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On a Stationary Transport Equation.

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1. – Introduction.

Let Ω be an open, bounded subset of R^n, $n \geqslant 2$, locally situated on one side of its boundary Γ, which is a C^1 differentiable manifold. We denote by ν the unit outward normal to Γ.

Let $v(x)$, $a(x)$ and μ be a vector field in Ω, a scalar field in Ω, and a real parameter, respectively. Let X be a Banach space of real functions defined in Ω. In the sequel we look for solutions $y \in X$ of the equation

$$\mu y + v \cdot \nabla y + a y = g, \tag{1.1}$$

where $g \in X$ is given. Without loss of generality, we assume that $\mu \geqslant 0$. More precisely, we look for $B \in \mathfrak{L}[X]$, such that $y = Bg$ is a solution of (1.1), for each $g \in X$. Here, $\mathfrak{L}[X]$ denotes the Banach space of all bounded linear operators in X. For convenience, we call this problem, the existence problem in X.

The above problem is more difficult to solve that its evolution counterpart $\partial y/\partial t + v \cdot \nabla y + a y = g$, since this last equation can be solved by using the characteristic's method. The title of our paper originates from the connection between the two problems.

Our interest in equation (1.1) is due to our recent study [2], in which we prove (among others) existence and uniqueness results in Sobolev spaces $W^{k,p}$ for the stationary solutions of the compressible, heat-conductive, Navier-Stokes equations in dimension n. However, the proofs given in reference [2] use some results on the existence problem for equation (1.1), in spaces $W^{-1,p}$. These results are stated here in theorem 5.2 and in corollary 5.3. In order to prove these two statements, we start by studying the

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existence problem in space $W_0^{1,p}$; see theorems 2.3 and 3.3 below. As a by-product we esatblish also existence results in spaces $W^{1,p}$; see theorem 4.1.

Existence theorems in Sobolev spaces where previously established only in the hilbertian case $p = 2$ (see K. O. FRIEDRICHS [5], P. D. LAX and R. S. PHILLIPS [7], J. J. KOHN and L. NIRENBERG [6]) and in the case $X = L^p(\Omega)$ (see G. FICHERA [3], [4]; see also O. A. OLEINIK and E. V. RADEKEVIC [11], and references therein).

In general, problem (1.1) is well posed in spaces $L^p(\Omega)$ if y is assigned on the set $\Gamma_1 = \{x \in \Gamma: v\cdot\nu < 0\}$. This was proved by FICHERA [3], [4], for a more general class of equations. However, in view of the applications given in [2], we are interested in studying equation (1.1) in spaces $W_0^{1,p}$ ($y = 0$ on Γ), under the assumption $v\cdot\nu = 0$ on Γ. It is worth noting that this last assumption is necessary (at least formally), and sufficient in order to solve equation (1.1) in $W_0^{1,p}$. In fact, equation (1.1) togheter with the assumption $g|_\Gamma = 0$ and the requirement $y|_\Gamma = 0$, implies that $(\nabla y)|_\Gamma$ is parallel to ν, and that $(v\cdot\nabla y)|_\Gamma = 0$. Hence, it is not reasonable looking for the existence result in $W_0^{1,p}$, without the assumption $v\cdot\nu = 0$. This last assumption is also sufficient for solving (1.1) in $W_0^{1,p}$, as follows from theorems 2.3 and 3.3 below. In particular, these theorems shows that the solution $y \in W^{1,p}$ of problem (1.1) (see section 4) must vanish on Γ, whenever g vanishes on Γ. Note that under the hypothesis $v\cdot\nu = 0$ on Γ, the solution of problem (1.1) is unique in the class $W^{1,p}$, for $\mu > (1/p)\,|\operatorname{div} v|_\infty + |a|_\infty$. Infact, by multiplying both sides of (1.1) by $|y|^{p-2}y$, and by doing obvious devices, one proves that

$$(\mu - (1/p)\,|\operatorname{div} v|_\infty - |a|_\infty)\,|y|_p \leq |g|_p\,. \tag{1.2}$$

For more general uniqueness results see [4], [11], [9], [10].

Let us now introduce some notations. We denote by L^p the Banach space $L^p(\Omega)$, $1 \leq p \leq +\infty$, endowed with the usual norm $|\ |_p$, and by $W^{k,p}$ the Sobolev space $W^{k,p}(\Omega)$, endoved with the usual norm $\|\ \|_{k,p}$. Moreover, $W_0^{1,p}$ is the closure of $\mathfrak{D}(\Omega)$ in $W^{1,p}$, and $W^{-1,q}$ is the dual space of $W_0^{1,p}$, $p \in]1, +\infty[$, $q = p/(p-1)$. Furthermore, $C^k = C^k(\bar{\Omega})$. These notations are also used for functional spaces whose elements are vector fields $v(x) = (v_1(x), \ldots, v_n(x))$.

We denote by $r = r(p)$ a real number such that

$$\begin{cases} r = p, & \text{if } p \in]n, +\infty[,\\ r > n, & \text{if } p = n,\\ r = n, & \text{if } p \in]1, n]. \end{cases} \tag{1.3}$$

In section 5 we will use a slightly different definition for $r(p)$, by setting $r > n$ if $p \in]1, n[$.

In the sequel we will assume that $v \in C^1$, $a \in W^{1,r}$ and we define

$$L(x) = \max_{|\xi|=1} \left| \sum_{i,k=1}^{n} D_k v_i(x) \xi_i \xi_k \right|,$$

where $D_k = \partial/\partial x_k$, $k = 1, \ldots, n$. Moreover,

$$\mu_p \equiv 1/p\, |\operatorname{div} v|_\infty + |L|_\infty + |a|_\infty + c_0 |\nabla a|_r\,, \tag{1.4}$$

where $c_0 = c(n, p, r)$ is a positive constant such that $|f|_s \leqslant c_0 |\nabla f|_p$, $\forall f \in W_0^{1,p}$. Here, s is defined by $1/s = (1/p) - (1/n)$ if $p < n$, $1/s = (1/n) - (1/r)$ if $p = n$, $s = +\infty$ if $p > n$. The existence of c_0 is guaranted by well known Sobolev inequalities.

In the sequel, c denote positive constants depending at most on Ω, n, p, r. The same symbol c will be utilized to denote distinct constants.

2. – The existence problem in $W_0^{1,p}$, $p \in]1, 2]$.

We start this section by stating the following preliminar existence result in $W_0^{1,2}$:

LEMMA 2.1. *Let $\Gamma \in C^2$, $v \in C^1(\bar{\Omega})$, $v \cdot \nu = 0$ on Γ, and $a \in L^\infty \cap W^{1,r}$, where $r = r(2)$. Assume that*

$$\mu > \mu_2\,. \tag{2.1}$$

Then, for each $g \in W_0^{1,2}$, the problem (1.1) has a (unique) solution $y \in W_0^{1,2}$. Moreover,

$$\begin{cases} (\mu - \mu_2)\, |\nabla y|_2 \leqslant |\nabla g|_2\,, \\ [\mu - (\tfrac{1}{2})\, |\operatorname{div} v|_\infty - |a|_\infty]\, |y|_2 \leqslant |g|_2\,. \end{cases} \tag{2.2}$$

PROOF. – For each $\varepsilon > 0$, let $y_\varepsilon \in W^{2,2} \cap W_0^{1,2}$ be the solution of

$$\begin{cases} -\varepsilon \Delta y_\varepsilon + \mu y_\varepsilon + v \cdot \nabla y_\varepsilon + a y_\varepsilon = g, & \text{in } \Omega, \\ (y_\varepsilon)|_\Gamma = 0. \end{cases} \tag{2.3}$$

By multiplying both sides of $(2.3)_1$ by Δy_ε, by integrating in Ω, and by doing a suitable integration by parts, since $v \cdot \nabla y_\varepsilon = 0$ on Γ, one easily gets

$$\varepsilon\, |\Delta y_\varepsilon|_2^2 + (\mu - \mu_2)\, |\nabla y_\varepsilon|_2^2 \leqslant |\nabla g|_2\, |\nabla y_\varepsilon|_2\,. \tag{2.4}$$

Hence, $(\mu-\mu_2)|\nabla y_\varepsilon|_2 \leqslant |\nabla g|_2$, and $\lim \varepsilon|\Delta y_\varepsilon|_2 = 0$ as $\varepsilon \to 0$. Consequently, there exists $y \in W_0^{1,2}$ such that (at least for a subsequence y_ε) $y_\varepsilon \to y$, weakly in $W_0^{1,2}$. Clearly, y is a solution of (1.1). □

We denote by $B_2 = B_2(\mu) \in \mathfrak{L}[W_0^{1,2}]$, the linear map $g \to y = B_2 g$, defined in lemma 2.1.

The next result, consists on a technical justification of an integration by parts formulae.

LEMMA 2.2. *Let* $p \in]1, 2[$, *and let* $\Gamma \in C^2$. *Assume that* $v \in C^1$, $y \in W_0^{1,2}$, *and* $v \cdot \nabla y \in W_0^{1,2}$. *Then*

$$\sum_{k=1}^{n} \int_\Omega D_k(v\cdot\nabla y)(D_k y)\,|\nabla y|^{p-2}\,dx = \tag{2.5}$$

$$= \sum_{i,k=1}^{n} \int_\Omega (D_k v_i)(D_i y)(D_k y)\,|\nabla y|^{p-2}\,dx - (1/p)\int_\Omega (\operatorname{div} v)\,|\nabla y|^p\,dx\,.$$

PROOF. – Let $\tilde v \in C^1(R^n)$ be a compact supported vector field such that $\tilde v = v$ in $\bar\Omega$. Next, extend y to all of R^n, by setting $y = 0$ in R^n/Ω. Denote by $y_\delta = \varphi_\delta * y$ the Friedrich's mollifier of y. A well known Friedrich's lemma (see for instance [8], corollary to lemma 6.1, page 315) establish that

$$\lim_{\delta\leftarrow 0^+}\left[\sum_{i=1}^{n}\varphi_\delta * \left(\tilde v_i \frac{\partial y}{\partial x_i}\right) - \sum_{i=1}^{n}\tilde v_i\left(\varphi_\delta * \frac{\partial y}{\partial x_i}\right)\right] = 0\,, \tag{2.6}$$

in the $W^{1,2}(R^n)$ norm. On the other hand,

$$\lim_{\delta\to 0^+}[\varphi_\delta * (\tilde v\cdot\nabla y)] = \tilde v\cdot\nabla y\,, \tag{2.7}$$

in the $W^{1,2}(R^n)$ norm. Note that $\tilde v\cdot\nabla y \in W^{1,2}(R^n)$. By using (2.6), (2.7) one gets

$$\lim_{\delta\to 0^+}\tilde v\cdot\nabla y_\delta = \tilde v\cdot\nabla y\,,$$

in the above norm. Consequently,

$$\lim_{\delta\to 0^+} D_k(\tilde v\cdot\nabla y_\delta) = D_k(\tilde v\cdot\nabla y)\,, \tag{2.8}$$

$(k = 1, \dots, n)$ in the $L^2(R^n)$ norm. On the other hand, a straight forward calculation shows that

$$(2.9)\qquad \sum_{k=1}^{n} \int_{R^n} D_k(\tilde{v}\cdot\nabla y_\delta)(D_k y_\delta)\,|\nabla y_\delta|^{p-2}\,dx = \\ = -(1/p)\int_{R^n}(\operatorname{div}\tilde{v})\,|\nabla y_\delta|^p\,dx + \sum_{i,k=1}^{n}\int_{R^n}(D_k\tilde{v}_i)(D_i\tilde{y}_\delta)(D_k\tilde{y}_\delta)\,|\nabla y_\delta|^{p-2}\,dx\,.$$

By passing to the limit in (2.9) as $\delta\to 0^+$, and by taking (2.8) in account, we show that (2.9) holds if y_δ is replaced by y. This proves (2.5), since $y\in W_0^{1,2}(\Omega)$.

THEOREM 2.3. *Let $p\in\,]1,2[$, and let the assumptions of lemma 2.1 be satisfied. If*

$$(2.10)\qquad \mu > \max\{\mu_p, \mu_2\}\,,$$

then the map $B_2(\mu)$ can be uniquely extended to a map $B_p = B_p(\mu)\in\mathfrak{L}[W_0^{1,p}]$ such that $y = B_p g$ is the (unique) solution of equation (1.1), for each $g\in W_0^{1,p}$.

Moreover,

$$(2.11)\qquad \begin{cases} (\mu-\mu_p)\,|\nabla y|_p \leq |\nabla g|_p\,, \\ (\mu-(1/p)\,|\operatorname{div} v|_\infty - |a|_\infty)\,|y|_p \leq |g|_p\,. \end{cases}$$

PROOF. Let $g\in W_0^{1,2}$, and let $y = B_2 g$. By multiplying both sides of the equation

$$(2.15)\qquad \mu D_k y + D_k(v\cdot\nabla y) + aD_k y + (D_k a)y = D_k g$$

by $|\nabla y|^{p-2}D_k y$, by adding with respect to k, and by integrating over Ω, we show that

$$(2.16)\qquad \mu|\nabla y|_p^p + \int_\Omega \sum_k D_k(v\cdot\nabla y)\,D_k y|\nabla y|^{p-2}\,dx + \int_\Omega a|\nabla y|^p\,dx + \\ + \int_\Omega y|\nabla y|^{p-2}\,\nabla y\cdot\nabla a\,dx = \int_\Omega |\nabla y|^{p-2}\,\nabla y\cdot\nabla g\,dx\,.$$

By using lemma 2.2, one shows that the inequality $(2.11)_1$ holds, for every $g\in W_0^{1,2}$. Finally, standard arguments yield the thesis. □

3. – The existence problem in $W_0^{1,p}$, $p\in\,]2,+\infty[$.

In this section we introduce a different approximation method, which allows us to consider the general case $p\in\,]1,+\infty[$, under the assumption that the mean curvature $\chi(x)$ of Γ at the point x is non-negative, for all $x \in \Gamma$.

LEMMA 3.1. *Let $p \in]1, +\infty[$, let $\Gamma \in C^3$, and assume that the mean curvature $\chi(x)$ of Γ is everywhere non-negative. Furthermore, let $y \in W^{3,p} \cap W_0^{1,p}$ be such that $\Delta y = 0$ on Γ. Then, for every $\delta > 0$, one has*

$$-\int_\Omega \Delta(\nabla y)\cdot\left[(|\nabla y|^2+\delta)^{(p-2)/2}\nabla y\right]dx \geq 0\,. \tag{3.1}$$

In particular,

$$-\int_\Omega \Delta(\nabla y)\cdot[|\nabla y|^{p-2}\nabla y]\,dx \geq 0\,. \tag{3.2}$$

PROOF. An integration by parts yields

$$\begin{aligned}(3.3)\quad -\int_\Omega \Delta(\nabla y)\cdot\nabla y(|\nabla y|^2+\delta)^{(p-2)/2}dx &= \\ = \sum_i\int_\Omega D_i(\nabla y)\cdot D_i[(|\nabla y|^2+\delta)^{(p-2)/2}\nabla y]\,dx &- \\ -\int_\Gamma \sum_{i,k}(D_{ik}^2 y)(D_k y)\,\nu_i(|\nabla y|^2+\delta)^{(p-2)/2}d\Gamma&\,.\end{aligned}$$

On the other hand, straightforward calculations show that identity

$$\begin{aligned}(3.4)\quad \sum_i D_i(\nabla y)\cdot D_i[(|\nabla y|^2+\delta)^{(p-2)/2}\nabla y] &= \\ = (|\nabla y|^2+\delta)^{(p-2)/2}\sum_{i,k}(D_{ik}^2 y)^2 + \frac{p-2}{4}&(|\nabla y|^2+\delta)^{(p-4)/2}\sum_i(D_i|\nabla y|^2)^2\,,\end{aligned}$$

holds a.e. in Ω. This proves that the first integral onthe right hand side of (3.3) is non-negative, if $p \in [2, +\infty[$. If $p \in]1, 2]$, straightforward calculations show that

$$\begin{aligned}(3.5)\quad \sum_i D_i(\nabla y)\cdot D_i\left[(|\nabla y|^2+\delta)^{(p-2)/2}\nabla y\right] &\geq \\ \geq [(p-1)\,|\nabla y|^2+\delta]\,(|\nabla y|^2+\delta)^{(p-4)/2}&\sum_{i,k}(D_{ik}^2 y)^2\,,\end{aligned}$$

a.e. in Ω. Consequently, the left hand side of (3.5) is non-negative in Ω, for every $p \in]1, +\infty[$.

In order to complete the proof, we show that the boundary integral that appears in equation (3.3) is less than or equal to zero. Since $\chi(x) \geq 0$

on Γ, it suffices to show that

$$-\sum_{i,k}(D^2_{ik}y)(D_k y)\nu_i = (n-1)\,\chi\left(\frac{\partial y}{\partial \nu}\right)^2, \tag{3.6}$$

in the usual trace's sense on Γ. We will prove that (3.6) holds pointwisely in Γ, for sufficiently regular functions y (say, $y \in C^2$). Then, standard devices show it in the above form.

Let $x_0 \in \Gamma$. By doing an orthonormal change of coordinates we assume, without loss of generality, that $\nu(x_0)$ points in the x_n direction, and that the principal directions of Γ at x_0 are parallel to the x_i directions, $i = 1, \ldots, n-1$. Since $y = \Delta y = 0$ on Γ, one has

$$-\sum_{i,k=1}(D^2_{ik}y)(D_k y)\nu_i = -(D^2_{nn}y)(D_n y) = (D_n y)\sum_{i=1}^{n-1} D^2_{ii}y\,.$$

By denoting with $\chi_i(x_0)$, $i = 1, \ldots, n-1$, the principal curvature in the x_i direction (considered negative if ν points toward the center of curvature) one proves, without difficulty, that

$$D^2_{ii}y = \chi_i(x_0)\,D_n y\,.$$

This yields (3.6) at x_0. □

LEMMA 3.2. *Let $p \in \,]1, +\infty[$ and let Ω and Γ be as in lemma 3.1. Assume that $v \in C^1$, $v\cdot\nu = 0$ on Γ, $a \in L^\infty \cap W^{1,r}$, where $r = r(p)$, $g \in W^{1,p}_0$, and*

$$\mu > \mu_p\,. \tag{3.7}$$

Then, for each $\varepsilon > 0$, the solution y_ε problem

$$\begin{cases} -\varepsilon\Delta y_\varepsilon + \mu y_\varepsilon + v\cdot\nabla y_\varepsilon + a y_\varepsilon = g, & \text{in } \Omega,\\ (y_\varepsilon)|_\Gamma = 0, \end{cases} \tag{3.8}$$

verifies the estimates

$$\begin{cases} (\mu-\mu_p)\,|\nabla y_\varepsilon|_p \leqslant |\nabla g|_p,\\ (\mu-(1/p)\,|\operatorname{div} v|_\infty - |a|_\infty)\,|y_\varepsilon|_p \leqslant |g|_p\,. \end{cases} \tag{3.9}$$

PROOF. Let $y_\varepsilon \in W^{3,p} \cap W^{1,p}_0$ be the solution of (3.8). From the assumptions on v and y_ε it follows that $v\cdot\nabla y_\varepsilon = 0$ on Γ. Consequently, $\Delta y_\varepsilon = 0$ on Γ, which is a crucial condition in order to apply lemma 3.1.

For convenience, we set here $\Lambda = (|\nabla y_\varepsilon|^2 + \delta)^{(p-2)/2}$. By taking the derivative of both sides of (3.8) with respect to x_k, by multiplying by $\Lambda D_k y_\varepsilon$, by adding with respect to k, and by integrating over Ω, one shows that

$$(3.10) \qquad -\varepsilon\int \Delta(\nabla y_\varepsilon)\cdot\Lambda\nabla y_\varepsilon\,dx + \mu\int\Lambda|\nabla y_\varepsilon|^2\,dx + \\ + \sum_{i,k}\int v_i(D^2_{ik}y_\varepsilon)\Lambda D_k y_\varepsilon\,dx + \sum_{i,k}\int(D_k v_i)(D_i y_\varepsilon)(D_k y_\varepsilon)\Lambda\,dx + \\ + \int a\Lambda|\nabla y_\varepsilon|^2\,dx + \int y_\varepsilon\Lambda\nabla y_\varepsilon\cdot\nabla a\,dx = \int\Lambda\nabla y_\varepsilon\cdot\nabla g\,dx\,.$$

By using lemma 3.1, and by noting that the third integral on the left hand side of (3.10) is equal to

$$-(1/p)\int(\operatorname{div} v)\,(|\nabla y_\varepsilon|^2 + \delta)^{p/2}\,dx\,,$$

one shows that

$$(3.11) \qquad \mu\int\Lambda|\nabla y_\varepsilon|^2\,dx \leqslant (1/p)\int(\operatorname{div} v)\,(|\nabla y_\varepsilon|^2 + \delta)^{p/2}\,dx + \\ + \int L(x)\,\Lambda|\nabla y_\varepsilon|^2\,dx + \int|a|\Lambda|\nabla y_\varepsilon|^2\,dx + \int|\nabla a|\,|y_\varepsilon|\Lambda|\nabla y_\varepsilon|\,dx + \int\Lambda|\nabla y_\varepsilon|\,|\nabla g|\,dx\,.$$

By passing to the limit as $\delta\to 0$, one finally gets $(3.9)_1$. □

Lemma 3.2 allows us to extend theorem 2.3 to all values $p\in\,]1, +\infty[$, under the additional assumption $\chi(x)\geqslant 0$, $\forall x\in\Gamma$. For $p\in\,]1,2[$ the proof becames more technical, moreover in that case, theorem 2.3 gives a stronger result. Hence, we will take into account only the case $p\in[2,+\infty[$. By the way, we remark that (by using Sobolev's embedding theorems) the proof given below applies as well, if $p\geqslant 2n/(n+4)$. In particular it applies for all $p\in\,]1,+\infty[$, if $n\leqslant 4$.

THEOREM 3.3. *Let $p\in[2,+\infty[$, let $\Gamma\in C^3$, and assume that the mean curvature $\chi(x)$ of Γ is non-negative, for all $x\in\Gamma$. Furthermore, let v, a and μ be as in lemma 3.2. Then, there exists a bounded linear map $B_p = B_p(\mu)\in \mathfrak{L}[W^{1,p}_0]$ such that $y = B_p g$ is the (unique) solution of problem (1.1), for each $g\in W^{1,p}_0$. Moreover*

$$(3.12) \qquad \begin{cases} (\mu-\mu_p)\,|\nabla y|_p \leqslant |\nabla g|_p\,, \\ (\mu-(1/p)\,|\operatorname{div} v|_\infty - |a|_\infty)\,|y|_p \leqslant |g|_p\,. \end{cases}$$

PROOF. From (3.9) it follows the existence of $y\in W^{1,p}_0$ such that (at least, for a subsequence y_ε) $y_\varepsilon\to y$, weakly in $W^{1,p}_0$. By multiplying both sides of equation (3.8) by y_ε, one easily shows that $\varepsilon|\nabla y_\varepsilon|_2^2$ is bounded by a constant independent of ε. Hence, $\varepsilon y_\varepsilon \to 0$ in $W_0^{1,2}$. In particular, $\varepsilon \Delta y_\varepsilon \to 0$ in $W^{-1,2}$ (1). By passing to the limit in equation (3.8), as $\varepsilon \to 0$, we show that y is a strong solution of (1.1). $\square$

4. – Existence results in $W^{1,p}$, $p \in]1, +\infty[$.

As a by-product of theorems 2.3 and 3.3, we will prove here an existence theorem in spaces $W^{1,p}$. In this section we assume that $\Gamma \in C^1$, and we associate to Ω a fixed open ball B, such that $\bar{\Omega} \subset B$. Moreover, we fix linear continuous maps $g \to \tilde{g}$, $v \to \tilde{v}$, $a \to \tilde{a}$, from $W^{1,p}(\Omega)$ into $W_0^{1,p}(B)$, from $C^1(\bar{\Omega})$ into $C^1(\bar{B})$, and from $L^\infty(\Omega) \cap W^{1,r}(\Omega)$ into $L^\infty(B) \cap W^{1,r}(B)$, respectively, and such that $\tilde{g}|_\Omega = g$, $\tilde{v}|_\Omega = v$, $\tilde{a}|_\Omega = a$. Moreover, $\tilde{g}$, $\tilde{a}$ and $\tilde{v}$ have support contained in a fixed compact subset of B.

We define $\tilde{\mu}_p$ by replacing in formulae (1.4), Ω, v, a with B, $\tilde{v}$, $\tilde{a}$, respectively. Clearly,

$$\tilde{\mu}_p \leqslant c(\Omega, n, p, r)\,\mu_p\,, \tag{4.1}$$

where $r = r(p)$. Here, $p \in]1, +\infty[$.

Let $\tilde{y} \in W_0^{1,p}(B)$ be the solution of $\mu\tilde{y} + \tilde{v}\cdot\nabla\tilde{y} + \tilde{a}\tilde{y} = \tilde{g}$, in B. Clearly, $y = \tilde{y}|_\Omega$ solves (1.1). Hence, one has the following result:

THEOREM 4.1. *Let $p \in]1, +\infty[$ be fixed, and set $r = r(2)$ if $p \in]1, 2]$, $r = r(p)$ if $p \in]2, +\infty[$. Assume that $\Gamma \in C^1$, $v \in C^1$, and $a \in L^\infty \cap W^{1,r}$. Then, if $\mu > c_1\mu_2$ (in case that $p \in]1, 2]$) or $\mu > c_1\mu_p$ (in case that $p \in]2, +\infty]$), where $c_1 = c_1(\Omega, n, p, r)$ is a suitable positive constant, there exists a linear continuous map $T \in \mathfrak{L}[W^{1,p}]$ such that $y = Tg$ is a solution of (1.1), for each $g \in W^{1,p}$. Moreover,*

$$(\mu - \tilde{\mu}_p)\,\|y\|_{1,p} \leqslant c(\Omega, p, n)\,\|g\|_{1,p}\,.$$

Note that the solution y is unique in the class $W^{1,p}$, if $v\cdot\nu = 0$ on Γ. Otherwise, the above statement should be completed in accordance with a result of [3], [4] refered in section 1. However, we didn't investigate in this direction.

We remark that the closure in L^p of the map $B_p \in \mathfrak{L}[W_0^{1,p}]$ solves the existence problem in L^p (under the hypothesis of theorem 2.3 if $p \in]1, 2]$, under the hypothesis of theorem 3.3 if $p \in]2, +\infty[$).

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(1) By multiplying (3.8) by Δy_ε, one could show that $\varepsilon \Delta y_\varepsilon \to 0$ in L^2.

By using this result, and by arguing as above, one easily proves an existence result in L^p-spaces, similar to theorem 4.1. However, existence theorems in L^p, for a more general class of problems, are given in [3], [4], [11]; see also the references in [11].

5. – Existence of weak solutions in spaces $W^{-1,p}$.

Finally, we will study equation (1.1) in spaces $W^{-1,p}$, which is our main concern, in view of [2]. In order to simplify the statements, we change in this section the definition of $r = r(p)$, by setting $r = p$ if $p > n$, $r > n$ if $p \leqslant n$. This modifies definition (1.3) only in case that $p < n$. Since the new value of $r(p)$ is greater than or equal to the old one, all the statements hold again if r is assumed to be defined as above. Furthermore, we assume here that $v \in W^{2,r}$, and that $a \in W^{1,r}$.

The following proposition is a readjustment of theorems 2.3 and 3.3.

PROPOSITION 5.1. *Let $p \in]1, +\infty[$ be fixed, and let Ω be as in theorem 2.3 if $p \in]1, 2]$; or as in theorem 3.3 if $p \in]2, +\infty[$. Let $r = r(p)$ be defined as above, and assume that $v \in W^{2,r}$, $a \in W^{1,r}$, and $v \cdot \nu = 0$ on Γ. Then, for a suitable positive constant $c = c(\Omega, n, p, r)$, if*

$$\mu > c(\|v\|_{2,r} + \|a\|_{1,r}), \tag{5.1}$$

there exists a map $B_p \in \mathcal{L}[W_0^{1,p}]$ such that $y = B_p g$ is the (unique) solution of the equation

$$\mu y - \operatorname{div}(yv) + ay = g, \tag{5.2}$$

for each $g \in W_0^{1,p}$. Moreover,

$$\left[\mu - c(\|v\|_{2,r} + \|a\|_{1,r})\right] \|y\|_{1,p} \leqslant \|g\|_{1,p}. \tag{5.3}$$

Clearly, the operated B_p is invertible. Let us denote by A_p the inverse of B_p, and by $D(A_p)$ the domain of A_p. The operated A_p is a closed operator, moreover

$$D(A_p) = \{y \in W_0^{1,p} : \mu y - \operatorname{div}(yv) + ay \in W_0^{1,p}\}. \tag{5.4}$$

In particular, $\mathcal{D}(\Omega) \subset D(A_p)$, and $D(A_p)$ is dense in $W_0^{1,p}$, for all $p \in]1, +\infty[$.

Set $q = p/(p-1)$, and consider the existence problems (1.1) in the space $W^{-1,q}$. We assume that Ω, v, and a are as in theorem 5.1, and that μ verifies (5.1).

If $g \in W^{-1,q}$, we say that $y \in W^{-1,q}$ is a weak solution of problem (1.1) if

$$\langle \mu\varphi - \operatorname{div}(\varphi v) + a\varphi, y\rangle = \langle \varphi, g\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{5.5}$$

where $\langle , \rangle$ denotes the duality pairing between $W_0^{1,p}$ and $W^{-1,q}$. In particular, $y \in W^{-1,q}$ is a weak solution of (1.1) if

$$\langle A_p \varphi, y\rangle = \langle \varphi, g\rangle, \quad \forall \varphi \in D(A_p)\,. \tag{5.6}$$

By deniting with A_p^* the adjoint of A_p, equation (5.6) is equivalent to

$$A_p^* y = g\,. \tag{5.7}$$

Since $B_p^* = (A_p^*)^{-1} \in \mathcal{L}[W^{-1,q}]$, and $\|B_p\| = \|B_p^*\|$, if follows that for each $g \in W^{-1,q}$ equation (5.7) has a unique solution $y \in W^{-1,q}$, given by $y = B_p^* g$. Moreover,

$$\big[\mu - c(\|v\|_{2,r} + \|a\|_{1,r})\big]\,\|y\|_{-1,q} \leqslant \|g\|_{-1,q}\,. \tag{5.8}$$

Hence, we have proved the following result:

THEOREM 5.2. *Let $p \in]1, +\infty[$, and assume that Ω is as in theorem 2.3 if $p \in]1, 2]$, or as in theorem 3.3 if $p \in]2, +\infty[$. Let $r = p$ if $p > n$, $r > n$ if $p \leqslant n$. Assume that $v \in W^{2,r}$, $v\cdot\nu = 0$ on Γ, and $a \in W^{1,r}$. Set $q = p/(p-1)$. Then, for a suitable positive constant $c = c(\Omega, n, p, r)$ the following statement holds:*

If μ verifies (5.1), then $y = B_p^* g$ is a weak solution of (1.1) (actually, (5.6) holds) for each $g \in W^{-1,q}$. Here, $B_p^* \in \mathcal{L}[W^{-1,q}]$ is defined as above. Moreover, (5.8) holds.

COROLLARY 5.3. *Let $q \in]1, +\infty[$, and $\Gamma \in C^2$. Set $p = q/(q-1)$, and let r be defined as in theorem 5.2. Assume that v and a are as in this last theorem. Then, for a suitable positive constant $c = c(\Omega, n, p, r)$ the following statement holds:*

If μ verifies (5.1), then there exists a bounded linear map $B_q \in \mathcal{L}[W^{-1,q}]$ such that $y = B_q g$ is a weak solution of (1.1), for each $g \in W^{-1,q}$. Moreover,

$$\big[\mu - c(\|v\|_{2,r} + \|a\|_{1,r})\big]\,\|y\|_{-1,q} \leqslant c_1 \|g\|_{-1,q}\,, \tag{5.9}$$

where $c_1 = c_1(\Omega, n, q)$ is a positive constant.

PROOF. Fix an open ball B such that $\bar{\Omega} \subset B$, and fix linear continuous maps $v \to \tilde{v}$, $a \to \tilde{a}$, $g \to \tilde{g}$, from $W^{2,r}(\Omega)$ into $W_0^{2,r}(B)$, from $W^{1,r}(\Omega)$ into $W_0^{1,r}(B)$, and from $W^{-1,q}(\Omega)$ into $W^{-1,q}(B)$, respectively, such that $\tilde{v}|_\Omega = v$, $\tilde{a}|_\Omega = a$, $\tilde{g}|_\Omega = g$. Let $\tilde{y} \in W^{-1,q}(B)$ be the weak solution of $\lambda\tilde{y} + \tilde{v}\cdot\nabla\tilde{y} + \tilde{a}\tilde{y} = \tilde{g}$ in B. The existence of $\tilde{y}$ is guaranteed by theorem 5.2, for $\mu > \tilde{c}(\|\tilde{v}\|_{2,r,B} + \|\tilde{a}\|_{1,r,B})$, where $\tilde{c} = \tilde{c}(B, n, q, r)$. This last condition holds if the constant $c = c(\Omega, n, p, r)$, appearing in equation (5.1), is defined in a suitable way. Obviously, the restriction y of $\tilde{y}$ to Ω, verifies (5.5). □

Finally, we show that the result used in [2] section 2, holds. Assume that Γ is of class C^2, $q > n$, and $v \in W^{2,q}$. Set $r = q$. By corollary 5.3, there exist positive constants $c = c(\Omega, n, q)$, $c_1 = c_1(\Omega, n, q)$ such that the following result holds:

If $\mu > c\|v\|_{2,q}$, then there exists a linear continuous map $B \in \mathfrak{L}[W^{-1,q}]$ such that $y = Bg$ is a weak solution of $\mu y + v\cdot\nabla y = g$, $\forall g \in W^{-1,q}$. Moreover, $(\mu - c\|v\|_{2,q})\ \|y\|_{-1,q} \leq c_1\|g\|_{-1,q}$.

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SUMMARY

Let Ω, Γ, v, a, and X be as described at the beginning of the introduction below, let $p \in]1, +\infty[$, and set $q = p/(p-1)$. If $p > 2$, we also assume that the mean curvature $\chi(x)$ of Γ is everywhere nonnegative. In this paper we solve the existence problem in spaces X, for equation (1.1) below, if $X = W_0^{1,q}$, or $X = W^{-1,p}$. As a by-product, the solvability of (1.1) in spaces $W^{1,p}$ and L^p follows (without any assumption on $\chi(x)$). For more general results on the above problem, see ref. [1].