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STATIONARY MOTIONS AND THE INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE VISCOUS FLUIDS

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ABSTRACT. We study the system of equations (1.1), describing the stationary motion of a compressible viscous fluid in a bounded domain Ω of $\mathbf{R}^3$. The total mass of fluid $m|\Omega|$, inside Ω, is fixed (condition (1.2)). We prove that for small f and g, there exists a unique solution (u,ρ) of the above system of equations, in a neighborhood of $(0,m)$. Moreover, by introducing a suitable parameter λ, we prove that the solution of the Navier-Stokes equations (1.14) are the incompressible limit of the solutions of the compressible Navier-Stokes equations (1.13). The proofs given here, apply, without supplementary difficulties, in the context of Sobolev spaces $H^{k,p}$, and other functional spaces. The results can be extended to the case when temperature dependence is taken into consideration.

1. Introduction and main results. In this paper we study the system

$$(1.1)\quad \begin{cases} -\mu\Delta u - \nu\nabla \mathrm{div}\, u + \nabla p(\rho) = \rho[f - (u\cdot\nabla)u],\ \text{in } \Omega, \\ \mathrm{div}(\rho u) = g,\ \text{in } \Omega, \\ u|_\Gamma = 0, \end{cases}$$

in a bounded, open domain in $\mathbf{R}^3$, locally situated on one side of its boundary Γ, a C^3 manifold. The case $n \neq 3$ can be studied by the same method. As usual,

$$(v\cdot\nabla)u = \Sigma_{i=1}^3 v_i \frac{\partial u}{\partial x_i}.$$

System (1.1) describes the stationary motion of a barotropic, compressible fluid; see Serrin [5]. In equation (1.1), $\rho(x)$ is the density of the fluid, $u(x)$ the velocity field, $f(x)$ the assigned external force field, and $p = p(\rho)$ the pressure. In the physical equation one has $g = 0$; however, on studying (1.1) from a mathematical point of view, it is not without interest to study the general case.

We assume that the total mass of fluid inside Ω is fixed, i.e., we impose to the solution of (1.1) the constraint

(1.2) $\frac{1}{|\Omega|}\int_\Omega \rho(x)dx = m,$

where the mean density m is a given positive constant. The function ρ will be written in the form $\rho = m + \sigma$, and the new unknown $\sigma(x)$ has to verify the constraint

(1.3) $\overline{\sigma} \equiv \frac{1}{|\Omega|}\int_\Omega \sigma(x)dx = 0.$

We assume that the real function $\rho \to p(\rho)$ is defined and has a Lipschitz continuous first derivative $p'(\rho)$ in a neighborhood $I \equiv [m - \ell, m + \ell]$ of m, for some positive $\ell < m/2$. We also assume the (unessential) physical condition $k = p'(m) > 0$. Clearly,

(1.4) $p'(\rho) = p'(m + \sigma) = k - \omega(\sigma), \quad \forall\, \sigma \in I,$

where $\omega(\sigma)$ is a Lipschitz continuous function, such that $\omega(0) = 0$. We set

$$S \equiv \sup_{\sigma,\tau\in[-\ell,\ell]} \frac{|\omega(\sigma) - \omega(\tau)|}{|\sigma-\tau|}.$$

Concerning the constants μ and ν, we only assume that

(1.5) $\mu > 0, \quad \nu > -\mu.$

We remark that, by obvious devices, the coefficients μ, ν can depend on u, ρ and x.

In the sequel, we write the system (1.1) in the equivalent form

$$(1.6)\quad \begin{cases} -\mu\Delta u - \nu\nabla \mathrm{div}\, u + k\nabla\sigma = \omega(\sigma)\nabla\sigma + (m+\sigma)[f - (u\cdot\nabla)u], \text{ in } \Omega, \\ m\, \mathrm{div}\, u + u\cdot\nabla\sigma + \sigma\, \mathrm{div}\, u = g, \text{ in } \Omega, \\ u_{|\Gamma} = 0. \end{cases}$$

Let us introduce some notation. We set

$$|\nabla v|^2 = \Sigma_{i,k=1}^3 \left(\frac{\partial v_i}{\partial x_k}\right)^2, \ \nabla v\colon \nabla^2\tau = \Sigma_{i,k=1}^3 \frac{\partial v_i}{\partial x_k}\frac{\partial^2\tau}{\partial x_i \partial x_k},$$

where v is a vector and τ a scalar.

We denote by H^k, k integer, the Sobolev space $W^{k,2}(\Omega)$, endowed with the usual norm $\|\ \|_k$, and by $|\ |_p$, $1 \leq p \leq +\infty$, the usual norm in $L^p = L^p(\Omega)$. Hence, $\|\ \|_0 = |\ |_2$. For convenience, we also utilize the same symbol H^k to denote the space of vector fields v in Ω such that $v_i \in W^{k,2}(\Omega)$, $i = 1,2,3$. This convention applies to all the functional spaces and norms utilized here.

For $k \geq 1$, we define

$$H_0^k = \{v \in H^k : v = 0 \text{ on } \Gamma\}.$$

Moreover,

$$\overline{H}^2 = \{\tau \in H^2 : \overline{\tau} = 0\}, \overline{H}_0^2 = H_0^2 \cap \overline{H}^2,$$

where $\overline{\tau}$ is the mean value in Ω of the scalar field $\tau(x)$. Finally, for vector fields we define

$$H_{0,d}^3 = \{v \in H_0^3 : \text{div } v = 0 \text{ on } \Gamma\}.$$

In the sequel, $c, c_0, c_1, c_2, \ldots$, denote positive constants depending at most on Ω. Moreover, $c', c_0', c_1', \ldots$, denote positive constants depending at most on Ω, μ, ν, k, m, ℓ, and S. The same symbol c (or c′) will be utilized to denote different constants, even in the same equation.

In Section 3 we prove the following result:

THEOREM A. *There exists positive constants* c_0' *and* c_1' *such that if* $f \in H^1$, $g \in \overline{H}_0^2$, *and*

(1.7) $\|f\|_1 + \|g\|_2 \leq c_0'$,

then there exists a unique solution $(u,\sigma) \in H_0^3 \times \overline{H}^2$ *of problem* (1.6), *in the ball*

(1.8) $\|u\|_3 + \|\sigma\|_2 \leq c_1'$.

A crucial tool in proving this result will be the study of the linear system

$$(1.9) \quad \begin{cases} -\mu\Delta u - \nu\nabla \text{div } u + k\nabla\sigma = F, \text{ in } \Omega, \\ m \text{ div } u + v \cdot \nabla\sigma + \sigma \text{ div } v = g, \text{ in } \Omega, \\ u_{|\Gamma} = 0, \end{cases}$$

for which we will prove the following result:

THEOREM B. *Let* $F \in \overline{H}^1$, $g \in \overline{H}_0^2$, *and* $v \in H_{0,d}^3$ *be given, and assume that* (2.14) *holds. Then, there exists a unique solution* $(u,\sigma) \in H_{0,d}^3 \times \overline{H}^2$ *of the linear system* (1.9). *Moreover,*

(1.10) $\mu\|u\|_3 + k\|\sigma\|_2 \leq c(1 + \frac{\mu+|\nu|}{\mu+\nu})\|F\|_1 + c_1 \frac{\mu+|\nu|}{m}\|g\|_2$.

In Section 4, we assume that the function $p(\rho,\lambda)$ depends, in a suitable way, on a parameter λ. By letting $\lambda \to +\infty$, we prove that the solution of the Navier-Stokes equation (1.14) is the incompressible limit of the solutions of system (1.13). For the justification of the physical aspects of the description (i.e., the behavior of $p(\rho,\lambda)$, as $\lambda \to +\infty$) we refer, for instance, to Klainerman and Majda [2].

We assume that for each value of the parameter $\lambda \in [\lambda_0,+\infty[$ the function $p(\rho,\lambda)$ is defined in a neighborhood $I_\lambda \equiv [m - \ell_\lambda, m + \ell_\lambda]$ of m, where $0 < \ell_\lambda < m/2$. The number $\lambda_0 \in \mathbf{R}$, has no special meaning. Moreover, for each fixed λ, the derivative $dp(\rho,\lambda)/d\rho \equiv p'(\rho,\lambda)$, is Lipschitz continuous on I_λ, with Lipschitz constant S_λ.

We set $k_\lambda \equiv p'(m,\lambda)$, and assume that $k_\lambda \geqslant k_0 > 0$. The constant k_0 has no special meaning, since we will let $k_\lambda \to +\infty$, as $\lambda \to +\infty$. We suppose that there exist positive constants ϕ and ℓ such that

$$(1.11)\quad S_\lambda \leqslant \phi k_\lambda^2, \forall \lambda \geqslant \lambda_0,$$

and

$$(1.12)\quad \ell_\lambda k_\lambda \geqslant \ell, \forall \lambda \geqslant \lambda_0.$$

By (eventually) defining a smaller ℓ_λ, we assume, without losing generality, that $\ell_\lambda k_\lambda = \ell$. Finally, let $\omega_\lambda(\sigma)$ be defined by $p'(m + \sigma,\lambda) = k_\lambda - \omega_\lambda(\sigma)$.

Consider the stationary compressible Navier-Stokes equation, with state function $p(\rho,\lambda)$,

$$(1.13)\quad \begin{cases} -\mu\Delta u_\lambda - \nu\nabla \operatorname{div} u_\lambda + \nabla p(\rho_\lambda,\lambda) = \rho_\lambda[f - (u_\lambda \cdot \nabla)u_\lambda], \\ \operatorname{div}(\rho_\lambda u_\lambda) = 0, \text{ in } \Omega, \\ (u_\lambda)_{|\Gamma} = 0, \end{cases}$$

and the incompressible Navier-Stokes equation

$$(1.14)\quad \begin{cases} -\mu\Delta u_\infty + \nabla\pi(x) = m[f - (u_\infty \cdot \nabla)u_\infty], \\ \operatorname{div} u_\infty = 0, \text{ in } \Omega, \\ (u_\infty)_{|\Gamma} = 0. \end{cases}$$

As above, we set $\rho_\lambda(x) = m + \sigma_\lambda(x)$, and we look for solutions of (1.13) verifying assumption (1.2), i.e., such that (1.3) holds.

We denote by $\tilde{c}, \tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots$, positive constants depending at most on Ω, μ, ν, m, ℓ, ϕ and k_0, and we say that a positive constant is of type $\tilde{c}$ if it depends at most on the above parameters.

In Section 4, we prove the following result:

THEOREM C. *There exists positive constants* $\tilde{c}_0$ *and* $\tilde{c}_1$ *such that the following* *statement holds*:

(i) *Let* $f \in H^1$, *belong to the ball*

(1.15) $\|f\|_1 \leqslant \tilde{c}_0$.

Then, for each $\lambda \geqslant \lambda_0$, *problem* (1.13) *has a unique solution* $(u_\lambda, \sigma_\lambda) \in H_0^3 \times \overline{H}^2$ *in the ball*

(1.16) $\|u_\lambda\|_3 + k_\lambda \|\sigma_\lambda\|_2 \leqslant \tilde{c}_1$.

(ii) *If* $\lim_{\lambda \to +\infty} k_\lambda = +\infty$, *then*

$$(1.17) \quad \begin{cases} u_\lambda \to u_\infty, \text{ weakly in } H_0^3, \text{ strongly in } H_0^s, \ \forall\, s < 3, \\ \operatorname{div} u_\lambda \to 0, \text{ weakly in } H_0^2, \text{ strongly in } H_0^s, \ \forall\, s < 2, \\ \sigma_\lambda \to 0, \text{ strongly in } \overline{H}^2, \\ \nabla p(\rho_\lambda, \lambda) \to \nabla \pi, \text{ weakly in } H^1, \text{ strongly in } H^s, \ \forall\, s < 1, \end{cases}$$

where $(u_\infty, \nabla\pi)$ *is the unique solution of problem* (1.14).

The existence of the solution $(u_\infty, \nabla\pi)$ of (1.14) is well known. However, it follows from our proof, too.

Note that both problems (1.13), (1.14) are invariant under addition of arbitrary constants to $p(\lambda, \rho)$ and π, respectively.

An existence result for system (1.1) was given first by Padula, in reference [4]. Unfortunately, the (quite simple) proof given there depends in a crucial way on a smallness condition on μ in respect to ν (μ and ν positive constants). This condition was dropped in Valli's paper [6], where a result similar to Theorem A is proved, by approximating the stationary solutions with periodic solutions of the corresponding evolution problem. This technique was applied in [7] to the heat-depending case, and to more general boundary conditions.

The proofs given in our paper are quite simple, and apply as well (without any supplementary difficulty) in the context of other spaces of functions, as for instance Sobolev spaces $H^{k,p}$, $1 < p < +\infty$.

In particular, for every $k \geqslant 0$, for small data $(f,g) \in H^{k+1} \times \overline{H}_0^{k+2}$, there exists a unique solution $(u,\sigma) \in H_0^{k+3} \times \overline{H}^{k+2}$, in a neighborhood of the origin. (Here we assume the derivative $p^{(k+1)}$ Lipschitz continuous, and Ω of class C^{k+3}.)

Furthermore, all the results hold again in any dimension of space although on dealing with the non-linear problem in $H^{k,p}$, k must be sufficiently large.

Statements and proofs, in the above general setting up, are given in a forthcoming paper [1], where (for completeness) we will consider the heat-conductive-case. In this paper, we state only the counterparts of Theorems A and B, in Appendix 2. The proofs can be easily done, by following those of Theorems A and B. Here, we have preferred to consider the main case (1.1) by itself, in order to avoid secondary technicalities. In fact, in the heat dependent case a third equation should be added to system (1.1) (see (6.1)) which is weakly coupled with its companion equations. As a matter of fact, the more interesting mathematical problems and the main difficulties, already appear on studying system (1.1).

It is a pleasure to thank Professor Robert Turner for his fruitful suggestions about the proof of Theorem C.

Finally, we notice that it has just come to our attention that another direct approach to the stationary problem is given by Valli in an independent paper [8].

2. **Proof of Theorem B.** We start by proving the uniqueness of the solution of the linear system (1.9), under the assumption (2.1) below. Let (u,σ) be a solution, with data $F=0$, $g=0$. By multiplying both sides of equation $(1.9)_1$ by mu and of equation $(1.9)_2$ by $k\sigma$, by integrating over Ω and by adding side by side the two equations, one easily shows that

$$m\mu_0\|\nabla u\|_0^2 \leqslant \frac{k}{2}|\operatorname{div} v|_\infty\|\sigma\|_0^2,$$

where

$$\mu_0 = \min\{\mu, \mu+\nu\}.$$

Hence,

$$\|u\|_1^2 \leqslant c\,\frac{k}{m\mu_0}|\operatorname{div} v|_\infty\|\sigma\|_0^2.$$

Moreover, from $(1.9)_1$ it follows that

$$k\|\sigma\|_0 \leqslant ck\|\nabla\sigma\|_{-1} \leqslant c(\mu+|\nu|)\|u\|_1,$$

since $\bar{\sigma}=0$. Consequently,

$$\|u\|_1^2 \leqslant c\,\frac{(\mu+|\nu|)^2}{m\mu_0 k}|\operatorname{div} v|_\infty\|u\|_1^2.$$

This proves that the uniqueness holds whenever

(2.1) $\|v\|_3 \leq \dfrac{m\mu_0 k}{c_0(\mu+|\nu|)^2}$,

for a suitable positive constant c_0; recall that $H^2 \subset L^\infty$.

In the remainder of this section we prove the existence of the solution of system (1.9). We assume that $v \in H^3_{0,d}$ verifies the condition

(2.2) $|\text{div } v|_\infty + 2|\nabla v|_\infty \leq mk/(\mu + \nu)$,

and that $F \in H^1$, $g \in \overline{H}^2_0$. Let $\tau \in \overline{H}^2$, and consider the linear problem

(2.3) $\dfrac{mk}{\mu+\nu}\lambda + v \cdot \nabla\lambda = G$,

where

(2.4) $G = \Delta g + \dfrac{m}{\mu+\nu}\text{div } F - [2\nabla v\colon \nabla^2\tau + \Delta v \cdot \nabla\tau + \Delta(\tau \text{ div } v)]$.

The significance of equation (2.3) is strongly related to the identity (2.20). It is well known (Lax-Phillips [3]) that there exists a linear map $G \to \lambda$, from all of L^2 into L^2, such that for each $G \in L^2$ the corresponding λ is a weak solution of (2.3), and verifies the estimate

(2.5) $\dfrac{1}{2}\dfrac{mk}{\mu+\nu}\|\lambda\|_0 \leq \|G\|_0$.

By a weak solution of (2.3), we mean here a function $\lambda \in L^2$ such that

(2.6) $\dfrac{mk}{\mu+\nu}\int_\Omega \lambda\varphi dx - \int_\Omega \lambda \text{ div}(\varphi v)dx = \int_\Omega G\varphi dx,\ \forall \varphi \in H^1$.

For the reader's convenience, we give a complete proof of this result in the Appendix I.

By using the embeddings $H^1 \subset L^4$ and $H^2 \subset L^\infty$, one verifies that $\|G\|_0$ is bounded by the right hand side of equation (2.7) below. Hence our solution λ of (2.3) verifies

(2.7) $\dfrac{mk}{\mu+\nu}\|\lambda\|_0 \leq c(\dfrac{m}{\mu+\nu}\|F\|_1 + \|g\|_2 + \|v\|_3\|\tau\|_2)$.

Let now $\theta \in H^2_0$ be the solution of the Dirichlet problem

(2.8) $\begin{cases} (\mu + \nu)\Delta\theta = k\lambda - \text{div } F, \text{ in } \Omega \\ \theta_{|\Gamma} = 0. \end{cases}$

By using (2.7), one has

(2.9) $(\mu + \nu)\|\theta\|_2 \leqslant c\frac{\mu+\nu}{m}(\|g\|_2 + \|v\|_3\|\tau\|_2) + c\|F\|_1$.

Now define

(2.10) $\theta_0(x) = \theta(x) - \bar{\theta}$.

Clearly, $\bar{\theta}_0 = 0$. Let (u,σ) be the unique solution in $H_0^3 \times \bar{H}^2$ of the following linear Stokes problem, in Ω:

$$(2.11)\quad \begin{cases} -\mu\Delta u + k\nabla\sigma = F + \nu\nabla\theta_0, \\ \operatorname{div} u = \theta_0, \\ u_{|\Gamma} = 0. \end{cases}$$

From the L^2 estimates for this problem one has

(2.12) $\mu\|u\|_3 + k\|\sigma\|_2 \leqslant c(\|F\|_1 + |\nu|\ \|\theta_0\|_2 + \mu\|\theta_0\|_2)$.

By taking in account that $\|\theta_0\|_2 \leqslant \|\theta\|_2$, one gets

(2.13) $\mu\|u\|_3 + k\|\sigma\|_2 \leqslant c(1 + \frac{\mu+|\nu|}{\mu+\nu})\|F\|_1 + c_1\frac{\mu+|\nu|}{m}(\|g\|_2 + \|v\|_3\|\tau\|_2)$.

Now let c_2 be a positive constant such that

$$|\operatorname{div} w|_\infty + 2|\nabla w|_\infty \leqslant c_2\|w\|_3,\ \forall\, w \in H_0^3.$$

In the remainder of this section we assume that the vector field v verifies the condition

(2.14) $\|v\|_3 \leqslant \gamma k$,

where, by definition,

$$(2.14')\quad \gamma \equiv \min\{\frac{\mu_0 m}{c_0(\mu+|\nu|)^2}, \frac{m}{2c_1(\mu+|\nu|)}, \frac{m}{c_2(\mu+\nu)}\}.$$

Assumption (2.14), implies, in particular, (2.1) and (2.2).

From (2.13) and (2.14), one gets

(2.15) $\mu\|u\|_3 + k\|\sigma\|_2 \leqslant \frac{1}{2}k\|\tau\|_2 + c(1 + \frac{\mu+|\nu|}{\mu+\nu})\|F\|_1 + c_1\frac{\mu+|\nu|}{m}\|g\|_2$.

At this point, we call attention to the sequence of *linear* maps, introduced above:

$$(F,g,\tau) \to (F,\lambda) \to (F,\theta) \to (F,\theta_0) \to (u,\sigma),$$

which were defined by equations (2.3) + (2.4), (2.8), (2.10), (2.11), respectively. The product map $(F,g,\tau) \to (u,\sigma)$ is linear and continuous, by (2.15). Hence, if (u_1,σ_1) is the solution corresponding to data (F,g,τ), it follows that $(u - u_1, \sigma - \sigma_1)$ is the solution corresponding to data $(0,0,\tau - \tau_1)$. Consequently, (2.15) yields, in particular,

$$\|\sigma - \sigma_1\|_2 \leq \frac{1}{2}\|\tau - \tau_1\|_2.$$

Hence, for fixed F and g, the map $\tau \to \sigma$ is a contraction in $\overline{H}^2$. Consequently, it has a (unique) fixed point $\sigma = \tau$.

In the sequel we prove that the pair (u,σ), corresponding to the fixed point $\sigma = \tau$, solves equation (1.9). Equations $(1.9)_1$ and $(1.9)_3$ follow from (2.11). In order to prove $(1.9)_2$, we start by substituting the expression of λ, obtained from equation $(2.8)_1$, in the first term on the left hand side of (2.3). This yields, since $\tau = \sigma$,

$$\text{(2.16)}\quad m\Delta\theta + v\cdot\nabla\lambda + 2\nabla v : \nabla^2\sigma + \Delta v\cdot\nabla\sigma + \Delta(\sigma \operatorname{div} v) = \Delta g.$$

On the other hand, by applying the divergence operator to both sides of equation $(2.11)_1$, and by utilizing $(2.11)_2$ one gets $-(\mu + \nu)\Delta\theta + k\Delta\sigma = \operatorname{div} F$, since $\Delta\theta_0 = \Delta\theta$. By comparison with $(2.8)_1$, one shows that $\lambda = \Delta\sigma$. By replacing λ by $\Delta\sigma$ in equation (2.16), it follows that

$$\text{(2.17)}\quad m\Delta \operatorname{div} u + v\cdot\nabla\Delta\sigma + 2\nabla v : \nabla^2\sigma + \Delta v\cdot\nabla\sigma + \Delta(\sigma \operatorname{div} v) - g = 0,$$

or equivalently,

$$\text{(2.18)}\quad \Delta[m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v - g] = 0, \text{ in } \Omega.$$

The function between square brackets (which belongs to H^1) is equal to the constant $-m\overline{\theta}$ on the boundary, by $(2.8)_2$, (2.10), $(2.11)_2$, and by the assumptions $v = 0$, $\operatorname{div} v = g = 0$ on Γ. Consequently,

$$m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v - g = -m\overline{\theta}, \text{ in } \Omega.$$

By integrating both sides of this equation in Ω, one shows that it must be $\overline{\theta} = 0$. Hence, equation $(1.9)_2$ is satisfied. Finally, the estimate (1.10) follows from (2.15).■

REMARK. One has to be careful on deducing (2.18) from (2.17), since both equations hold only in a weak sense. The point is to prove the identity

$$\text{(2.19)}\quad -\int_\Omega \nabla(v\cdot\nabla\sigma)\cdot\nabla\varphi dx = -\int_\Omega \Delta\sigma \operatorname{div}(\varphi v)dx + \int_\Omega [2\nabla v : \nabla^2\sigma + \Delta v\cdot\nabla\sigma]\varphi dx,$$

$$\forall\, \varphi \in C_0^\infty,$$

which is a weak formulation of

(2.20) $\Delta(v\cdot\nabla\sigma) = v\cdot\nabla\Delta\sigma + 2\nabla v\colon \nabla^2\sigma + \Delta v\cdot\nabla\sigma.$

For $\sigma \in H^3$, this last identity holds, and yields (2.19). If $\sigma \in H^2$, we approximate it (in the H^2 norm) by a sequence of functions $\sigma_n \in H^3$, and we pass to the limit in equation (2.19) (written with σ replaced by σ_n) as $n \to +\infty$.

3. Proof of Theorem A. For convenience, in this section we will not make explicit the dependence of the positive constants on the parameters. However, all the constants depend at most on Ω, μ, ν, k, m, ℓ and S.

Let c_3 be a constant such that $|\tau|_\infty \leq c_3\|\tau\|_2$, for every $\tau \in \overline{H}^2$. We will utilize here the condition

(3.1) $\|\tau\|_2 \leq \dfrac{\ell}{c_3}$,

which guarantees that $m + \sigma(x)$ belongs to the domain of p, for every $x \in \Omega$, since $-\ell \leq \tau(x) \leq \ell$.

Let $v \in H_0^3$ verify (2.14), and $\tau \in \overline{H}^2$ verify (3.1), define

(3.2) $F(v,\tau) = (m + \tau)[f - (v\cdot\nabla)v] + \omega(\tau)\nabla\tau,$

and consider the linearized system (1.9) with F(x) given by $F(v,\tau)$, i.e., the system

$$(3.3)\quad \begin{cases} -\mu\Delta u - \nu\nabla\ \mathrm{div}\ u + k\nabla\sigma = F(v,\tau),\ \text{in } \Omega, \\ m\ \mathrm{div}\ u + (v\cdot\nabla)\sigma + \sigma\ \mathrm{div}\ v = g,\ \text{in } \Omega, \\ u_{|\Gamma} = 0. \end{cases}$$

Since $H^1 \subset L^4$, $H^2 \subset L^\infty$, and $|\omega(\tau)|_\infty \leq S|\tau|_\infty \leq c_3S\|\tau\|_2$, one easily shows that

(3.4) $\|F(v,\tau)\|_1 \leq c(\frac{3}{2}m + \frac{\ell}{c_3})(\|f\|_1 + \|v\|_2^2) + cS\|\tau\|_2^2.$

This last estimate, together with (1.10), yields the following result:

THEOREM 3.1. *Let* $v \in H^3_{0,d}$, $\tau \in \overline{H}^2$ *and let* (2.1), (2.14) *and* (3.1) *be satisfied. Then, the unique solution* $(u,\sigma) \in H^3_{0,d} \times \overline{H}^2$ *of system* (3.3), *verifies the estimate*

(3.5) $\|u\|_3 + \|\sigma\|_2 \leq a(\|\tau\|_2 + \|v\|_2)^2 + b(\|f\|_1 + \|g\|_2),$

where the positive constants a *and* b *depend only on* Ω, μ, ν, k, m, ℓ *and* S.

The existence and uniqueness of the solution (u,σ) of system (3.3), enables us to define the corresponding map $(u,\sigma) = T(v,\tau)$. The fixed points of the map T are just the solutions of the non-linear system (1.6). In order to prove the existence of these fixed points we assume that

(3.6) $\|f\|_1 + \|g\|_2 \leq \frac{1}{2b}\min\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\}$,

and that

(3.7) $\|v\|_3 + \|\tau\|_2 \leq \min\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\}$.

The parameter $\delta \in]0,1]$, will be fixed later on. Consider the ball

$$B_\delta \equiv \{(v,\tau) \in H^3_{0,d} \times \overline{H}^2 \colon (3.7) \text{ holds}\}.$$

This is a compact set in $H^1_0 \times L^2$. Moreover, by using (3.5), one shows that $TB_\delta \subset B_\delta$, for every $\delta \leq 1$. We want to prove that, for a sufficiently small δ, depending only on Ω, μ, ν, k, m, ℓ and S, the map T is a contraction in B_δ. Hence, T has a (unique) fixed point in B_δ, and Theorem A is proved.

Let $(u,\sigma) = T(v,\tau)$, $(u_1,\sigma_1) = T(v_1,\tau_1)$, $F = F(v,\tau)$, $F_1 = F(v_1,\tau_1)$. One has, in Ω,

$$(3.8) \begin{cases} -\mu\Delta(u-u_1) - \nu\nabla\,\mathrm{div}(u-u_1) + k\nabla(\sigma-\sigma_1) = F - F_1, \\ m\,\mathrm{div}(u-u_1) + v\cdot\nabla(\sigma-\sigma_1) + (v-v_1)\cdot\nabla\sigma_1 \\ \qquad + \sigma_1\mathrm{div}(v-v_1) + (\sigma-\sigma_1)\mathrm{div}\,v = 0. \end{cases}$$

By multiplying both sides of equation $(3.8)_1$ by $m(u-u_1)$ and both sides of equation $(3.8)_2$ by $k(\sigma-\sigma_1)$, by integrating in Ω, and by adding side by side the two equations obtained in that way, one shows that

$$(3.9)\quad m(\mu-|\nu|)\|\nabla(u-u_1)\|_0^2 \leq \frac{1}{2}k|\mathrm{div}\,v|_\infty\|\sigma-\sigma_1\|_0^2 + ck\|\sigma_1\|_2\|v-v_1\|_1\|\sigma-\sigma_1\|_0 + m\|F-F_1\|_{-1}\|u-u_1\|_1.$$

In proving (3.9), we utilized the Sobolev's embedding theorems $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^4$, and also the inequality $\|\mathrm{div}(u-u_1)\|_0^2 \leq \|\nabla(u-u_1)\|_0^2$.

From (3.9) one has

$$(3.10)\quad \|u-u_1\|_1^2 \leq c'|\mathrm{div}\,v|_\infty\|\sigma-\sigma_1\|_0^2 + c'\|\sigma_1\|_2\|v-v_1\|_1\|\sigma-\sigma_1\|_0 + c'\|F-F_1\|_{-1}^2.$$

On the other hand $\|\sigma-\sigma_1\|_0 \leq c\|\nabla(\sigma-\sigma_1)\|_{-1}$, since $\sigma-\sigma_1$ has mean value zero. Hence, by using the expression of $\nabla(\sigma-\sigma_1)$ obtained from equation $(3.8)_1$, (or L^2 estimates for the linear Stokes problem) we show that

$$(3.11)\quad \|\sigma-\sigma_1\|_0^2 \leq c_2'\|u-u_1\|_1^2 + c'\|F-F_1\|_{-1}^2.$$

By multiplying both sides of equation (3.11) by $1/(2c_2')$, by adding (side by side) this equation to equation (3.10), and by using standard devices, we prove that

$$(3.12)\quad \frac{1}{2}\|u - u_1\|_1^2 + c_3'(1 - c_4'|\text{div } v|_\infty)\|\sigma - \sigma_1\|_0^2 \leq c'\|\sigma_1\|_2^2\|v - v_1\|_1^2 + c'\|F - F_1\|_{-1}^2,$$

for some suitable positive constants c_3', c_4' and c'.

On the other hand

$$(3.13)\quad \|F - F_1\|_{-1} \leq \|f\|_1\|\tau - \tau_1\|_0 + c(1 + \|\tau\|_2)(\|v\|_1 + \|v_1\|_1)\|v - v_1\|_1 + c[\|v_1\|_2^2 + S(\|\tau\|_2 + \|\tau_1\|_2)]\|\tau - \tau_1\|_0.$$

In fact, by using the embedding $H^1 \hookrightarrow L^4$, one easily shows that

$$\|(\tau - \tau_1)f\|_{-1} \leq \|f\|_1\|\tau - \tau_1\|_0.$$

Similarly,

$$\|(v\cdot\nabla)v - (v_1\cdot\nabla)v_1\|_{-1} \leq c(\|v\|_1 + \|v_1\|_1)\|v - v_1\|_1,$$

and

$$\|\tau(v\cdot\nabla)v - \tau_1(v_1\cdot\nabla)v_1\|_{-1} \leq c\|\tau\|_2\|(v\cdot\nabla)v - (v_1\cdot\nabla)v_1\|_{-1} + c\|v_1\|_2^2\|\tau - \tau_1\|_0.$$

Furthermore,

$$\|\omega(\tau)\nabla\tau - \omega(\tau_1)\nabla\tau_1\|_{-1} = \|\nabla_x \int_{\tau_1(x)}^{\tau(x)} \omega(\xi)d\xi\|_{-1} \leq \|\int_{\tau_1(x)}^{\tau(x)} \omega(\xi)d\xi\|_0 \leq S(|\tau|_\infty + |\tau_1|_\infty)\|\tau - \tau_1\|_0.$$

The above inequalities yield (3.13).

For $\delta \leq a/c_1c_4'$, one has $c_4'|\text{div } v|_\infty \leq c_4'c_1\|v\|_3 \leq \frac{1}{2}$, by (3.7). Hence, from (3.12), (3.13) one gets

$$(3.14)\quad \|u - u_1\|_1^2 + \|\sigma - \sigma_1\|_0^2 \leq c'\|f\|_1^2\|\tau - \tau_1\|_0^2 + c'[(1 + \|\tau\|_2)^2(\|v\|_1 + \|v_1\|_1)^2 + \|\sigma_1\|_2^2]\|v - v_1\|_1^2 + c[\|v_1\|_2^2 + S(\|\tau\|_2 + \|\tau_1\|_2)]^2\|\tau - \tau_1\|_0^2.$$

By choosing δ sufficiently small, depending only on Ω, μ, ν, k, m, ℓ and S, one has

$$\|u - u_1\|_1^2 + \|\sigma - \sigma_1\|_0^2 \leq \frac{1}{2}(\|v - v_1\|_1^2 + \|\tau - \tau_1\|_0^2).$$

Hence T is a contraction in B_δ, which proves Theorem A.■

REMARK. B_1 is a compact and convex subset of $H_0^1 \times L^2$, $T: B_1 \to B_1$ is continuous with respect to that topology, and $TB_1 \subset B_1$. Hence, we can prove the existence of (at least) a fixed point in B_1 by using Schauder's theorem. The uniqueness follows by using (3.14) (actually, it is quite trivial to obtain more stringent uniqueness results).

4. Proof of Theorem C. During the proof of part (i) of Theorem C, μ_λ, σ_λ, k_λ, ω_λ, will be denoted by u, σ, k, ω respectively. Theorem B states that if $F \in H^1$, $g = 0$, and if $v \in H^3_{0,d}$ verifies the condition

(4.1) $\|v\|_3 \leqslant \gamma k$,

then there exists a unique solution $(u,\sigma) \in H^3_{0,d} \times \overline{H}^2$ of the linear system (1.9). Moreover,

(4.2) $\mu\|u\|_3 + k\|\sigma\|_2 \leqslant c(1 + \frac{\mu+|\nu|}{\mu+\nu})\|F\|_1$.

Let us now fix $\tau \in \overline{H}^2$ in the ball

(4.3) $\|\tau\|_2 \leqslant \frac{\ell_\lambda}{c_3}$, or equivalently, $\|k\tau\|_2 \leqslant \frac{\ell}{c_3}$,

where c_3 was defined in Section 3, and ℓ is the positive constant defined in (1.12). Condition (4.3) guarantees that $|\tau(x)| \leqslant \ell_\lambda$, $\forall\, x \in \Omega$. In particular $m/2 \leqslant m + \tau(x) \leqslant (3m)/2$.

By defining $F(v,\tau)$ as in (3.2) (recall that, now, $\omega = \omega_\lambda$) one has, as in Section 3,

$$\|F(v,\tau)\|_1 \leqslant c(\frac{3}{2}m + \frac{\ell_\lambda}{c_3})(\|f\|_1 + \|v\|_2^2) + cS_\lambda\|\tau\|_2^2.$$

Hence,

(4.4) $\|F(v,\tau)\|_1 \leqslant c(\frac{3}{2}m + \frac{\ell}{c_3 k_0})(\|f\|_1 + \|v\|_2^2) + c\phi k^2\|\tau\|_2^2$,

(recall $k = k_\lambda$). If v and τ verify assumptions (4.1) and (4.3), it follows from (4.2) and (4.4) that the unique solution (u,σ) of system (3.3) verifies the estimate

(4.5) $\|u\|_3 + \|k\sigma\|_2 \leqslant a(\|k\tau\|_2 + \|v\|_2)^2 + b\|f\|_1$,

where now a and b are constants of type $\tilde{c}$. The above result corresponds to Theorem 3.1 in Section 3.

The proof goes on as in Section 3, by now utilizing $k\sigma$ and $k\tau$ instead of σ and τ, respectively. In this way, inequalities (3.5) and (3.1) become (4.5) and $(4.3)_2$, respectively; condition (4.1) remains unchanged.

Following Section 3, we denote by T the map $(u,\sigma) = T(v,\tau)$, where the data $(v,\tau) \in H^3_{0,d} \times \overline{H}^2$ verify (4.1), (4.3) and (u,σ) is the (corresponding) solution of system (3.3).

We fix $f \in H^1$ verifying (3.6) (here, $g = 0$), and we consider the restriction of T to the ball B_δ, $0 < \delta \leqslant 1$, defined by the condition

$$(4.6)\quad \|v\|_3 + \|k\tau\|_2 \leqslant \min\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\}.$$

The substitution of τ by $k\tau$ transforms (3.7) on (4.6). Arguing as in Section 3, and recalling that $k \geqslant k_0$, we prove inequalities (3.10), (3.11) and (3.12), provided that in these inequalities we replace σ, σ_1, τ, τ_1 by $k\sigma$, $k\sigma_1$, $k\tau$, $k\tau_1$ respectively. The constants c', c_2', c_3', c_4' are now of type $\widetilde{c}$, hence independent of k.

Inequality (3.13) holds, as written in Section 3. Recalling that $S_\lambda \leqslant \phi k^2$, and that $k \geqslant k_0$, we show that (3.13) holds again, if τ, τ_1, and S are replaced by $k\tau$, $k\tau_1$, and ϕ, respectively, and if the right hand side of the inequality is multiplied by $1 + (1/k_0)$.

By choosing δ as in Section 3, i.e., $\delta \leqslant a/(c_1 c_4')$, we get an inequality similar to (3.14), where now τ, σ, τ_1 and σ_1 are multiplied by k, and the constants are of type $\widetilde{c}$. By choosing δ sufficiently small (depending only on Ω, μ, ν, m, ℓ, ϕ, k_0) one gets

$$(4.7)\quad \|u - u_1\|_1^2 + \|k\sigma - k\sigma_1\|_0^2 \leqslant \frac{1}{2}(\|v - v_1\|_1^2 + \|k\tau - k\tau_1\|_0^2).$$

Hence T is a contraction in B_δ, which proves the first part of Theorem C.

We now prove part (ii) of that theorem. Condition (1.15) guarantees the uniqueness of the solution of problem (1.17), for a sufficiently small $\widetilde{c}_0$.

Let us write system (1.13) in the form (1.6), i.e.,

$$(4.8)\quad \begin{cases} -\mu\Delta u_\lambda - \nu\nabla \operatorname{div} u_\lambda + k_\lambda \nabla\sigma_\lambda = \omega_\lambda(\sigma_\lambda)\nabla\sigma_\lambda + (m + \sigma_\lambda)[f - (u_\lambda \cdot \nabla)u_\lambda], \\ m \operatorname{div} u_\lambda + u_\lambda \cdot \nabla\sigma_\lambda + \sigma_\lambda \operatorname{div} u_\lambda = 0, \text{ in } \Omega, \\ (u_\lambda)_{|\Gamma} = 0. \end{cases}$$

From (1.16), it follows that there exists $u_\infty \in H^3_0$, such that $(1.17)_1$ holds. Here, we consider subsequences of u_λ; the convergence of all the u_λ to u_∞, as $\lambda \to +\infty$, will follow from the uniqueness of the limit u_∞, since we will show that u_∞ is the solution of (1.14).

The bound (1.16), and the hypothesis $k_\lambda \to +\infty$ as $\lambda \to +\infty$, imply $(1.17)_3$. Furthermore, equation $(4.8)_2$, together with (1.16) and $(1.17)_3$, shows that div $u_\lambda \to 0$, strongly in H_0^1, as $\lambda \to +\infty$. Since $\|\text{div } u_\lambda\|_2$ is bounded, $(1.17)_2$ follows. In particular, div $u_\infty = 0$.

Now, we pass to the limit in equation $(1.13)_1$, as $\lambda \to +\infty$. One has $\mu\Delta u_\lambda \to \mu\Delta u_\infty$ and $-\nu\nabla$ div $u_\lambda \to 0$, weakly in H^1 and strongly in H^s, $0 \leq s < 1$; and $\rho_\lambda \to m$, strongly in H^2. Moreover, $\rho_\lambda(u_\lambda \cdot \nabla)u_\lambda \to m(u_\infty \cdot \nabla)u_\infty$, weakly in H^2 and strongly in H^s, $0 \leq s < 2$. By using equation $(1.13)_1$, it follows that $\nabla p(\rho_\lambda,\lambda) \to \mu\Delta u_\infty + m[f - (u_\infty \cdot \nabla)u_\infty]$, weakly in H^1, strongly in H^s, $0 \leq s < 1$. Obviously, the limit function must be of the form $\nabla\pi(x)$. Theorem C is completely proved.■

Appendix I. For the readers convenience we prove here the result stated at the beginning of Section 2, concerning equation (2.3). We assume that the function $v \in H^3$, verifies $v \cdot n = 0$ on Γ, and assumption (2.2). This last condition could be replaced by $1/2|\text{div } v|_\infty + |\nabla v|_\infty < M$. Let $G \in H_0^1$, and λ_ϵ be the solution of

$$(5.1) \quad \begin{cases} -\epsilon\,\Delta\lambda_\epsilon + M\lambda_\epsilon + v\cdot\nabla\lambda_\epsilon = G, \text{ in } \Omega, \\ \lambda_{\epsilon|\Gamma} = 0, \end{cases}$$

where ϵ is a positive constant. By multiplying both sides of (5.1) by $\Delta\lambda_\epsilon$, and by integrating over Ω, one easily shows that

$$\epsilon\|\Delta\lambda_\epsilon\|_0^2 + [M - (\tfrac{1}{2}|\text{div } v|_\infty + |\nabla v|_\infty)]\,\|\nabla\lambda_\epsilon\|_0^2 \leq \|\nabla G\|_0\|\nabla\lambda_\epsilon\|_0.$$

This estimate, together with (2.2), gives

$$(5.2) \quad \|\nabla\lambda_\epsilon\|_0 \leq (2/M)\|\nabla G\|_0,$$

and also

$$(5.3) \quad \epsilon\|\Delta\lambda_\epsilon\|_0 \leq \sqrt{2\epsilon/M}\,\|\nabla G\|_0.$$

Hence, there exists a subsequence λ_ϵ such that $\lambda_\epsilon \to \lambda \in H_0^1$, weakly in H_0^1 and strongly in L^2. Moreover, $\epsilon\Delta\lambda_\epsilon \to 0$ in L^2. By passing to the limit in (5.1), as $\epsilon \to 0$, one proves that λ is a strong solution of (2.3). In particular, λ verifies (2.6). By multiplying both sides of (2.3) by λ, and by integrating over Ω, one shows that

$$(5.4) \quad (M - \tfrac{1}{2}|\text{div } v|_\infty)\|\lambda\|_0 \leq \|G\|_0.$$

This gives, in particular, the uniqueness of the solution λ, in H^1.

Since the linear map $T: H_0^1 \to H_0^1$, defined by $TG = \lambda$, is continuous with respect to the L^2 norm, there exists a unique linear continuous map $\widetilde{T}$ extending T to all of L^2. Clearly, (5.4) holds again. Furthermore, $\lambda = \widetilde{T}G$ is a solution of (2.6).

REMARK. The rèsult holds again without assuming that $v \cdot n = 0$ on Γ, and with condition (2.2) replaced by the weaker condition $|\text{div}\, v|_\infty \leqslant M$ (or, more generally, by $|\text{div}\, v|_\infty < 2M$). In that case, equation (2.6) holds for every test function $\varphi \in H_0^1$. The proof of this case starts by proving the existence of a solution λ for data belonging to the linear space H generated by an arbitrarily fixed basis $\{G^\ell\}$, $\ell = 1,2,\ldots$, on L^2. Then we extend the map $G \to \lambda$ to all of L^2, by continuity.

Appendix II.

6. **The heat-dependent case.** In the heat-dependent case the equations are

$$(6.1)\quad \begin{cases} -\mu\Delta u - \nu\nabla\, \text{div}\, u + \nabla p(\rho,\theta) = \rho[f - (u\cdot\nabla)u], \\ \text{div}(\rho u) = g, \\ -\chi\Delta\theta + c_v\rho u\cdot\nabla\theta + \theta p_\theta(\rho,\theta)\text{div}\, u = \rho h + \psi(u,u), \text{ in } \Omega, \\ u_{|\Gamma} = 0,\ \theta_{|\Gamma} = n, \end{cases}$$

where

$$(6.2)\quad \psi(u,u) = \lambda\, \Sigma_{i,j}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big)^2 + \lambda'(\text{div}\, u)^2.$$

For convenience, we will assume that $\mu > 0$, $\mu + \nu > 0$, $\chi > 0$, c_v, λ, λ' and $n > 0$, are constants. We also impose here, as for system (1.1), the additional condition (1.2).

The function $p(\rho,\theta)$ is defined, and has Lipschitz continuous first derivatives, in an ℓ-neighborhood $[m - \ell, m + \ell] \times [n - \ell, n + \ell]$ of (m,n). By setting $k = p_\rho(m,n)$, $\gamma = p_\theta(m,n)$, one has $p_\rho(m + \sigma, n + \alpha) = k - \omega_1(\sigma,\alpha)$, $p_\theta(m + \sigma, n + \alpha) = \gamma - \omega_2(\sigma,\alpha)$, where ω_i, $i = 1,2$, are Lipschitz continuous (with norms $\leqslant S$) in the ℓ-neighborhood of $(0,0)$. Moreover $\omega_i(0,0) = 0$. We assume the physical condition $k > 0$.

By setting

$$\rho = m + \sigma,\ \theta = n + \alpha,$$

the system (6.1) becomes

$$
(6.3)\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma + \gamma\nabla\alpha = \rho[f - (u\cdot\nabla)u] + \omega_1(\sigma,\alpha)\nabla\sigma + \omega_2(\sigma,\alpha)\nabla\alpha, \\ m \operatorname{div} u + u\cdot\nabla\sigma + \sigma \operatorname{div} u = g, \\ -\chi\Delta\alpha + \gamma n \operatorname{div} u = \rho h - c_v(m+\alpha)u\cdot\nabla\alpha + \psi(u,u) - \gamma\alpha \operatorname{div} u \\ \qquad + \omega_2(\sigma,\alpha)(\alpha+n)\operatorname{div} u, \text{ in } \Omega, \\ u_{|\Gamma} = 0,\ \alpha_{|\Gamma} = 0. \end{cases}
$$

The additional constraint is given by (1.13).

The linearized system is now

$$
(6.4)\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma + \gamma\nabla\alpha = F, \\ m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v = g, \\ -\chi\Delta\alpha + \gamma n \operatorname{div} u = H, \text{ in } \Omega, \\ u_{|\Gamma} = 0,\ \alpha_{|\Gamma} = 0. \end{cases}
$$

In the sequel, c', c_0', c_1', β, denote positive constants, depending at most on Ω, on μ, ν, γ, m, χ, n, λ, λ', ℓ and S. The dependence of the constants c' on the above parameters can be easily checked.

One has the following result:

THEOREM A′. *There exist positive constants* c_0' *and* c_1' *such that if* $f \in H^1$, $g \in \overline{H}_0^2$, $h \in L^2$, *and*

(6.5) $\|f\|_1 + \|g\|_2 + \|h\|_0 \leq c_0'$,

then there exists a unique solution $(u,\sigma,\alpha) \in H_0^3 \times \overline{H}^2 \times H_0^2$ *of problem* (6.1), *in the ball*

(6.6) $\|u\|_3 + \|\sigma\|_2 + \|\alpha\|_2 \leq c_1'$.

The proof relies on the following result, for the linearized system (6.4):

THEOREM B′. *Let* $F \in H^1$, $g \in \overline{H}_0^2$, $H \in L^2$, *and* $v \in H_{0,d}^3$. *There exists a positive constant* β, *such that if*

(6.7) $\|v\|_3 \leq \beta$,

then there exists a unique solution $(u,\sigma,\alpha) \in H_0^3 \times \overline{H}^2 \times H_0^2$ *of the linear system* (6.4). *Moreover,*

(6.8) $\|u\|_3 + \|\sigma\|_2 + \|\alpha\|_2 \leq c'(\|F\|_1 + \|g\|_2 + \|H\|_0)$.

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