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Kato's Perturbation Theory and Well-Posedness for the Euler Equations in Bounded Domains

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1. Introduction

As pointed out by KATO & LAI in reference [12] "the continuous dependence in 'strong' topology of the solution on the data is the most difficult part in a theory dealing with nonlinear equations of evolution. As far as we know, [8] is the only place where continuous dependence (in the strong sense) has been proved for the Euler equation in a bounded domain". Below I provide a new and simple proof of the above property (well-posedness) for the Euler equations in a bounded domain Ω, in Sobolev spaces $W^m \equiv W^{m,p}(\Omega)$ (see Theorem 5.3). I prove this result by using KATO's perturbation theory [10]. This requires, in particular, the construction of "Kato's operator S". The existence of such an operator is not a trivial matter when dealing with the Euler equations in domains with boundary. Let me quote again from the introduction of [12]. "The general theory developed in [11] by one of the authors for quasi-linear equations is unfortunately not applicable, since it is difficult to find the operator S with the required properties in the case of a bounded domain". In the sequel I succed in proving (by introducing a suitable device) that KATO's general perturbation theory does apply to the above problem.

It is worth noting that the method developed here applies to other interesting problems. We consider the Euler equations just to fix the ideas; an application to non-homogeneous fluids is given in reference [4].

This paper is organized as follows. In Section 2 some notations are fixed. In Section 3 the stationary equation (3.4) is studied. In Section 4 we establish the main result; namely the perturbation Theorem 4.2. In Section 5 an application of Theorem 4.2 to the study of the well-posedness of the Euler equations (5.1) is given.

2. Notations

Let Ω be an open bounded subset of R^n, $n \geqq 2$, that lies (locally) on one side of its boundary Γ, a C^4 manifold. Denote by ν the unit outward normal to Γ.

For $h(x) = \{h_{rs}(x)\}$, $r = 1, \ldots, R$, $s = 1, \ldots, S$, where h_{rs} are real functions defined on Ω, define

$$|D^l h(x)|^2 = \sum_{|\alpha|=l} \sum_{r=1}^{R} \sum_{s=1}^{S} |D^\alpha h_{rs}(x)|^2, \tag{2.1}$$

where l is a nonnegative integer, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Set $|h| = |D^0 h|$, $|Dh| = |D^1 h|$. If for each pair r, s of indices $h_{rs} \in X$, where X is a function space, we simply write $h \in X$.

For $u = (u_1, \ldots, u_N)$, $w = (w_1, \ldots, w_N)$, $v = (v_1, \ldots, v_n)$, define

$$u \cdot w = \sum_{j=1}^{N} u_j w_j, \quad |u|^2 = u \cdot u, \quad (v \cdot \nabla)\, u = \sum_{i=1}^{n} v_i\, D_i u. \tag{2.2}$$

We will use the abbreviated notations

$$D_i h = \frac{\partial h}{\partial x_i}, \qquad \int h = \int_\Omega h(x)\, dx, \qquad (u, w) = \int u \cdot w.$$

In general, if X and Y are Banach spaces, $\mathscr{L}(X, Y)$ denotes the Banach space of all bounded linear maps from X into Y. We set $\mathscr{L}(X) = \mathscr{L}(X, X)$. We denote by L^p the Banach space $L^p(\Omega)$, and by $|\ |_p$ its canonical norm (see below). The real number $p \in]1, +\infty[$, and the domain Ω are fixed once and for all. For convenience these symbols will be dropped even from some standard notations. According to this convention, W^k denotes the Sobolev space $W^{k,p}(\Omega)$ and $\|\ \|_k$ denotes the canonical norm $\|\ \|_{k,p}$ defined below.

Define $\overset{0}{W}{}^l$, $l \geqq 1$, as the closure of $C_0^\infty(\Omega)$ in W^l, and set

$$W_1^k = W^k \cap \overset{0}{W}{}^1 \equiv W^{k,p}(\Omega) \cap \overset{0}{W}{}^{1,p}(\Omega).$$

For convenience set $W_0^k = W^k$.

The above notation will also be used to denote function spaces whose elements are vector fields or matrices. For instance, both L^p and $L^p \times \ldots \times L^p$ (N times) will be denoted by the same symbol L^p, and the corresponding norms by the same symbol $|\ |_p$. Finally, for $h = (h_{rs}) \in W^k$, $k \geqq 0$, define

$$|D^l h|_p = \left(\int |D^l h(x)|^p\, dx\right)^{1/p}, \qquad \|h\|_k = \sum_{l=0}^{k} |D^l h|_p.$$

T will denote a fixed positive real number, and $I = [-T, T]$. Standard notations will be used for functional spaces consisting of functions defined on I with values in a Banach space. In particular, the canonical norm in the Banach space $L^\infty(I; W^k)$ is denoted by $\|\ \|_{I,k}$.

The symbol c henceforth denotes any positive constant. The symbol $c(\Omega, n, N, p, k)$ means that c depends *at most* on the variables inside brackets. In this context the symbol n always denotes the dimension of R^n (the symbol n will be used also to enumerate sequences of functions).

3. The stationary case

Let $a=(a_{jk})$ be a $N\times N$ matrix, $N\geqq 1$, and $v=(v_1,\dots,v_n)$ be a vector field, both defined on Ω. We assume that

$$v\cdot\nu=0 \quad \text{on } \Gamma, \tag{3.1}$$

that

$$v, a\in W^m, \qquad \text{where } m>1+(n/p), \tag{3.2}$$

and we define the differential operator

$$\mathscr{A}u\equiv(v\cdot\nabla)u+au, \tag{3.3}$$

acting on vector fields $u=(u_1,\dots,u_N)$ defined in Ω. By definition $\mathscr{A}$ acts in the distributional sense. For each pair of integers (l,k) such that $0\leqq l\leqq 1\leqq k\leqq 2$, we define an operator A_l^k by setting

$$D_l^k\equiv\{u\in W_l^k:(v\cdot\nabla)u\in W_l^k\}, \qquad A_l^k\equiv\mathscr{A}_{|D_l^k}.$$

It is immediate to verify that each A_l^k is a closed operator in W_l^k, and is preclosed in L^p. Sometimes the symbol l will be dropped when $l=0$.

Since $W^3\subset D^2$, it follows that D^2 is dense in W^2. On the other hand, the vectors ∇u_j and ν have the same direction, if $u\in W_1^3$. Hence, $v\cdot\nabla u_j=0$ on Γ, $\forall j=1,\dots,N$. It follows that $W_1^3\subset D_1^2$. Hence D_1^2 is dense in W_1^2. Similarly, D^1 is dense in W^1, and D_1^1 is dense in W_1^1.

A denotes the closure of A_1^2 in L^p, and D denotes the domain of A. Clearly D is dense in L^p. Moreover, $D\subset\{u\in L^p:\mathscr{A}u\in L^p\}$.

The results stated in this section are particular cases of results proved in the preceding paper [2], [3]. However, for the sake of completeness, I give here the corresponding proofs. In the sequel, Sobolev's embedding theorems and Hölder's inequality will be freely used.

Let λ be a real number, and consider the equation

$$\lambda u+(v\cdot\nabla)u+au=f. \tag{3.4}$$

In this section our main concern is to prove the following result.

Theorem 3.1. *Let the conditions* (3.1), (3.2) *be satisfied, and denote by Z any one of the function spaces W_l^k, $0\leqq l\leqq k\leqq 2$, $l\leqq 1$. Then, if $|\lambda|>\theta$, where by definition*

$$\theta\equiv c(\Omega,n,N,p,m)\,(\|v\|_m+\|a\|_m) \tag{3.5}$$

and c is a suitable positive constant, equation (3.4) *has a unique solution $u\in Z$ for each $f\in Z$. Moreover,*

$$\|u\|_Z\leqq\frac{1}{|\lambda|-\theta}\|f\|_Z. \tag{3.6}$$

The following is an elementary but important auxiliary result.

Lemma 3.2. *Let* $p \in [1, +\infty[$, $w = (w_1, \ldots, w_N) \in C^1$, *and set* $\Lambda = (\delta + |w|^2)^{1/2}$, *where* $\delta > 0$. *Then*

$$\sum_{i=1}^{n} (D_i w) \cdot D_i(\Lambda^{p-2} w) = \Lambda^{p-2} |Dw|^2 + \frac{p-2}{4} \Lambda^{p-4} |\nabla(|w|^2)|^2, \tag{3.7}$$

and

$$\begin{aligned} \sum_{i=1}^{n} (D_i w) \cdot D_i(\Lambda^{p-2} w) = \Lambda^{p-4} \Big\{ &[(p-1)|w|^2 + \delta] |Dw|^2 \\ &+ (2-p) \Big[|w|^2 |Dw|^2 - \sum_{i=1}^{n} (w \cdot (Dw))^2 \Big] \Big\}. \end{aligned} \tag{3.8}$$

In particular, for each $p \in]1, +\infty[$,

$$-\int \Delta w \cdot \Lambda^{p-2} w \geqq 0 \qquad \forall\, w \in W_1^2. \tag{3.9}$$

Proof. Proof of the identities (3.7) and (3.8) is effected by direct computation. If w belongs to $C^2(\bar{\Omega})$ and vanishes on Γ, equation (3.9) follows by an integration by parts, and by using (3.7) (if $p \geqq 2$) or (3.8) (if $p \leqq 2$). Finally, if $w \in W_1^2$, we use the density of $\{w \in C^2(\bar{\Omega}): w = 0 \text{ on } \Gamma\}$ in W_1^2.

Lemma 3.3. *Let* $w = (w_1, \ldots, w_N) \in C^1$. *Then*

$$\Lambda^{p-2} D_i w \cdot w = (1/p)\, D_i \Lambda^p, \qquad i = 1, \ldots, n. \tag{3.10}$$

In particular, if $v \in C^1(\bar{\Omega})$ *satisfies* (3.1), *one has*

$$\int [(v \cdot \nabla)\, w] \cdot \Lambda^{p-2} w = -\frac{1}{p} \int (\operatorname{div} v)\, \Lambda^p, \qquad \forall\, w \in W^1. \tag{3.11}$$

The proof is left to the reader.

Lemma 3.4. *Assume that* $v \in C^1(\bar{\Omega})$ *verifies* (3.1), *that* $a \in C(\bar{\Omega})$, *and that* $f \in L^p$. *Let* $u \in W^1$ *be a solution of equation* (3.4). *Then, for* $|\lambda| > \tilde{\theta} \equiv c(p, N)(\|v\|_{C^1} + \|a\|_{C^0})$, *one has* $(|\lambda| - \tilde{\theta})\, |u|_p \leqq |f|_p$. *In particular, the above solution* u *(if it exists) is unique.*

The proof is classical and well known. Multiply both sides of (3.4) by $(\delta + |u|^2)^{(p-2)/2} u$, integrate over Ω, and pass to the limit as $\delta \to 0^+$.

Theorem 3.5. *Under the assumptions of Theorem 3.1, the equation* (3.4) *has a unique solution* $u \in W_1^2$ *for each* $f \in W_1^2$. *Moreover,*

$$\|u\|_2' \leqq \frac{1}{|\lambda| - \theta} \|f\|_2' \qquad \forall\, f \in W_1^2,$$

where, by definition, $\|u\|_2' = |u_p| + |\Delta u|_p$. *Note that, in* W_1^2, *the norms* $\| \|_2$ *and* $\| \|_2'$ *are equivalent. In particular*

$$\|u\|_2 \leqq \frac{c(\Omega, n, N, p)}{|\lambda| - \theta} \|f\|_2 \qquad \forall f \in W_1^2. \tag{3.12}$$

Furthermore, Theorem 3.1 holds for $Z = L^p$.

Proof. Let $\varepsilon > 0$ if $\lambda > 0$, $\varepsilon < 0$ if $\lambda < 0$, and consider the elliptic Dirichlet problem

$$\begin{aligned} -\varepsilon \Delta u_\varepsilon + \lambda u_\varepsilon + (v \cdot \nabla) u_\varepsilon + a u_\varepsilon = f \quad \text{in } \Omega, \\ (u_\varepsilon)|_\Gamma = 0. \end{aligned} \tag{3.13}$$

In order to fix the ideas, assume that $\lambda > 0$. For a sufficiently large λ, the above problem has a unique solution $u_\varepsilon \in W_1^4$. Moreover (a crucial point!) (3.13) and (3.1) yield

$$(\Delta u)_\varepsilon|_\Gamma = 0. \tag{3.14}$$

Hence $\Delta u_\varepsilon \in W_1^2$. Set $\Lambda = (\delta + |\Delta u_\varepsilon|^2)^{\frac{1}{2}}$, where $\delta > 0$. Equations (3.9) and (3.11) imply

$$\begin{aligned} -\int \Delta(\Delta u_\varepsilon) \cdot \Lambda^{p-2} \Delta u_\varepsilon \geqq 0, \\ \int [(v \cdot \nabla) \Delta u_\varepsilon] \cdot \Lambda^{p-2} \Delta u_\varepsilon = -\frac{1}{p} \int (\operatorname{div} v) \Lambda^p, \end{aligned} \tag{3.15}$$

for each $\varepsilon > 0$. Equation $(3.15)_2$ together with the identity $\Delta[(v \cdot \nabla) u] = (v \cdot \nabla) \Delta u + 2\nabla v : \nabla^2 u + (\Delta v \cdot \nabla) u$ yields

$$\begin{aligned} \int \Delta[(v \cdot \nabla) u_\varepsilon] \cdot \Lambda^{p-2} \Delta u_\varepsilon = -\frac{1}{p} \int (\operatorname{div} v) \Lambda^p \\ + 2 \int (\nabla v : \nabla^2 u_\varepsilon) \cdot \Lambda^{p-2} \Delta u_\varepsilon + \int [(\Delta v \cdot \nabla) u_\varepsilon] \cdot \Lambda^{p-2} \Delta u_\varepsilon, \end{aligned} \tag{3.16}$$

where $\nabla v : \nabla^2 u = \sum_{i,l=1}^{n} (D_l v_i)(D_i D_l u)$. By applying the operator Δ to both sides of equations $(3.13)_1$, by taking the scalar product in R^n with $\Lambda^{p-2} \Delta u_\varepsilon$, by integrating in Ω, by taking into account $(3.15)_1$ and (3.16), it follows that

$$\begin{aligned} \lambda \int |\Delta u_\varepsilon|^2 \Lambda^{p-2} - \frac{1}{p} \int (\operatorname{div} v) \Lambda^p \leqq \int \big(2 |\nabla v : \nabla^2 u_\varepsilon| + |(\Delta v \cdot \nabla) u_\varepsilon| \\ + |\Delta (a u_\varepsilon)| + |\Delta f|\big) |\Delta u_\varepsilon| \Lambda^{p-2}. \end{aligned}$$

Since $0 \leqq |\Delta u_\varepsilon| \Lambda^{p-2} \leqq \Lambda^{p-1}$, the Lebesgue's dominated convergence theorem applies, as $\delta \to 0^+$. Hence, the last inequality holds if Λ is replaced by $|\Delta u_\varepsilon|$. In particular $(\lambda - \theta_2) |\Delta u_\varepsilon|_p \leqq |\Delta f|_p$ for a suitable value of the constant c in definition (3.5). Consequently, $\varepsilon \Delta u_\varepsilon \to 0$ in L^p as $\varepsilon \to 0$, and (on the other hand) there is a subsequence u_ε weakly convergent in W_1^2 to a limit u. It follows that u is a solution of (3.4). Clearly, $(\lambda - \theta)(|\Delta u|_p + |u|_p) \leqq |\Delta f|_p + |f|_p$ (use also Lemma 3.4).

The last assertion of the theorem is proved as follows. Let $f \in L^p$, and let $f_n \in W_1^2$ be a sequence such that $f_n \to f$ in L^p. Let $u_n \in W_1^2$ be the solution of the equation $A_1^2 u_n = f_n$. By Lemma 3.4, $(|\lambda| - \theta)\,|u_n|_p \leq |f_n|_p$; moreover $(|\lambda| - \theta)\,|u_n - u_m|_p \leq |f_n - f_m|_p$. Hence u_n is a Cauchy sequence in L^p, and the limit u satisfies the estimate $(|\lambda| - \theta)\,|u|_p \leq |f|_p$. Since A is the closure of A_1^2 in L^p, one has $Au = f$. □

Proposition 3.6. *Assume that the conditions* (3.1), (3.2) *hold, and let* $f \in W^k$, *for* $k = 0, 1,$ *or* 2. *Assume that* $u \in W^{k+1}$ *is a solution of* (3.4). *Then* $|\lambda|\,|D^k u|_p \leq c(\|v\|_m + \|a\|_m)\,\|u\|_k + |D^k f|_p$, *for a suitable positive constant* $c = c(\Omega, n, N, p, m)$. *In particular, if* $|\lambda| > 0$, *one has*

$$\|u\|_k \leq \frac{1}{|\lambda| - \theta}\,\|f\|_k. \tag{3.17}$$

Finally, Theorem 3.1 holds for $Z = W_1^1$.

The proof of the *a priori* estimate (3.17) is quite immediate (and well known). See, for instance, [2] Proposition 3.1, or [3] Lemma 3.7 (however, the *a priori* estimate is quite far from providing an existence theorem. In this regard, see [3] Remark 2.4).

The last assertion of the proposition is proved as follows. Let $f \in W_1^1$, and let $f_n \in W_1^2$, $f_n \to f$ in W_1^1. Let $u_n \in W_1^2$ be the solution of the equation $\lambda u_n + (v \cdot \nabla) u_n + a u_n = f_n$ (denoted here by $(3.4)_n$). The estimate (3.17) (for $k = 1$) shows that $\|u_n\|_1 \leq (|\lambda| - \theta)^{-1}\,\|f_n\|_1$, and also that u_n is a Cauchy sequence in W_1^1. By passing to the limit in equation $(3.4)_n$, as $n \to \infty$, we show that u is the solution of (3.4).

Theorem 3.7. *Theorem 3.1 holds for* $Z = W^1$ *and for* $Z = W^2$ *if in equation* (3.6) *the constant* 1 *is replaced by a suitable constant* $c = c(\Omega, n, N, p, m)$.

Proof. Let B be an open ball such that $\bar{\Omega} \subset B$, and let E map functions defined on Ω into functions defined on B, in such a way that $(Ew)_\Omega = w$ (i.e. E is a extension map). Assume that $E \in \mathscr{L}(W^k, \mathring{W}^k(B))$, either for $k = 0, 1, 2$ and for $k = m$. Set for convenience $\tilde{w} = Ew$. Let $f \in W^2$, and let $\hat{u} \in W_1^2(B)$ be the solution of the equation $\lambda \hat{u} + (\tilde{v} \cdot \nabla)\,\hat{u} + \tilde{a}\hat{u} = \tilde{f}$ in B, whose existence is guaranteed by Theorem 3.5. Clearly, $u = \hat{u}|_\Omega$ is a solution of (3.4) in Ω. By Lemma 3.4, this solution is unique. Moreover, the estimate (3.12) holds in Ω for u, since it holds in B for $\hat{u}$. A similar proof can be given for W^1, by using the existence theorem in $W_1^1(B)$. □

In order to complete the proof of Theorem 3.1, it only remains to show that in the statement of Theorem 3.7 the constant c can be taken equal to 1. Let $f \in W^1$ and let $f_n \in W^2$, $f_n \to f$ in W^1 as $n \to +\infty$. Let $u_n \in W^2$ be the solution of problem $\lambda u_n + (v \cdot \nabla)\,u_n + a u_n = f_n$, whose existence is guaranteed by Theorem 3.7. By Proposition 3.6 the pair u_n, f_n satisfies the estimate (3.17) for the value $k = 1$. In particular, u_n is a Cauchy sequence in W^1, and the limit u is the desired solution of problem (3.4).

Let us now prove the result for W^2. Let $f \in W^2$, and let $u \in W^2$ be the solution of (3.4), provided by Theorem 3.7. One has, for each $i = 1, \ldots, n$,

$$\lambda D_i u + (v \cdot \nabla) D_i u + a D_i u = D_i f - (D_i v \cdot \nabla) u - (D_i a) u.$$

This is again a system of the type (3.4) on the nN variables $D_i u_j$ (the right-hand side of the above equation is treated here as a datum), whose (unique) solution belongs to W^1. Hence, from the result just proved for the case W^1, it readily follows that

$$(|\lambda| - \theta)(|D^2 u|_p + |Du|_p) \leq |D^2 f|_p + |Df|_p + \bar{c}(\Omega, n, N, p, m)(\|v\|_m + \|a\|_m)\|u\|_2.$$

Moreover, $(|\lambda| - \theta)|u|_p \leq |f|_p$. Hence satisfies u the estimate (3.6) with respect to the norm $\| \|_2$, for a suitable constant c in equation (3.5). □

Remark. Let $\bar{A}_l^k$ denote the closure of A_l^k in L^p. Arguing as at the end of the proof of Theorem 3.5, we show that $\lambda + \bar{A}_l^k$ is an invertible map from its domain onto L^p, for a sufficiently large λ. Since $A_1^2 \subset A_l^k$, it follows that $A = \bar{A}_l^k$.

4. The evolution case

Let us now assume that v an a are defined on $I \times \Omega$, that v satisfies condition (3.1) for each $t \in I$, and that

$$v, a \in L^\infty(I; W^m) \cap C(I; W^{m-1}), \quad \text{where } m > 1 + (n/p). \tag{4.1}$$

In particular, v and a are well defined as elements of W^m for each $t \in I$. Here we will use definitions and notations introduced in Section 3 for the stationary case. The meaning of symbols like $f(t)$, $u(t)$, $A_l^k(t)$, $D_l^k(t)$, and so on, is obvious.

Let us now consider the evolution equation

$$D_t u + \mathscr{A}(t) u = f(t), \quad t \in I, \tag{4.2}$$
$$u(0) = u_0.$$

By "the evolution equation (4.2) in W_l^k" we mean the above equation for $A_l^k(t)$ instead of $\mathscr{A}(t)$. Furthermore, we denote by $U_l^k(t, s)$ the corresponding evolution operator in the Banach space W_l^k. One has the following result.

Theorem 4.1. *Assume that* (3.1) *and* (4.1) *holds. Let* (l, k) *be a pair of integers such that* $0 \leq l \leq 1 \leq k \leq 2$, *and set* $X = L^p$, $Y = W_l^k$. *Then the evolution operator* $U_l^k(t, s)$, $t, s \in I$, *exists and has the properties described in the Theorem 5.2 of reference* [9]. *Remarks 5.3 and 5.4* (*in* [9]) *also apply.*

Proof. The main hypothesis required on Theorem 5.2 of reference [9] is the $(1, \theta)$-stability of the families of operators $\{A_l^k(t)\}_{t \in I}$, which was proved in Theorem 3.1. The other hypotheses required on the above theorem are easily verified, and we leave it to the reader (see [3] for details). □

Some remarks. Below we prove (*cf.*, in particular, (4.11)) that the existence and the perturbation theorem for the evolution operator $U^2(t, s)$ can be directly obtained from the existence and the perturbation theorem for $U_1^2(t, s)$; consequently, we need to show existence and perturbation *only* for U_1^2. In particular, the existence Theorem 4.1 is needed only for U_1^2. In this regard it turns out to be more convenient to prove the existence of U_1^2 by using Theorem I of [10] instead of the Theorem 5.2 of [9] (as done above), since we have to apply other results of [10] (Theorem VI) in order to establish the perturbation theorem for U_1^2.

Finally, we remark that the above argument requires the proof of the $(1, \theta)$-stability only for $Y = W_1^2$, and for $X = L^p$. Theorem 3.5 is sufficient for this purpose. In particular, the additional results stated in Theorem 3.1 are not necessary here.

Assume that $\{v_n\}$, $\{a_n\}$ are two sequences satisfying (3.1) and (4.1), for each positive integer n. Assume also that

$$\|v_n\|_{I,m} \quad \text{and} \quad \|a_n\|_{I,m} \quad \text{are uniformly bounded,} \tag{4.3}$$

and that

$$v_n \to v, \quad a_n \to a, \quad \text{in} \quad C(I; W^{m-1}). \tag{4.4}$$

By using the coefficients v_n, a_n instead of v, a, we define in an obvious way operators $\mathscr{A}^{(n)}(t)$ and $A_l^{k,(n)}(t)$, domains $D_l^{k,(n)}(t)$, and evolution operators $U_l^{k,(n)}(t, s)$.

One has the following result where, as throughout this paper, we assume that $m > 1 + (n/p)$.

Theorem 4.2. *Let* $m \geqq 3$. *Under the above hypothesis on the coefficients* v, a, v_n, a_n

$$\lim_{n\to\infty} U_l^{2,(n)}(t, s) = U_l^2(t, s) \tag{4.5}$$

strongly in $\mathscr{L}(W_l^2)$ *and uniformly on* $(t, s) \in I \times I$. *Here* $l = 0$ *or* $l = 1$. *A similar result holds in* L^p, *for the evolution operators* $U^0(t, s)$ *and* $U^{0,(n)}(t, s)$.

Proof of Theorem 4.2 (*Case* $l = 1$). Set $Y = W_1^2$, $X = L^p$, and define

$$S = \Delta, \quad D(S) = Y. \tag{4.6}$$

The linear map S is an isomorphism from Y onto X. Define, for each $\varphi \in X$ and for each $t \in I$, the linear operator

$$B(t)\varphi = (\Delta v \cdot \nabla) u + 2 \sum_{i,l=1}^{n} (D_l v_i)(D_i D_l u) + (\Delta a) u + 2 \sum_{l=1}^{n} (D_l a)(D_l u), \tag{4.7}$$

where $u = S^{-1}\varphi$. I wish to prove that the hypotheses of Theorem VI [10] are satisfied. For convenience I will refer to the equations in Kato's paper [10] by adding the symbol "-K" to the reference numbers in [10]. The hypotheses (i′)-K holds, since A^n and A are $\{1, \theta\}$-stable, and θ is independent of n. Let us prove that (*ii‴*)-K holds. By using Sobolev's embedding theorems, one shows that

$$|||B(t)|||_X \leqq c(\|v(t)\|_m + \|a(t)\|_m). \tag{4.8}$$

Furthermore, the function $t \to B(t)$ is strongly measurable, since the map $t \to \int B(t)\varphi\Psi$ is measurable for each $\varphi \in L^p$, $\Psi \in L^q$, $p^{-1} + q^{-1} = 1$. Equation $(4.8)_n$ shows that $|||B^n(\cdot)|||_X$ is upper integrable on I, and the integral is uniformly bounded with respect to n. Clearly $\dot{S} \equiv 0$. Let us prove that

$$SA(t)S^{-1} = A(t) + B(t) \quad \text{for } a.\,a.\; t \in I. \tag{4.9}$$

If $\varphi \in L^p$, it follows that $u = S^{-1}\varphi \in W_1^2$. Hence the vectors ∇u_j, $j = 1, \ldots, N$, belong to W^1 and are orthogonal to Γ. Then, by (3.1), one has $(v \cdot \nabla)u = 0$ on Γ. Consequently, for each fixed $t \in I$, $A(t)S^{-1}\varphi \in W_1^1$, $\forall \varphi \in L^p$. In particular, $u = S^{-1}\varphi \in W_1^3 \cap D_1^2$ and $A(t)S^{-1}\varphi \in W_1^2 \equiv D(S)$ if $\varphi \in W^1$. It readily follows that

$$SA(t)S^{-1}\varphi = A(t)\varphi + B(t)\varphi \quad \forall \varphi \in W^1. \tag{4.10}$$

Hence (4.10) holds for all $\varphi \in D_1^2$. Note that D_1^2 forms a core of $A(t)$. Equation (4.9) follows now from (4.10) exactly as (8.5) follows from (8.6) in reference [9].

Conditions (iii)-K, (11.1)-K, and (11.2)-K are trivially satisfied; recall that $W^{m-1} \subset C^0$. Let us prove (11.4)-K. Fix a positive constant ε such that $m - \varepsilon > 2$* and $m - \varepsilon > 1 + (n/p)$. By using Sobolev's well known embedding theorems it readily follows that

$$|||B^n(t) - B(t)|||_X \leqq c(\|v_n(t) - v(t)\|_{m-\varepsilon} + \|a_n(t) - a(t)\|_{m-\varepsilon}).$$

Since $\|v_n(t)\|_m$ is uniformly bounded with respect to n, since the embedding $W^m \subset W^{m-\varepsilon}$ is compact, and since $v_n(t) \to v(t)$ in W^{m-1}, it follows that $v_n(t) \to v(t)$ in $W^{m-\varepsilon}$. Finally, the condition (11.5)-K is a consequence of $(4.8)_n$ and (4.3). □

Proof of Theorem 4.2. (*Case* $l = 0$). Let B denote a fixed open ball such that $\bar{\Omega} \subset B$, and denote by R the operator defined by $Rw = w|_\Omega$. Assume for convenience that $\Gamma \in C^m$, and fix a linear continuous map E from $W^l(\Omega)$ into $\mathring{W}^l(B)$, for each $l = 0, 1, \ldots, m$, such that $(Eu)|_\Omega = u$. Let $\tilde{A}_1^2(t)$, $t \in I$, be the family of operators defined by replacing in the definition of $A_1^2(t)$ the domain Ω by B and the coefficients v and a by Ev and Ea. Let $\tilde{U}_1^2(t, s)$ be the evolution operator in $W_1^2(B)$, generated by the family $\tilde{A}_1^2$. In an obvious way, we define $\tilde{A}_1^{2,(n)}(t)$ and $\tilde{U}_1^{2,(n)}(t, s)$.

Now let $\varphi \in W^2(\Omega)$. By the definition of evolution operator, the function $w(t) = \tilde{U}_1^2(t, s)E\varphi$ belongs to $C(I; W_1^2(B))$, and is the solution of the problem

$$D_t w + ((Ev) \cdot \nabla)w + (Ea)w = 0 \quad \text{in } I,$$
$$w(s) = E\varphi.$$

Hence $Rw(t) = [R\tilde{U}_1^2(t, s)E]\varphi$ belongs to $C(I; W^2(\Omega))$; moreover**

$$D_t(Rw) + (v \cdot \nabla)(Rw) + a(Rw) = 0 \quad \text{in } I,$$
$$(Rw)(s) = \varphi.$$

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* This is the only point where the assumption $m \geqq 3$ is used.

** Since R commutes with differentiation, and $R(uv) = (Ru)(Rv)$, $\forall u, v$.

This shows that the evolution operator generated by the family $A^2(t)$ in the space $W^2(\Omega)$ is given by

$$U^2(t,s) = R\tilde{U}_1^2(t,s)\,E. \tag{4.11}$$

Similarly, $U^{2,(n)}(t,s) = R\tilde{U}_1^{2,(n)}(t,s)\,E$. By the first part of Theorem 4.2 it follows that

$$\lim_{n\to+\infty} \tilde{U}_1^{2,n}(t,s) = \tilde{U}_1^2(t,s)$$

strongly in $W_1^2(B)$, uniformly in t, s. Hence,

$$\lim_{n\to+\infty} U^{2,n}(t,s) = U^2(t,s)$$

strongly in $W_1^2(\Omega)$, uniformly in t, s. □

5. Well-posedness for the Euler equations

We start by making some remarks. The persistence property (see [13]) for the Cauchy problem $D_t v + A(t)\,v = f(t)$, $v(0) = v_0$ (in the function space X) means that the solution v belongs to $C(I; X)$ if $v_0 \in X$, $f \in E(I; X)$, for some function space E. Well-posedness means that the map $(v_0, f) \to v$ is continuous from $X \times E(I; X)$ into $C(I; X)$. For the Euler equations in a bounded domain (in case $f \equiv 0$) the well-posedness was first proved by EBIN & MARSDEN [8], for $X = W^{k,p}$ and for $X = C^{k,\alpha}$, by using techniques of Riemannian geometry on infinite dimensional manifolds. We refer also to EBIN [6], for the proof when $X = C^{k,\alpha}$. In reference [1] I prove, for $n = 2$, the well-posedness in the Banach space $X \equiv \{u \in C(\bar{\Omega})\colon \operatorname{rot} u \in C(\bar{\Omega})\}$, in the general case $E(I; X) = L^1(I; X)$. Note that the X-norm of the solution, namely $\|v(t)\|_X = \|\operatorname{rot} v(t)\|_{C(\Omega)} + \|v(t)\|_{L^2(\Omega)}$ is time-invariant when $f \equiv 0$. KATO & LAI [12] prove well-posedness when $X = H^{s,2}(\Omega)$, $E(I; X) = C(I; X)$*. For well-posedness when $\Omega = R^n$, see KATO & PONCE [13] and references given therein. All the above results are local in time, when $n \geqq 3$.

Reference [8] was followed by a series of papers in which the main goal was not to give more general results but to furnish simpler proofs. BOURGUIGNON & BREZIS [5] prove existence and the persistence property in Sobolev spaces $W^{s,p}$, when $f \in C(I; W^{s+1,p})$, by reducing the problem to an ordinary differential equation on a Banach space. TEMAM [14] proves the existence of a local solution $v \in L^\infty(I^*; W^{m,p})$, if $v_0 \in W^{m,p}$ and $f \in L^1(I; W^{m,p})$. Under these assumptions on the data, and by using a completely different method, I proved in [3] the existence of a solution $v \in C(I^*; W^{m,p})$. Below, I wish to prove well-posedness for this solution. Our departure point will be the existence theorem for local solutions $v \in L^\infty(I^*; W^m)$ which is assumed as well-know here; see [14], [3]. We need not assume that $v \in C(I^*; W^m)$ since this will follow from the arguments developed below. However, by using the existence theorem proved in reference [3], the reader can replace L^∞ by C.

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* Professor T. KATO informs me that he believes that KATO & LAI's method also applies if $p \neq 2$.

Before going on, I wish cite an interesting result obtained recently by EBIN. He shows [7] that the free boundary problem for the Euler equations is not (in general) well-posed.

Henceforth it is assumed that the reader is familiar with the Euler equations describing the motion of an inviscid, incompressible fluid in a bounded domain $\Omega \subset R^n$, namely

$$\begin{aligned} D_t v + (v \cdot \nabla) v &= -\nabla \pi + f \quad \text{in } I \times \Omega, \\ \operatorname{div} v &= 0 \quad \text{in } I \times \Omega, \\ v \cdot \nu &= 0 \quad \text{on } \Gamma \times \Omega, \\ v(0) &= v_0, \end{aligned} \tag{5.1}$$

where $v_0(x)$ and $f(t, x)$ are given. We assume that $\operatorname{div} v_0 = 0$ in Ω, that $v_0 \cdot \nu = 0$ on Γ, and that $v_0 \in W^m$ and $f \in L^1(I; W^m)$, where $m > 1 + (n/p)$, and $m \geq 3$. Here, $p \in]1, +\infty[$. Let $T^* \in]0, T]$ be a positive integer such that the problem (5.1) has a (unique) local solution $v \in L^\infty(I^*; W^m)$. The proof of this local existence theorem (see [14], [3]) also furnishes a lower bound for T^*, depending only on the norms of the data v_0 and f (see, for instance, [3] Theorem 6.1). Assume now that the one has a sequence of data v_0^n, f_n satisfying the hypothesis required above for the data v_0, f, and such that

$$\lim_{n \to +\infty} v_0^n = v_0 \quad \text{in } W^m, \quad \lim_{n \to +\infty} f_n = f \quad \text{in } L^1(I; W^m). \tag{5.2}$$

Let $v_n \in L^\infty(I^*; W^m)$ be the solution of the equation $(5.1)_n$ which is obtained from equation (5.1) by replacing v, π, f, v_0 by v_n, π_n, f_n, v_0^n respectively. For convenience I denote by I^* an arbitrary common time-interval of existence for both the solution v and the solutions v_n with large values of n. It is also assumed that the norms of the solutions v_n in the space $L^\infty(I^*; W^m)$ are uniformly bounded. Note that such an interval I^* exists. In fact, from (5.2), it follows that $\|v_0^n\|_m \leq M$, and that $\|f_n\|_{L^1(I; W^m)} \leq M_f$, where the constants M, M_f are independent of n. From Theorem 6.1 in reference [3] (we may also use reference [14]) it follows that there are positive constants c, c', depending only on Ω, $n\,(= \dim \mathbb{R}^n)$, p, m, such that the solutions v, v_n exist in the interval $I^* = [-T^*, T^*]$ for $T^* = c(M + M_f)^{-1}$, and belong to $C(I^*; W^m)$. Moreover $\|v_n\|_{I^*, m} \leq c'(M + M_f)$.

One has the following result.

Lemma 5.1. *Under the above hypothesis, the solution v of problem* (5.1) *and the solutions v_n of problems* $(5.1)_n$ *belong to $C(I^*; W^m)$. Moreover*

$$\lim_{n \to +\infty} v_n = v \quad \text{in } C(I^*; W^{m-1}). \tag{5.3}$$

Proof. From equation (5.1) it readily follows, by using a well known device, that $\nabla \pi = \nabla \pi^{(1)} + \nabla \pi^{(2)}$, where

$$\begin{aligned} \Delta \pi^{(1)} &= -\sum_{i,j} (D_i v_j)(D_j v_i) \quad \text{in } \Omega, & \Delta \pi^{(2)} &= \operatorname{div} f \quad \text{in } \Omega, \\ \frac{\partial \pi^{(1)}}{\partial \nu} &= \sum_{i,j} \frac{\partial \nu_i}{\partial x_j} v_i v_j \quad \text{on } \Gamma, & \frac{\partial \pi^{(2)}}{\partial \nu} &= f \cdot \nu \quad \text{on } \Gamma. \end{aligned} \tag{5.4}$$

Similarly, one has a family of equations $(5.4)_n$. From these equations it follows that $\nabla\pi, \nabla\pi_n \in L^1(I^*; W^m)$. Hence, by using equations (5.1) and $(5.1)_n$, it follows that $D_t v,\ D_t v_n \in L^1(I^*; W^{m-1})$, and that $v,\ v_n \in C(I^*; W^{m-1})$. Let us show that

$$\lim_{n\to+\infty} v_n = v \quad \text{in } C(I^*; W^{m-1}). \tag{5.5}$$

From $(5.4)_n$ it follows that $\|\nabla\pi_n^{(1)}\|_{I^*,m-1} \leqq C$, uniformly with respect to n, since the norms $\|v_n\|_{I^*,m}$ are uniformly bounded*. Similarly $\|(v_n\cdot\nabla)\,v_n\|_{I^*,m-1} \leqq C$. On the other hand, from $(5.4)_n$ and (5.4), it follows that $\nabla\pi_n^{(2)} \to \nabla\pi^{(2)}$ in $L^1(I^*; W^m)$. Hence, from equation $(5.1)_n$ one gets $(\tau < t)$

$$\|v_n(t) - v_n(\tau)\|_{m-1} \leqq C\,|t-\tau| + \int_\tau^t \|f(s) + \nabla\pi^{(2)}(s)\|_{m-1}\,ds$$
$$+ \int_{I^*} \|(f+\nabla\pi^{(2)})(s) - (f_n + \nabla\pi_n^{(2)})(s)\|_{m-1}\,ds.$$

It readily follows that given $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ and a positive real number $\delta(\varepsilon)$ such that if $n > N(\varepsilon)$ and $|t-\tau| < \delta(\varepsilon)$ then $\|v_n(t) - v_n(\tau)\|_{m-1} < \varepsilon$. Since $v_n \in C(I^*; W^{m-1})$, $\forall\, n \geqq 1$, it follows that the above property holds for each $n \geqq 1$. Consequently, the v_n are equicontinuous on I^* with values in W^{m-1}. Since the embedding $W^{m-1} \subset W^m$ is compact, the theorem of Ascoli and Arzelà shows that the sequence v_n is relatively compact in $C(I^*; W^{m-1})$. Limits of convergent subsequences of the sequence $\{v_n\}$ are solutions of (5.1). Since the solution v of (5.1) is unique, the property (5.5) holds.

Remark. If we also assume that $f_n \to f$ in $L^q(I; W^{m-1})$, $q > 1$, the proof of (5.5) is simpler, and the decomposition $\nabla\pi = \nabla\pi^{(1)} + \nabla\pi^{(2)}$ is unnecessary. In fact, from the equation satisfied by π_n it follows that $\nabla\pi_n$ is uniformly bounded in $L^q(I^*; W^{m-1})$. From $(5.1)_n$ it follows then that $D_t v_n$ is uniformly bounded in $L^q(I^*; W^{m-1})$. Hence the v_n are equicontinuous in $C(I^*; W^{m-1})$ and (5.5) follows as above.

Now let D^α be any space derivative such that $0 \leqq |\alpha| \leqq m-2$. From (5.1) one has

$$\begin{gathered} D_t(D^\alpha v) + (v\cdot\nabla)(D^\alpha v) = F^\alpha[v] - D^\alpha(\nabla\pi) + D^\alpha f \equiv G^\alpha, \\ (D^\alpha v)(0) = D^\alpha v_0, \end{gathered} \tag{5.6}$$

where

$$\begin{aligned} F^\alpha[v] &\equiv (v\cdot\nabla)(D^\alpha v) - D^\alpha[(v\cdot\nabla)\,v] \\ &= -(D^\alpha v\cdot\nabla)\,v - (D^{\alpha-1}v\cdot\nabla)\,Dv - \ldots - (Dv\cdot\nabla)D^{\alpha-1}v. \end{aligned}$$

The above right-hand side is written in a quite compact but appropriate notation. By using Sobolev's embedding theorems and Hölder's inequality, one easily proves that (with obvious notations)

$$\begin{aligned} \|F^\alpha[v](t) - F^\alpha[v_n](t)\|_2 &\leqq c(\|v(t)\|_m + \|v_n(t)\|_m)\,\|v(t) - v_n(t)\|_m \\ &\leqq C\,\|v(t) - v_n(t)\|_m, \end{aligned}$$

* In this section C denotes any positive constant that depends at most on Ω, n, p, m, M, M_f, T^*.

since the norms $\|v_n\|_{I^*,m}$ are uniformly bounded. On the other hand, since $\Delta\pi = -\Sigma\,(D_i v_j)\,(D_j v_i) + \operatorname{div} f$ in Ω, $\partial\pi/\partial\nu = \Sigma\,(\partial\nu_i/\partial x_j)\,v_i v_j + f\cdot\nu$ in Γ, and since similar equations hold for π_n, one easily gets

$$\|D^\alpha \nabla(\pi - \pi_n)(t)\|_2 \leqq C\,\|v(t) - v_n(t)\|_m + c\,\|f(t) - f_n(t)\|_m.$$

Hence

$$\|G^\alpha(t) - G_n^\alpha(t)\|_2 \leqq C\,\|v(t) - v_n(t)\|_m + c\,\|f(t) - f_n(t)\|_m. \tag{5.7}$$

Moreover, $G^\alpha,\ G_n^\alpha \in L^1(I^*; W^2_\sigma)$.

For convenience, $U(t, s)$ will denote henceforth the evolution operator $U^2(t, s)$ generated by the family of operators $\{A^2(t)\}_{t\in I^*}$ in the Banach space W^2. Here $A^2(t)\,u \equiv (v(t)\cdot\nabla)\,u$, i.e. $a \equiv 0$. Similarly $U_n(t, s)$ will denote the evolution operator $U^{2,(n)}(t, s)$ generated by the family of operators $\{A^{2,(n)}(t)\}_{t\in I^*}$ in the Banach space W^2. Here $A^{2,n}(t)\,u \equiv (v_n(t)\cdot\nabla)\,u$, i.e. $a_n \equiv 0$. From Theorem 4.2 it follows that $U_n(t, s) \to U(t, s)$ strongly in $\mathscr{L}(W^2)$, uniformly for $(t, s) \in I^* \times I^*$. From equation (5.6) one gets*

$$D^\alpha v(t) = U(t, 0)\,D^\alpha v_0 + \int_0^t U(t, s)\,G^\alpha(s)\,ds. \tag{5.8}$$

Consequently, by subtracting equation $(5.8)_n$** from equation (5.8) one easily verifies that (assume, for convenience, that $t \in [0, T^*]$)

$$\begin{aligned}\|D^\alpha v(t) - D^\alpha v_n(t)\|_2 &\leqq \|(U(t, 0) - U_n(t, 0))\,D^\alpha v_0\|_2 + |||U_n(t, 0)|||\\ &\quad \|D^\alpha v_0 - D^\alpha v_0^n\|_2 + \int_0^t \|(U(t, s) - U_n(t, s))\,G^\alpha(s)\|_2\,ds\\ &\quad + \int_0^t |||U_n(t, s)|||\;\|G^\alpha(s) - G_n^\alpha(s)\|_2\,ds,\end{aligned}$$

where $|||\cdot|||$ denotes the norm in the Banach space $\mathscr{L}(W^2)$. By recalling that $|||U_n(t, s)||| \leqq e^{\beta_n T^*} \leqq e^{cMT}$ (by (3.5)), and by using (5.7), it readily follows that

$$\begin{aligned}\|D^\alpha v(t) - D^\alpha v_n(t)\|_2 &\leqq \|(U(t, 0) - U_n(t, 0))\,D^\alpha v_0\|_2 + C\,\|v_0 - v_0^n\|_m\\ &\quad + \int_0^\tau \|(U(t, s) - U_n(t, s))\,G^\alpha(s)\|_2\,ds + C\int_0^\tau \|f(s) - f_n(s)\|_m\,ds\\ &\quad + C\tau\,\|v - v_n\|_{[0,\tau],m},\end{aligned} \tag{5.9}$$

for every $t \in [0, \tau]$. By adding side by side the above inequalities, for every index α such that $0 \leqq |\alpha| \leqq m - 2$, one easily gets

$$\begin{aligned}\|v - v_n\|_{[0,\tau],m} &\leqq \sum_\alpha \sup_{t\in[0,\tau]} \|(U(t, 0) - U_n(t, 0))\,D^\alpha v_0\|_2 + C\,\|v_0 - v_0^n\|_m\\ &\quad + c\sum_\alpha \int_0^\tau \sup_{t\in[0,\tau]} \|(U(t, s) - U_n(t, s))\,G^\alpha(s)\|_2\,ds + C\int_0^\tau \|f(s) - f_n(s)\|_m\,ds,\end{aligned}$$

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* This shows that $v, v_n \in C(I^*; W^m)$.

** Replace, in equation (5.8), U, v, v_0, f, by U_n, v_n, v_0^n, f_n, respectively.

for a sufficiently small positive value of τ which depends only on $\Omega, n, p, m, M, M_f, T^*$. It readily follows, by using in particular the dominated convergence theorem, that $v_n \to v$ in $C(0, \tau; W^m)$. By applying this result successively to the intervals $[j\tau, (j+1)\tau] \cap [0, T^*]$, one establishes the uniform convergence of v_n to v in all of $[0, T^*]$. □

In the following I assume without loss of generality, that $I =]-\infty, +\infty[$. One has the following existence result.

Theorem 5.2. *Let* $m \in [3, +\infty[$ *and* $p \in]1, +\infty[$ *satisfy the condition* $m > 1 + (n/p)$. *Let* $v_0 \in W^m$ *satisfy the assumptions* $\operatorname{div} v_0 = 0$ *in* Ω *and* $v_0 \cdot \nu = 0$ *on* Γ, *and let* $f \in L^1(I; W^m)$. *Assume that there is a solution* $v \in L^\infty(]-\tau_1, \tau_2[; W^m)$ *of the Euler equations* (5.1) *in an interval* $]-\tau_1, \tau_2[$*. *Let* $v_0^n \in W^m$ *and* $f_n \in L^1(I; W^m)$ *be sequences of data, such that* $\operatorname{div} v_0^n = 0$ *in* Ω, $v_0^n \cdot \nu = 0$ *on* Γ, *and such that* (5.2) *holds. Then, for sufficiently large values of* n, *the problem* $(5.1)_n$ (i.e., *the problem* (5.1) *with data* v_0^n, f_n) *has a solution* $v_n \in C([-\tau_1, \tau_2]; W^m)$ *in all of* $[\tau_1, \tau_2]$. *Moreover,*

$$\lim_{n\to+\infty} v_n = v \quad \text{in } C([-\tau_1, \tau_2]; W^m),$$

and

$$\lim_{n\to+\infty} \nabla\pi_n = \nabla\pi \quad \text{in } L^1(]-\tau_1, \tau_2[; W^m).$$

If $f_n \to f$ *in* $C([-\tau_1, \tau_2]; W^m)$ [*respectively in* $L^q(]-\tau_1, \tau_2[; W^m)$, *for* $q \in [1, +\infty]$] *then* $\nabla\pi_n \to \nabla\pi$ *in this same space.*

Proof. It is sufficient to present the argument for $[0, \tau_2]$. Hence we assume that $I = [0, +\infty[$. By equation (5.8), v must belong to $C([0, \tau_2[; W^m)$. Moreover, by Theorem 5.1 in reference [3], it follows that the solution v can be extended to an interval larger than $[0, \tau_2[$. In particular, $v \in C([0, \tau_2]; W^m)$. Set $M = \|v\|_{[0,\tau_2],m}$, and let M_f be a uniform upper bound for the norms of the functions f_n in the space $L^1(0, \tau_2; W^m)$. Let $c = c(\Omega, n, p, k)$ be the constant that appears in [3] Theorem 5.1, and set

$$\tau^* = c(1 + M + M_f)^{-1}.$$

Let $[0, t]$ be an arbitrary interval such that the solutions v_n exist in $[0, t]$ and are uniformly bounded in $C([0, t]; W^m)$, for large values of n. Theorem 5.1 in reference [3] guarantees the existence of such intervals.

From Lemma 5.1 above it follows, in particular, that $\|v_n(t)\|_m \leqq 1 + M$, for sufficiently large values of n. Again by Theorem 5.1 in reference [3] the solutions v_n can be extended to all of $[t, t + \tau^*]$, and are uniformly bounded in the space $C([0, t + \tau^*]; W^m)$. Since τ^* is independent of t, it readily follows that

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* Such a solution exists if the positive real numbers τ_1 and τ_2 are sufficiently small. However, in this theorem, $]-\tau_1, \tau_2[$ is any arbitrary interval on which such a solution exists.

(for sufficiently large values of n) the solutions v_n exist in all of $[0, \tau_2]$ and are uniformly bounded in the space $C([0, \tau_2]; W^m)$. The conclusion then follows by applying the Lemma 5.1 to the sequence of solutions v_n.

Theorem 5.3. *Let v_0, f and v be as in Theorem 5.2 Then, for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that the following statement holds.*

Let $u_0 \in W^m$ satisfy $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot \nu = 0$ on Γ, and let $g \in L^1(]-\tau_1, \tau_2[; W^m)$. If

$$\|u_0 - v_0\|_m < \delta, \qquad \|g - f\|_{L^1(]-\tau_1,\tau_2[;W^m)} < \delta \tag{5.10}$$

then the solution u, $\nabla\pi'$ of the Euler equations (5.1) *with data u_0, g exists on all of $[-\tau_1, \tau_2]$. Moreover $u \in C([-\tau_1, \tau_2]; W^m)$, $\nabla\pi' \in L^1(]-\tau_1, \tau_2[; W^m)$ and (continuous dependence on the data)*

$$\begin{aligned} &\|u - v\|_{C([-\tau_1,\tau_2];W^m)} < \varepsilon, \\ &\|\nabla(\pi - \pi')\|_{L^1(]-\tau_1,\tau_2[;W^m)} < \varepsilon. \end{aligned} \tag{5.11}$$

In the above statement, one can replace everywhere L^1 by L^q, $q \in]1, +\infty]$, or by C.

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