_i v_j)(\partial_j v_i) \Big\} .$$

By differentation with respect to x we get an expression for $\partial_x^3 g$ on E. Clearly, this exression is the same as that obtained for $\partial_x^3 g_n$ if one simply replaces v_n, g_n and $\bar{l}_n \equiv h(g_n)$ by v, g and $l \equiv h(g)$ respectively. Next we form the difference between $\partial_x^3 g$ and $\partial_x^3 g_n$ and estimate its L^2-norm over S (for each t) by comparing the difference between each single pair of homologous terms; here estimates of the form $|h''(g_n) - h''(g)|_\infty \leqq c \| g - g_n \|_2$ are used at each place. Trivial calculations now show that

$$\| \partial_x^3 g - \partial_x^3 g_n \|_{C_T(L^2(S))} \leqq c \Big(\sum_{j=0}^{1} \| \partial_t^j (v - v_n) \|_{2-j,T} + \sum_{j=1}^{2} \| \partial_t^j (g - g_n) \|_{3-j,T} + \sum_{i=1}^{2} \| \partial_i (g - g_n) \|_{2,T} \Big) .$$

Since $\| v - v_n \|_{2,T}$ and $\| \partial_t (v - v_n) \|_{1,T}$ tend to zero as $n \to \infty$, it follows that there is an integer $N(\varepsilon)$ such that

(7.9)
$$\| \partial_x^3 g - \partial_x^3 g_n \|_{C_T(L^2(S))} \leqq c\varepsilon + \sum_{j=1}^{2} \| \partial_t^j (g - g_n) \|_{3-j,T} + c \sum_{i=1}^{2} \| \partial_1 (g - g_n) \|_{2,T}$$

for $n \geqq N(\varepsilon)$.

Finally, we estimate the L^2-norm of $\partial_x^3 g - \partial_x^3 g_n$ on $\mathbb{R}_+^3 \backslash S$. Fix a function $\vartheta \in C^\infty(\mathbb{R}^+)$, $0 \leqq \vartheta(x) \leqq 1$, such that $\vartheta(x) = 0$ if $0 \leqq x \leqq c_3^{-1}/2$ and $\vartheta(x) = 1$ if $x \geqq c_3^{-1}$. By applying the operator $(\partial_t + v \cdot \nabla)^2 - \nabla \cdot (\bar{l} \nabla)$ to the function ϑg and by taking equation (7.2) into account, we obtain

$$(\partial_t + v \cdot \nabla)^2 (\vartheta g) - \nabla \cdot [\bar{l} \nabla (\vartheta g)] = H[\vartheta, \bar{l}, v, g] \quad \text{in } Q_T,$$

$$\partial_x (\vartheta g) = 0 \text{ on } \Sigma_T, \qquad (\vartheta g)(0) = \vartheta g_0, \qquad \partial_t (\vartheta g)(0) = \vartheta g_1,$$

where $H = \vartheta \Sigma (\partial_i v_j)(\partial_j v_i) + H_0$ and $H_0[\vartheta, \bar{l}, v, g]$ is the "commutator"

$$H_0 = (\partial_t + v \cdot \nabla)^2 (\vartheta g) - \vartheta (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot [\bar{l} \nabla (\vartheta g)] + \vartheta \nabla \cdot (\bar{l} \nabla g) .$$

By taking derivatives with respect to x, we then get

(7.10)
$$(\partial_t + v \cdot \nabla)^2 (\vartheta g)_x - \nabla \cdot [\bar{l} \nabla (\vartheta g)_x] = G[\vartheta, \bar{l}, v, g] \quad \text{in } Q_T,$$

$$\partial_x (\vartheta g)_x = 0 \text{ on } \Sigma_T, \qquad (\vartheta g)_x (0) = \partial_x (\vartheta g_0), \qquad \partial_t (\vartheta g)_x (0) = \partial_x (\vartheta g_1),$$

yields on the set E the relation

$$\partial_x^2 g = (\bar{l} - v_3^2)^{-1} \Big\{ \partial_t^2 g + \partial_t (v \cdot \nabla g) + v \cdot (\nabla\, \partial_t g)$$

$$+ \sum_{i=1}^{2} v_i \partial_i (v \cdot \nabla g) + v_3 \partial_x \Big(\sum_{i=1}^{2} v_i \partial_i g \Big) + v_3 (\partial_x v_3)(\partial_x g)$$

$$- \bar{l}(\partial_1^2 g + \partial_2^2 g) - \nabla \bar{l} \cdot \nabla g - \Sigma(\partial_i v_j)(\partial_j v_i) \Big\}.$$

By differentation with respect to x we get an expression for $\partial_x^3 g$ on E. Clearly, this exression is the same as that obtained for $\partial_x^3 g_n$ if one simply replaces v_n, g_n and $\bar{l}_n \equiv h(g_n)$ by v, g and $l \equiv h(g)$ respectively. Next we form the difference between $\partial_x^3 g$ and $\partial_x^3 g_n$ and estimate its L^2-norm over S (for each t) by comparing the difference between each single pair of homologous terms; here estimates of the form $|h''(g_n) - h''(g)|_\infty \leqq c\|g - g_n\|_2$ are used at each place. Trivial calculations now show that

$$\|\partial_x^3 g - \partial_x^3 g_n\|_{C_T(L^2(S))} \leqq c \Big(\sum_{j=0}^{1} \|\partial_t^j (v - v_n)\|_{2-j,T}$$

$$+ \sum_{j=1}^{2} \|\partial_t^j (g - g_n)\|_{3-j,T} + \sum_{i=1}^{2} \|\partial_i (g - g_n)\|_{2,T} \Big).$$

Since $\|v - v_n\|_{2,T}$ and $\|\partial_t (v - v_n)\|_{1,T}$ tend to zero as $n \to \infty$, it follows that there is an integer $N(\varepsilon)$ such that

(7.9)
$$\|\partial_x^3 g - \partial_x^3 g_n\|_{C_T(L^2(S))} \leqq c\varepsilon + \sum_{j=1}^{2} \|\partial_t^j (g - g_n)\|_{3-j,T} + c \sum_{i=1}^{2} \|\partial_1 (g - g_n)\|_{2,T}$$

for $n \geqq N(\varepsilon)$.

Finally, we estimate the L^2-norm of $\partial_x^3 g - \partial_x^3 g_n$ on $\mathbb{R}_+^3 \backslash S$. Fix a function $\vartheta \in C^\infty(\mathbb{R}^+)$, $0 \leqq \vartheta(x) \leqq 1$, such that $\vartheta(x) = 0$ if $0 \leqq x \leqq c_3^{-1}/2$ and $\vartheta(x) = 1$ if $x \geqq c_3^{-1}$. By applying the operator $(\partial_t + v \cdot \nabla)^2 - \nabla \cdot (\bar{l}\nabla)$ to the function ϑg and by taking equation (7.2) into account, we obtain

$$(\partial_t + v \cdot \nabla)^2 (\vartheta g) - \nabla \cdot [\bar{l} \nabla (\vartheta g)] = H[\vartheta, \bar{l}, v, g] \quad \text{in } Q_T,$$

$$\partial_x (\vartheta g) = 0 \text{ on } \Sigma_T, \quad (\vartheta g)(0) = \vartheta g_0, \quad \partial_t (\vartheta g)(0) = \vartheta g_1,$$

where $H = \vartheta \Sigma(\partial_i v_j)(\partial_j v_i) + H_0$ and $H_0[\vartheta, \bar{l}, v, g]$ is the "commutator"

$$H_0 = (\partial_t + v \cdot \nabla)^2 (\vartheta g) - \vartheta(\partial_t + v \cdot \nabla)^2 g - \nabla \cdot [\bar{l} \nabla (\vartheta g)] + \vartheta \nabla \cdot (\bar{l} \nabla g).$$

By taking derivatives with respect to x, we then get

(7.10)
$$(\partial_t + v \cdot \nabla)^2 (\vartheta g)_x - \nabla \cdot [\bar{l} \nabla (\vartheta g)_x] = G[\vartheta, \bar{l}, v, g] \quad \text{in } Q_T,$$
$$\partial_x (\vartheta g)_x = 0 \text{ on } \Sigma_T, \quad (\vartheta g)_x (0) = \partial_x (\vartheta g_0), \quad \partial_t (\vartheta g)_x (0) = \partial_x (\vartheta g_1),$$

where $G = \partial_x H + G_0$, and $G_0[\vartheta, \bar{l}, v, g]$ is the "commutator"

$$G_0 = (\partial_t + v \cdot \nabla)^2 \partial_x (\vartheta g) - \partial_x (\partial_t + v \cdot \nabla)^2 (\vartheta g) - \nabla \cdot [l \nabla\, \partial_x (\vartheta g)] + \partial_x \nabla \cdot [l \nabla (\vartheta g)].$$

Analogously, $(\vartheta g_n)_x$ satisfies

(7.10)^n the equation obtained from (7.10) by replacing g, v, $\bar{l}$, g_0, g_1 by g_n, v_n, $\bar{l}_n$, g_0^n, g_1^n.

Next we apply Corollary 6.4 to the solutions $\phi = (\vartheta g)_x$ and $\phi_n = (\vartheta g_n)_x$ of equations (7.10) and (7.10)^n. Note that $\partial_x(\vartheta g_0^n) \to \partial_x(\vartheta g_0)$ in H^2 and that $\partial_x(\vartheta g_1^n) \to \partial_x(\vartheta g_1)$ in H^1. It readily follows that

(7.11)
$$\|\partial_x (g - g_n)\|^2_{C_T(H^2(\mathbb{R}_+^3 \backslash S))} \leqq c\varepsilon + c|||G - G_n|||^2_{1,T}$$

for $n \geqq N(\varepsilon)$. Note that the left-hand side of the equation in Corollary 6.4 has been restricted to $\mathbb{R}_+^3 \backslash S$.

Now by taking into account the particular form of G, we easily show that

(7.12)
$$\|G[\vartheta, \bar{l}, v, g] - G[\vartheta, \bar{l}_n, v_n, g_n]\|_1$$
$$\leqq c \sum_{j=0}^{1} (\|\partial_t^j (v - v_n)\|_{3-j} + \|\partial_t^j (g - g_n)\|_{3-j}), \quad t \in [0, T].$$

From (7.9), (7.11) and (7.12) it follows that

(7.13)
$$\|\partial_x^3 (g - g_n)\|^2_{0,T} \leqq c\varepsilon + c \sum_{j=1}^{2} \|\partial_t^j (g - g_n)\|^2_{3-j,T} + c \sum_{i=1}^{2} \|\partial_i (g - g_n)\|^2_{2,T}$$
$$+ cT \sum_{j=0}^{1} (\|\partial_t^j (v - v_n)\|^2_{3-j,T} + \|\partial_t^j (g - g_n)\|^2_{3-j,T}).$$

Furthermore, (7.13) and (7.8) yield

(7.14)
$$\sum_{j=0}^{2} \|\partial_t^j (g - g_n)\|^2_{3-j,T} \leqq c\varepsilon + cT \sum_{j=0}^{2} \|\partial_t^j (v - v_n)\|^2_{3-j,T} + cT \sum_{j=0}^{1} \|\partial_t^j (g - g_n)\|^2_{3-j,T}.$$

On the other hand, from equations (3.7) and (3.7)^n we get

(7.15)
$$\sum_{j=1}^{2} \|\partial_t^j (v - v_n)\|^2_{3-j,T} \leqq c\|v - v_n\|^2_{3,T} + c \sum_{j=0}^{1} \|\partial_t^j (g - g_n)\|^2_{3-j,T}.$$

By imposing the additional restriction that $T \leqq c_4$ for a suitable constant c_4, we deduce from equations (7.1), (7.14) and (7.15) that to each $\varepsilon > 0$ there corresponds an integer $N(\varepsilon)$ such that

(7.16)
$$\sum_{j=0}^{2} \|\partial_t^j (g - g_n)\|^2_{3-j,T} \leqq c\varepsilon \quad \text{for } n \geqq N(\varepsilon).$$

Finally, equations (7.1), (7.15), and (7.16) show that

$$\sum_{j=0}^{2} \| \partial_t^j (v - v_n) \|_{3-j,T}^2 \leqq c\varepsilon. \tag{7.17}$$

In order to express the third derivatives with respect to time in terms of the other derivatives, we apply equations (3.7) and (3.7)″. This shows that (3.12) holds. The proof of Theorem 3.4 is completed.

Appendix

Here we prove the equivalence of (3.7) and (5.1). Assume that (v, g, ζ) is a solution of (5.1). For each fixed t, define the vector field

$$V \equiv \partial_t v + (v \cdot \nabla) v + h(g) \nabla g.$$

Since $\zeta = \nabla \times v$ and since $h(g)\nabla g$ is a gradient field on $\mathbb{R}^3_+$, it follows from $(5.1)_1$ that $\nabla \times V = 0$ on $\mathbb{R}^3_+$. On the other hand,

$$\nabla \cdot V = \partial_t \delta + (v \cdot \nabla)\delta + \Sigma(\partial_i v_j)(\partial_j v_i) + \nabla \cdot (h(g) \nabla g),$$

where $\delta \equiv \nabla \cdot v$. Since $\delta = -(\partial_t g + v \cdot \nabla g)$, it follows from $(5.1)_2$ that $\nabla \cdot V = 0$. The orthogonality of the vector fields v and $\nabla(v \cdot \nu)$ on Γ shows that

$$0 = \sum_{i,j} v_i\, \partial_i (v_j \nu_j) = [(v \cdot \nabla)\, v] \cdot \nu + \sum_{i,j} (\partial_i \nu_j)\, v_i v_j.$$

Hence $[(v \cdot \nabla) v] \cdot \nu = 0$ on Γ. On the other hand, $\partial_t (v \cdot \nu) = 0$ and $h(g)\, \partial_x g = 0$ on Γ. Consequently, $V \cdot \nu = 0$ on Γ.

Since $\nabla \times V = 0$ and $\nabla \cdot V = 0$ on $\mathbb{R}^3_+$ and since $V \cdot \nu = 0$ on Γ, it follows that $V = 0$ on $\mathbb{R}^3_+$. Hence $(3.7)_1$ holds. Moreover $v(0) = v_0$, since both these vector fields have in $\mathbb{R}^3_+$ the same divergence and the same curl, and both are tangential to Γ.

Conversely, if (v, g) is a solution of (3.7), set $\zeta = \nabla \times v$ and apply the operators curl, divergence, and $\nu \cdot$, to the equation $(3.7)_1$ and the identity $v(0) = v_0$. This yields (5.1).

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Finally, equations (7.1), (7.15), and (7.16) show that

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