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ON THE SUITABLE WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACE

By H. BEIRÃO DA VEIGA

Introduction

In the absence of an uniqueness theorem for the Navier-Stokes equations, the properties proved for solutions obtained by different methods could not coincide. Hence, it is interesting to construct solutions with useful "additional" properties, solutions which are (very likely) those having a physical meaning.

A classical property proved for the solutions of Navier-Stokes equations is the energy inequality (0.4). Recently Scheffer ([7] to [10]) and Caffarelli, Kohn and Nirenberg [3] proved very interesting results on the Hausdorff dimension of the singular set for *suitable* weak solutions of the Navier-Stokes equations. In [3] the authors proved that the singular set of a suitable weak solution has Hausdorff 1-dimensional measure (in space-time) zero. After this result it becomes natural to require this additional property for the weak solutions. It is not clear if "suitability" holds for solutions constructed by Faedo-Galerkin's method. In fact, suitable weak solutions must verify the local energy inequality (0.5), which is difficult to be proved if a well defined pressure doesn't exist for the approximating problem.

Scheffer [8] constructed suitable weak solutions in the case $\Omega=\mathbb{R}^3$, $f=0$, $u_0\in H$. Caffarelli, Kohn and Nirenberg [3] constructed suitable weak solutions also when $f\in L^2(0, T; \mathbb{H}^{-1}(\mathbb{R}^3))$. These last authors also considered the case of a bounded domain Ω.

We will present a quite natural and easy construction of suitable weak solution in the case:

$$\Omega=\mathbb{R}^3, \qquad u_0\in H, \qquad f\in L^2(0, T; \mathbb{H}^{-1}(\mathbb{R}^3)).$$

The method utilize only L^2 theory for the space variables. (*see* also [5], Chap. I, Remarque 6.11, and [1]). The corresponding problem for a bounded open set $\Omega\subset\mathbb{R}^3$ is studied in [2], by introducing a new approximation theorem for non-linear problems.

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Let now $u_0 \in H$, $f \in L^2(0, T; \mathbb{H}^{-1})$, and consider sequences $u_{0,\varepsilon} \in W$, $f_\varepsilon \in L^2(0, T; \mathbb{L}^2)$ such that:

$$(0.9) \qquad \begin{cases} \lim_{\varepsilon \to 0} u_{0,\varepsilon} = u_0, & \text{in } H, \\ \lim_{\varepsilon \to 0} f_\varepsilon = f, & \text{in } L^2(0, T; \mathbb{H}^{-1}). \end{cases}$$

One has the following result:

THEOREM *B*. — *Let* u_0, f, $u_{0,\varepsilon}$ *and* f_ε *be as in* (0.2), (0.9) *and consider the solution* u_ε, ∇p_ε *of problem* $(0.6)_\varepsilon$. *When* $\varepsilon \to 0$ *there exists a subsequence, still denoted by* u_ε, ∇p_ε, *converging* (see *below*) *to a solution* u, ∇p *of the Navier-Stokes equations* (0.1), *verifying* (0.3) *and* (0.4). *Moreover, the local energy inequality* (0.5) *holds.*

Convergence, and regularity of u, p are as follows:

(i) u_ε converges to u in $L^2(0, T; V)$ weak; in $L^\infty(0, T; H)$ weak-*; almost everywhere in Q_T; and in $L^2(]0, T[\times B_R)$, $\forall R > 0$.

(ii) u'_ε converges weakly to u' in $L^2(0, T; W')$ and in $L^{4/3}(0, T; \mathbb{H}^{-1}) + L^2(0, T; \mathbb{H}^{-2})$. Moreover $u' \in L^{4/3}(0, T; \mathbb{H}^{-1})$.

(iii) $u^i_\varepsilon u^j_\varepsilon$ converges weakly to $u^i u^j$ in $L^{4/3}(0, T; L^2)$, and $(u_\varepsilon . \nabla) u_\varepsilon$ converges weakly to $(u . \nabla) u$ in $L^{4/3}(0, T; \mathbb{H}^{-1})$.

(iv) p_ε converges weakly to p in $L^{4/3}(0, T; L^2 + L^6)$ and ∇p_ε converges weakly to ∇p in $L^{4/3}(0, T; \mathbb{H}^{-1})$. If $\nabla . f = 0$ then p_ε converges to p in $L^{4/3}(0, T; L^2)$.

Note that by using equation (2.9) other regularity results on p and ∇p follow.

1. *Proof of Theorem A*. — Let $f \in H$, $\operatorname{Re} \lambda > 0$(1) and consider the problem:

$$(1.1) \qquad \lambda u + \varepsilon \Delta^2 u - \Delta u = f.$$

By Fourier transform the unique solution $u \in H$ of (1.1) is given by:

$$(1.2) \qquad \hat{u}(\xi) = (\lambda + \varepsilon |\xi|^4 + |\xi|^2)^{-1} \hat{f}(\xi).$$

It follows that $(\operatorname{Re}\lambda + \varepsilon |\xi|^4 + |\xi|^2) |\hat{u}(\xi)| \leq |\hat{f}(\xi)|$, hence $u \in \mathbb{H}^4$. On the other hand:

$$|\lambda| \, |\hat{u}(\xi)| \leq |\lambda + \varepsilon |\xi|^4 + |\xi|^2| . |\hat{u}(\xi)| \leq |\hat{f}(\xi)|,$$

consequently:

$$(1.3) \qquad |u| \leq |\lambda|^{-1} |f|, \qquad \forall f \in H.$$

Define $D(A) = \mathbb{H}^4 \cap W$ and $A u = \varepsilon \Delta^2 u - \Delta u$. It easily follows that A is a closed operator, $D(A)$ is dense in H, the resolvent set $\rho(A)$ contains $\{\lambda : \operatorname{Re}\lambda > 0\}$ and:

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(H;\, H)} \leq |\lambda|^{-1},$$

(1) Here we consider the complex valued case.

for every λ such that $\mathrm{Re}\,\lambda > 0$. By using well known results on linear evolution equations (*see* [6], Vol. II, Chap. 4, Thm. 3.2) one gets the following statement:

THEOREM 1.1. — *Let* $u_0 \in W$ ([2]) *and* $g \in L^2(0, T; H)$. *There exists a unique solution* u *of problem:*

$$(1.4)_\varepsilon \qquad \begin{cases} u' + \varepsilon \Delta^2 u - \Delta u = g, \\ \nabla . u = 0, \\ u_{/t=0} = u_0. \end{cases}$$

Moreover:

$$(1.5) \qquad \begin{cases} u \in L^2(0, T; \mathbb{H}^4) \cap C(0, T; W), \\ u' \in L^2(0, T; H). \end{cases}$$

Let P be the ortogonal projection of $\mathbb{L}^2$ onto the closed subspace H. It is well known that $H^\perp = \{p \in L^2_{\mathrm{loc}} : \nabla p \in \mathbb{L}^2\}$. In terms of Fourier transform, for every $u \in \mathbb{L}^2$ one has $u = v + \nabla q$, where $v \in H$ and $\nabla q \in H^\perp$ are given by:

$$(1.6) \qquad \begin{cases} \hat{v} = \hat{u} - |\xi|^{-2} (\xi . \hat{u})\, \xi, \\ \hat{q} = |\xi|^{-2} \xi . \hat{u}. \end{cases}$$

Consider now the non linear problem:

$$(1.7)_\varepsilon \qquad \begin{cases} u' + \varepsilon \Delta^2 u + P[(u . \nabla) u] - \Delta u = g, \\ \nabla . u = 0, \\ u_{|t=0} = u_0. \end{cases}$$

The following result holds:

THEOREM 1.2. — *Let* $g \in L^2(0, T; H)$, $u_0 \in W$. *There exists a unique solution of problem* $(1.7)_\varepsilon$, *which verifies* (1.5).

Proof. — By using Hölder's inequality and Sobolev embedding theorem one easily obtains:

$$(1.8) \qquad |P(v . \nabla) v|^2 \leqq |(v . \nabla) v|^2 \leqq |v|_6^2 \, |\nabla v|_2 \, |\nabla v|_6 \leqq c \|v\|^3 \|v\|_2, \qquad \forall v \in \mathbb{H}^2.$$

Consider now the problem $(1.7)_\varepsilon$ with $P(v . \nabla) v$ in the place of $P(u . \nabla) u$. By Theorem 1.1 there exists a unique solution u, verifying (1.5). Define a map S by $Sv = u$, and let $\mathbb{K}$ be the set:

$$\mathbb{K} = \{ v \in C(0, \tau; V) \cap L^2(0, \tau; W) : v_{|t=0} = u_0, \|v\|_{C(0, \tau; V)} \leqq B, \|v\|_{L^2(0, \tau; W)} \leqq B \},$$

([2]) $W = [D(A), H]_{1/2}$, in the notation of [6].

where B and τ will be fixed later. $\mathbb{K}$ is a closed subset of $C(0, \tau; V) \cap L^2(0, \tau; W)$. From (1.8) it follows that:

$$(1.9) \qquad \| P(v.\nabla)v \|^2_{L^2(0,T;\,H)} \leqq c\,B^4\,\tau^{1/2}, \qquad \forall v \in \mathbb{K}.$$

We want to show that S is a contraction in $\mathbb{K}$, if τ is sufficiently small. By taking the scalar product in $\mathbb{L}^2$ of equation $(1.7)_\varepsilon$ [with $P(v.\nabla)v$ in the place of $P(u.\nabla)u$] with u and Δu one easily obtains:

$$(1.10) \qquad \| u \|^2_{C(0,\tau;\,V)} + \| u \|^2_{L^2(0,\tau;\,W)} \leqq c_0^2 \{ \| u_0 \|^2_V + \| g \|^2_{L^2(0,\tau;\,H)} + B^4\,\tau^{1/2} \},$$

where c_0 is a numerical constant ([3]). By fixing $B = 2c_0 \| u_0 \|_V$, and by choosing τ sufficiently small, one shows that $S(\mathbb{K}) \subset \mathbb{K}$. Let $u = S\,v$, $\bar{u} = S\,\bar{v}$, $w = u - \bar{u}$. Clearly:

$$(1.11) \qquad \begin{aligned} & w' + \varepsilon\,\Delta^2 w - \Delta w = P[(\bar{v}.\nabla)\bar{v} - (v.\nabla)v], \\ & w_{|t=0} = 0. \end{aligned}$$

On the other hand, as in (1.8),

$$|(v.\nabla)v - (\bar{v}.\nabla)\bar{v}|^2 \leqq c\,\| v \|.\| v \|_2 \| v - \bar{v} \|^2 + c\,\| \bar{v} \|^2 \| v - \bar{v} \|.\| v - \bar{v} \|_2.$$

Hence:

$$(1.12) \quad \| P[(v.\nabla)v - (\bar{v}.\nabla)\bar{v}] \|^2_{L^2(0,\tau;\,H)} \leqq c\,\tau^{1/2}\,B^2 \{ \| v - \bar{v} \|^2_{C(0,\tau;\,V)} + \| v - \bar{v} \|^2_{L^2(0,T;\,W)} \}.$$

From (1.11) and (1.12) it follows that S is a contraction on $\mathbb{K}$ if τ is sufficiently small. Hence there exists a unique local solution of $(1.7)_\varepsilon$. Finally, by taking the scalar product of $(1.7)_\varepsilon$ with u and $\Delta^2 u$ it follows a global *a priori* bound for $\| u(t) \|^2_2$. □

We now prove Theorem A. Let u_ε be the solution of $(1.7)_\varepsilon$ with $g = P\,f_\varepsilon$. Define $w_\varepsilon = u'_\varepsilon + \varepsilon\,\Delta^2 u_\varepsilon - \Delta u_\varepsilon + (u_\varepsilon.\nabla)u_\varepsilon - f_\varepsilon$. Clearly $w_\varepsilon \in L^2(0, T; \mathbb{L}^2)$ and $P\,w_\varepsilon = 0$. Hence there exists a unique ∇p_ε such that $w_\varepsilon = -\nabla p_\varepsilon$, and $(0.6)_\varepsilon$, (0.7) hold. Estimate (0.8) is obvious.

Finally, let $\varphi(t, x)$ be a non-negative C^∞ function with compact support in the space variables. By taking the scalar product of $(0.6)_\varepsilon$ with $\varphi\,u_\varepsilon$ and by integrating on Q_t one obtains:

$$(1.15) \quad \int_\Omega |u_\varepsilon(t)|^2 \varphi + 2 \iint_{Q_t} |\nabla u_\varepsilon|^2 \varphi + 2\varepsilon \iint_{Q_t} |\Delta u_\varepsilon|^2 \varphi$$
$$\leqq \int_\Omega |u_{0,\varepsilon}|^2 \varphi + \iint_{Q_t} |u_\varepsilon|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right)$$
$$+ \iint_{Q_t} (|u_\varepsilon|^2 + 2p_\varepsilon)\,u_\varepsilon.\nabla\varphi + 2 \iint_{Q_t} f_\varepsilon.u_\varepsilon\,\varphi$$
$$+ 4\varepsilon \| \nabla\varphi \|_{L^\infty(Q_t)} \| \nabla u_\varepsilon \|_{L^2(Q_t)} \| \Delta u_\varepsilon \|_{L^2(Q_t)}$$
$$+ 2\varepsilon \| \Delta\varphi \|_{L^\infty(Q_t)} \| u_\varepsilon \|_{L^2(Q_t)} \| \Delta u_\varepsilon \|_{L^2(Q_t)}.$$

([3]) Assume $\tau < 1$, in order to get $\| v \|_{L^2(0,\tau;\,H)} \leqq \| v \|_{C(0,\tau;\,H)}$.

2. *Proof of Theorem B.* — Let u_0, $u_{0,\varepsilon}$, f, f_ε be as in Theorem B and let u_ε, ∇p_ε be the solution of $(0.6)_\varepsilon$. From (0.8) it follows that:

$$\lim_{\varepsilon\to 0} \varepsilon \| \Delta u_\varepsilon \|_{L^2(0,T;\,\mathbb{L}^2)} = 0, \tag{2.1}$$

hence $\varepsilon \Delta^2 u_\varepsilon \to 0$ in $L^2(0, T; \mathbb{H}^{-2})$. On the other hand it is not difficult to verify that (since $\mathbb{H}^2 \subsetneq \mathbb{L}^\infty$):

$$\| P(u_\varepsilon . \nabla) u_\varepsilon \|_{L^2(0,T;\,W')} \leqq c \| u_\varepsilon \|_{C(0,T;\,H)} \| u_\varepsilon \|_{L^2(0,T;\,V)},$$

and by utilizing (0.8) one gets:

$$\| P(u_\varepsilon . \nabla) u_\varepsilon \|_{L^2(0,T;\,W')} \leqq c_T (| u_{0,\varepsilon} |^2 + \| f_\varepsilon \|^2_{L^2(0,T;\,\mathbb{H}^{-1})}). \tag{2.2}$$

Consequently:

$$\| u'_\varepsilon \|_{L^2(0,T;\,W')} \leqq c_T (1+\sqrt{\varepsilon}) (| u_{0,\varepsilon} | + \| f_\varepsilon \|_{L^2(0,T;\,\mathbb{H}^{-1})}) + c_T (| u_{0,\varepsilon} |^2 + \| f_\varepsilon \|^2_{L^2(0,T;\,\mathbb{H}^{-1})}). \tag{2.3}$$

For the sake of convenience we denote by "constant", quantities that remain bounded when ε goes to zero, as for instance the right hand side of (2.3).

Using notations of [6], one has $\mathbb{L}^2 = [\mathbb{H}^1, \mathbb{H}^{-2}]_{1/3}$ (*see* [6], Vol. I, Chap. I, Thm 12.3) and similarly:

$$H = [V, W']_{1/3}.$$

Since u_ε is bounded in $L^2(0, T; V)$ and in $H^1(0, T; W')$ it follows that:

$$\| u_\varepsilon \|_{H^{1/3}(0,T;\,H)} \leqq \text{Const.}\ (^4), \tag{2.4}$$

by the theorem of intermediate derivatives ([6], Vol. I, Chap. I, Thms 4.1 and 2.3). Let $B_R = \{ x \in \mathbb{R}^3 : |x| < R \}$. The sequence u_ε is bounded in $L^2(0, T; \mathbb{H}^1(B_R)) \cap H^{1/3}(0, T; \mathbb{L}^2(B_R))$, and by a compactness argument there exists a subsequence converging in $L^2(0, T; \mathbb{L}^2(B_R))$, *see* [5], Chap. I, Thm 5.2. At this stage of the proof, all the statements in (i) and the first statement in (ii) follow easily. On the other hand, $L^2(0, T; \mathbb{L}^6) \cap L^\infty(0, T; \mathbb{L}^2) \subsetneq L^{8/3}(0, T; \mathbb{L}^4)$; consequently:

$$\| u_\varepsilon \|_{L^{8/3}(0,T;\,\mathbb{L}^4)} \leqq \text{Const.}, \tag{2.5}$$

and a well known argument (use Lemma 1.3 in [5] or argue as in p. 76) leads to the first statement in (iii). The second statement follows from the first one, since $\sum_i u^i (D_i u^j) = \sum_i D_i (u^i u^j)$.

Now we prove statement (iv). From $(0.6)_\varepsilon$ it follows that:

$$-\Delta p_\varepsilon = \nabla . [(u_\varepsilon . \nabla) u_\varepsilon] - \nabla . f_\varepsilon. \tag{2.6}$$

(^{4}) $u \in H^s(0, T; X)$, means that u and the fractional derivative $d^s u/dt^s$, $s \in \mathbb{R}^+$, belong to $L^2(0, T; X)$. *See* [6], for details.

Since $f_\varepsilon \in \mathbb{H}^{-1}$, there exist (non unique) $f_{\varepsilon, j} \in \mathbb{L}^2$, $j=0, 1, 2, 3$, such that:

$$f_\varepsilon = f_{\varepsilon, 0} + \sum_{j=1}^{3} \frac{\partial f_{\varepsilon, j}}{\partial x_j}. \tag{2.7}$$

The norms of $f_{\varepsilon, j}$, $j=0, 1, 2, 3$, in the space $L^2(0, T; \mathbb{L}^2)$ remain bounded when ε tends to zero, since the same holds for the $L^2(0, T; \mathbb{H}^{-1})$ norm of f_ε. From (2.6) and (2.7) it follows that:

$$p_\varepsilon = p_\varepsilon^{(1)} + p_\varepsilon^{(2)} + p_\varepsilon^{(3)}, \tag{2.8}$$

where:

$$\begin{cases} -\Delta p_\varepsilon^{(1)} = \displaystyle\sum_{i, j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} (u_\varepsilon^i u_\varepsilon^j), \\ -\Delta p_\varepsilon^{(2)} = - \displaystyle\sum_{i, j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} f_{\varepsilon, j}^i, \\ -\Delta p_\varepsilon^{(3)} = - \displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial x_j} f_{\varepsilon, 0}^j. \end{cases} \tag{2.9}$$

By Fourier transform $|\xi|^2 \hat{p}^{(1)} = -\sum \xi_i \xi_j \widehat{u_\varepsilon^i u_\varepsilon^j}$. Taking in account the estimate (2.5) one obtains:

$$\begin{cases} \|p_\varepsilon^{(1)}\|_{L^{4/3}(0, T; L^2)} \leqq \text{Const.}, \\ \|\nabla p_\varepsilon^{(1)}\|_{L^{4/3}(0, T; \mathbb{H}^{-1})} \leqq \text{Const.} \end{cases} \tag{2.10}$$

Analogously,

$$\begin{cases} \|p_\varepsilon^{(2)}\|_{L^2(0, T; L^2)} \leqq \text{Const.}, \\ \|\nabla p_\varepsilon^{(2)}\|_{L^2(0, T; \mathbb{H}^{-1})} \leqq \text{Const.} \end{cases} \tag{2.11}$$

Finally, $|\xi| \hat{p}_\varepsilon^{(3)} = -i \sum_j |\xi|^{-1} \xi_j \hat{f}_{j, 0}$, consequently:

$$\begin{cases} \|p_\varepsilon^{(3)}\|_{L^2(0, T; L^6)} \leqq \text{Const.}, \\ \|\nabla p_\varepsilon^{(3)}\|_{L^2(0, T; L^2)} \leqq \text{Const.} \end{cases} \tag{2.12}$$

The first statement in (2.12) follows from a well known theorem on Riesz potentials [[11], Chap. 5, Thm 1, (*b*)].

From (2.8), (2.10), (2.11), (2.12) it follows that:

$$\begin{cases} \|p_\varepsilon\|_{L^{4/3}(0, T; L^2 + L^6)} \leqq \text{Const.}, \\ \|\nabla p_\varepsilon\|_{L^{4/3}(0, T; \mathbb{H}^{-1})} \leqq \text{Const.} \end{cases} \tag{2.13}$$

This proves statement (iv). If $\nabla . f = 0$ one choose f_ε verifying $\nabla . f_\varepsilon = 0$. Hence $p_\varepsilon \equiv p_\varepsilon^{(1)}$, and p_ε verifies (2.10).

The properties proved until now allow us to pass to the limit in $(0.6)_\varepsilon$, in order to obtain (0.1) and the remaining properties in (ii). Weak continuity of $u(t)$, $(0.1)_3$ and (0.4) are proved in the usual way.

Proof of the local energy inequality. — We start by proving (0.5) for almost all $t \in [0, T]$. Since $u_\varepsilon \to u$ strongly in $L^2(0, T; \mathbb{L}^2(B_R))$, for every $R>0$, it follows that (*a* subsequence) u_ε converges strongly to u in $\mathbb{L}^2(B_R)$, for every $R>0$, and for almost all $t \in [0, T]$. For this values of t the first term on the left hand side of (1.15) converges to the corresponding term in (0.5). The lim inf of the second term is less or equal to the corresponding term in (0.5), and the third one is non-negative. From the inclusion $L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{L}^6) \subsetneq \mathbb{L}^{10/3}(Q_T)$, and from the convergence of u_ε to u almost everywhere in Q_T it follows that $|u_\varepsilon|^2 u_\varepsilon \rightharpoonup |u|^2 u$ weakly in $\mathbb{L}^{10/9}(Q_T)$.

Hence:

$$\iint_{Q_t} |u_\varepsilon|^2 u_\varepsilon . \nabla \varphi \to \iint_{Q_t} |u|^2 u . \nabla \varphi.$$

On the other hand from:

$$\| u_\varepsilon - u \|_{L^4(0, T;\, \mathbb{L}^2(B_R))} \leqq \| u_\varepsilon - u \|^{1/2}_{L^\infty(0, T;\, \mathbb{L}^2(B_R))} \times \| u_\varepsilon - u \|^{1/2}_{L^2(0, T;\, \mathbb{L}^2(B_R))}$$

it follows that $u_\varepsilon \to u$ strongly in $L^4(0, T; \mathbb{L}^2(B_R))$, $R>0$. This is sufficient to pass to the limit on the pressure term. Finally the two last terms on the right hand side of (1.15) tend to zero, as a consequence of (2.1).

To finish the proof it remains to show that (0.5) holds for every $t \in [0, T]$. Let $t_n \to t$ be a sequence of values for which (0.5) holds. Write (0.5) for t_n and take the lim inf when $t_n \to t$. This yields (0.5) with the first integral on the left hand side replaced by the right hand side of (2.14) below.

However,

$$\int_{\Omega_t} |u|^2 \varphi \, dx \leqq \liminf_{t_n \to t} \int_{\Omega_{t_n}} |u|^2 \varphi \, dx, \tag{2.14}$$

since $u(t)\sqrt{\varphi(t)}$ is weakly continuous in $[0, T]$ with values in $\mathbb{L}^2(\mathbb{R}^3_x)$. □

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ON THE CONSTRUCTION OF SUITABLE WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS VIA A GENERAL APPROXIMATION THEOREM

By H. BEIRÃO DA VEIGA (*)

Introduction

In this paper we continue the study (initiated in [2]) of methods of construction of suitable weak solutions to the Navier-Stokes equations.

Basically, a *suitable weak solution* is a weak solution $u \in L^2(0, T; V) \cap C_{deb}(0, T; H)$ which verifies the local energy inequality (0.3); other properties requested in the definition (*see* [3]) follow directly from the equations if the data are smooth enough.

Caffarelli, Kohn and Nirenberg proved in [3] that the one dimensional Hausdorff measure of the set of the interior singularities of a suitable weak solution is zero. Weaker results were proved previously by Sheffer; *see* [7], [8], [9]. At the light of that result it seems quite natural to require the local energy inequality as an additional property to be verified for the weak solutions of the Navier-Stokes equations. In fact, on deducing the various differential equations of Mathematical Physics from physical principles it is generally assumed that the functions describing the physical quantities are "sufficiently smooth". Under this assumption, physical principles and differential equations are more or less equivalent. On considering weak solutions this equivalence could disappear. In that case, in order to maintain the physical meaning of the description, one has to complement the differential equations with the lost physical principles.

There is no evidence that solutions obtained by Faedo-Galerkin method verifies the local energy estimate. Scheffer [7] constructed suitable weak solutions in the whole space. Caffarelli, Kohn and Nirenberg [3] constructed them also in bounded domains. For the whole space case, we proved in [2] that by adding $\varepsilon\Delta^2 u_\varepsilon$ $(\varepsilon>0)$ to the main equation and by letting ε go to zero one obtains a suitable weak solution as limit of the u_ε. In the case of a bounded domain the same approach gives a weak

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solution to the Navier-Stokes system (*see* [1]). We are convinced that (as for the whole space) the same approach gives a suitable weak solution. However, the proof would require long calculations in order to obtain for the linearized equation with the term $\Delta^2 u$ results similar to those of Solonnikov [10] for the linearized equation with $-\Delta u$. The aim of this paper is to introduce a different approach, which seems us particularly simple and elegant, based on an abstract approximation result (*see* theorem A). It allows us to prove the local energy estimates *up to the boundary*, i. e., without assuming the text functions in equation (0.3) with compact support in Ω (*see* Theorem B).

We give a *complete* proof of Theorem B below, without assuming the reader familiar with the Navier-Stokes equations. A certain length of this paper is due to these facts.

NOTATIONS AND RESULTS. — One has the following approximation theorem:

THEOREM *A*. — *Let* $\mathbb{K}$ *be a non-empty, convex and compact subset of a Banach space* X, *and let* $\mathbb{Q}\subset\mathbb{K}$ *be a dense convex subset in* $\mathbb{K}$. *Assume that* $S:\mathbb{Q}\to\mathbb{K}$ *is a map such that its restriction to the convex hull of every finite number of elements of* $\mathbb{Q}$ *is continuous.*

Then, given $\varepsilon>0$ *there exists a couple of elements* $v_\varepsilon\in\mathbb{Q}$, $u_\varepsilon\in\mathbb{K}$ *such that* $u_\varepsilon=S v_\varepsilon$ *and* $\|u_\varepsilon-v_\varepsilon\|<\varepsilon$.

Let us briefly illustrate this result. In general we want to solve a non-linear equation, say $\varphi(u, u)=f$, where $\varphi(v, u)=f$ is solvable in u for each fixed smooth v (e. g., for the construction of weak solutions of the Navier-Stokes equations, replace in (0.1) the term $(u.\nabla)u$ by $(v.\nabla)u$). Let Y be a Banach space in which an *a priori* estimate is known (e. g., for the Navier-Stokes equations set:

$$Y=\{u: u\in L^\infty(0, T;H)\cap L^2(0, T; V),\ u'\in L^{4/3}(0, T; V')\}\,(^2));$$

let $\mathbb{K}$ be the corresponding ball, and let $\mathbb{Q}$ be the set of smooth elements of $\mathbb{K}$. Let X be a larger space, with respect to which $\mathbb{K}$ is a compact subset [e. g., for N.S. equations set $X=L^2(Q_T)$ (2)]. Denote by $u=S v$ the solution of $\varphi(v, u)=f$, for each fixed $v\in\mathbb{Q}$.

For the Navier-Stokes equation the map S is not continuous, except for quite strong topologies. In this last case, however, we loose the inclusion $S\mathbb{Q}\subset\mathbb{K}$, unless small values of T are choosen (local solutions in time). A similar situation appears very often, for non-linear problems. However, S is continuous on finite dimensional subspaces, since all norms are then equivalent. Hence theorem A applies. Hence, from v_ε, $u_\varepsilon\in\mathbb{K}$, $\|v_\varepsilon-u_\varepsilon\|_X<\varepsilon$ it follows the existence of suitable subsequences $u_\varepsilon\to u$, $v_\varepsilon\to u$, weakly in Y. Moreover $\varphi(v_\varepsilon, u_\varepsilon)=f$, since $S v_\varepsilon=u_\varepsilon$. By going to the limit as $\varepsilon\to 0$, and under natural assumptions on φ, one gets $\varphi(u, u)=f$.

The proof of Theorem A will be given in section 1.

We present now the main notations:

Ω, an open, bounded subset of $\mathbb{R}^3$, locally situated on one side of his boundary Γ, a differentiable manifold of class C^2.

$$\Omega_t\equiv\{t\}\times\Omega;\qquad Q_T\equiv]0, T[\times\Omega;\qquad \Sigma_T\equiv]0, T[\times\Gamma,\qquad \text{for}\quad T\in]0, +\infty].$$

(2) In the next section a different choice will be taken, since we want to construct weak solutions verifying the additional properties (0.3) and u', u, $\nabla p\in L^{5/4}(Q_T)$.

L^p, $| \ |_p$, usual $L^p(\Omega)$ space $(1 \leqq p \leqq +\infty)$, and usual norm in L^p.

W^s_p, $\| \ \|_{s,p}$, Sobolev space $W^s_p(\Omega)$, $1 \leqq p < +\infty$, $s \in \mathbb{R}$ (*see* [6] for definition and properties), and usual norm in W^s_p.

$\overset{\circ}{W}{}^k_p$, Closure of $C^\infty_0(\Omega)$ in W^k_p, k positive integer.

The norm in $W^k_p(\Omega)$, k non-negative integer, is:

$$\| f \|_{k,p} \equiv \left(\sum_{l=0}^{k} \sum_{|\alpha|=l} | D^\alpha f |_p^p \right)^{1/p}.$$

As done for scalar functions, we define for vector functions $v=(v_1, v_2, v_3)$ the spaces $\mathbb{L}^p$, $\mathbb{W}^s_p$, $\overset{\circ}{\mathbb{W}}{}^k_p$, and so on. Norms will be denoted by the same symbol in both cases.

For vector functions we also define:

$$| \nabla v |^2 = \sum_{i,j=1}^{3} \left(\frac{\partial v_j}{\partial x_i} \right)^2,$$

$$\| v \|_V = | \nabla v |_2 = \left(\int_\Omega | \nabla v |^2 \, dx \right)^{1/2},$$

and:

$$(w \,.\, \nabla) v = \sum_{i,j=1}^{3} w_i \frac{\partial v_j}{\partial x_i}.$$

As usual we define:

$$\mathscr{V} \equiv \{ v \in [C^\infty_0(\Omega)]^3 : \nabla \,.\, v = 0 \text{ in } \Omega \},$$
$$H \equiv \{ v \in \mathbb{L}^2 : \nabla \,.\, v = 0 \text{ in } \Omega, \ v \,.\, n = 0 \text{ on } \Gamma \},$$
$$V \equiv \{ v \in \overset{\circ}{\mathbb{W}}{}^1_2 : \nabla \,.\, v = 0 \text{ in } \Omega \}.$$

H is the closure of $\mathscr{V}$ in $\mathbb{L}^2$ and V is the closure of $\mathscr{V}$ in $\mathbb{W}^1_2$.

$L^p(0, T; \mathscr{X})$, Banach space of strongly measurable functions in $]0, T[$ with values in the Banach space $\mathscr{X}$, for which:

$$\| u \|^p_{L^p(0,T;\mathscr{X})} \equiv \int_0^T \| u(\tau) \|^p_{\mathscr{X}} \, d\tau < +\infty,$$

with the usual modification if $p = +\infty$.

$C(0, T; \mathscr{X})$; $C_{deb}(0, T; \mathscr{X})$, space of continuous [resp. weakly continuous] functions in $[0, T]$ with values in $\mathscr{X}$.

$L^p_{loc}(0, +\infty; \mathscr{X})$, space of functions defined in $]0, +\infty[$ with values in $\mathscr{X}$, whose restrictions to $]0, T[$ belong to $L^p(0, T; \mathscr{X})$, for every $T > 0$.

For convenience we adopt the notation:

$$\| u \|_{q,p,T} \equiv \| u \|_{L^p(0,T;L^q(\Omega))}; \qquad \| u \|_{q,T} \equiv \| u \|_{q,q,T}.$$

We denote by c positive constants depending at most on Ω and on the fixed parameter p. For convenience we denote different constants by the same symbol c. Otherwise, we will write $c_0, c_1, c_2, \ldots$

The Navier-Stokes equations describing the motion of a viscous incompressible fluid are $(0<\mathrm{T}\leqq +\infty)$:

$$(0.1)\qquad \begin{cases} u'+(u.\nabla)u-\Delta u=f-\nabla p & \text{in } Q_T,\\ \nabla . u=0 & \text{in } Q_T,\\ u=0 & \text{on } \Sigma_T,\\ u|_{t=0}=u_0(x) & \text{in } \Omega,\end{cases}$$

where $u'=\partial u/\partial t$. We assume, without loosing generality, that the density ρ and the viscosity μ are equal to one. The initial data $u_0(x)$ and the external force field $f(t, x)$ are given. The velocity $u(t, x)$ and the pressure $p(t, x)$ are unknowns. In this paper we prove the following result:

THEOREM B. — *Let* $u_0 \in \mathrm{H} \cap \mathbb{W}_p^{2-2/p}$ *and* $f \in \mathrm{L}^1_{\mathrm{loc}}(0, +\infty; \mathbb{L}^2) \cap \mathrm{L}^p_{\mathrm{loc}}(0, +\infty; \mathbb{L}^p)$, *with* $10/9<p\leqq 5/4$. *Then there exists a weak solution* u, p *of system* (0.1) *in* $Q_{+\infty}$, *such that:*

$$(0.2)\qquad \begin{aligned} &u \in \mathrm{L}^2_{\mathrm{loc}}(0, +\infty; \mathrm{V}) \cap \mathrm{C}_{\mathrm{deb}}(0, +\infty; \mathrm{H}) \cap \mathrm{L}^p_{\mathrm{loc}}(0, +\infty; \mathbb{W}^2_p),\\ &u' \in \mathrm{L}^{4/3}_{\mathrm{loc}}(0, +\infty; \mathbb{W}_2^{-1}) \cap \mathrm{L}^p_{\mathrm{loc}}(0, +\infty; \mathbb{L}^p),\\ &p \in \mathrm{L}^p_{\mathrm{loc}}(0, +\infty; \mathbb{W}^1_p).\end{aligned}$$

Moreover, u, p verifies the local energy estimate up to the boundary:

$$(0.3)\qquad \int_{\Omega_t} |u|^2 \varphi + 2\iint_{Q_t} |\nabla u|^2 \varphi \leqq \int_{\Omega_0} |u_0|^2 \varphi + \iint_{Q_t} |u|^2 (\varphi' + \Delta\varphi) + \iint_{Q_t} (|u|^2+2p)\, u.\nabla\varphi + 2\iint_{Q_t} f.u\varphi,$$

for every $t>0$ *and for every* $\varphi \in \mathrm{C}^2(\overline{Q}_{+\infty})$, $\varphi \geqq 0$ on $\overline{Q}_{+\infty}$.

Finally, for every $\mathrm{T}>0$ *one has:*

$$\|u\|_{\infty, 2, \mathrm{T}} \leqq |u_0|_2 + \|f\|_{1, 2, \mathrm{T}},$$

$$\|\nabla u\|^2_{2, \mathrm{T}} \leqq |u_0|^2_2 + 2\|f\|^2_{1, 2, \mathrm{T}},$$

$$\|u'\|_{p, \mathrm{T}} + \|u\|_{\mathrm{L}^p(0, \mathrm{T}; \mathbb{W}^2_p)} + \|\nabla p\|_{p, \mathrm{T}} \leqq c_0 \|u_0\|_{2-(2/p), p} + c_1 \|f\|_{p, \mathrm{T}} + c_3 (|u_0|^2_2 + \|f\|^2_{1, 2, \mathrm{T}}).$$

Remark 0.1. — As in [10] equation (172), we assume the pressure $p(t, x)$ determined by the supplementary condition:

$$(0.5)\qquad \int_\Gamma p(t, x)\, d\Gamma = 0, \quad \text{for almost all } t>0.$$

Alternatively, we could utilize the condition:

$$(0.6)\qquad \int_\Omega p(t, x)\,dx=0, \quad \text{for almost all } t>0.$$

1. The approximating problem

We begin by defining the set of vectors:

$$\Lambda\equiv\{v\in C^\infty(\bar{Q}_T): v(t)\in\mathscr{V}, \text{ for all } t\in[0, T]\},$$

where T is an arbitrary positive real number, and by considering the linear system:

$$(1.2)\qquad \begin{cases} u'+(v.\nabla)u-\Delta u+\nabla p=\tilde{f} & \text{in } Q_T,\\ \nabla.u=0 & \text{in } Q_T,\\ u=0 & \text{on } \Sigma_T,\\ u|_{t=0}=\tilde{u}_0 & \text{in } \Omega, \end{cases}$$

for $v\in\Lambda$. For convenience we define:

$$(1.3)\qquad \begin{cases} A(u_0, f)\equiv|u_0|_2+\|f\|_{1,\,2,\,T},\\ A_1^2(u_0, f)\equiv|u_0|_2^2+2\|f\|_{1,\,2,\,T}^2,\\ B(u_0, f)\equiv c_0\|u_0\|_{2-(2/p),p}+c_1\|f\|_{p,\,T}, \end{cases}$$

where the constants c_0 and c_1 will be defined in the proof of Theorem 1.2. One has the following result:

THEOREM 1.1. — *Let $v\in\Lambda$, $\tilde{u}_0\in V$ and $\tilde{f}\in L^2(Q_T)$. Then there exists a unique solution u, ∇p of problem* (1.2), *which verifies:*

$$(1.4)\qquad \begin{cases} u\in L^2(0, T; \mathbb{W}_2^2)\cap C(0, T; V),\\ u'\in L^2(Q_T),\\ \nabla p\in L^2(Q_T), \end{cases}$$

and:

$$(1.5)\qquad \begin{cases} \|u\|_{\infty,\,2,\,T}\leq A(\tilde{u}_0, \tilde{f}),\\ \|\nabla u\|_{2,\,T}\leq A_1(\tilde{u}_0, \tilde{f}). \end{cases}$$

Proof. — Existence, uniqueness and regularity follows as for the usual linearized Navier-Stokes equation; alternatively, one can use Theorem 4.2 in [11], page 487. Estimates (1.5) follow easily by multiplying equation $(1.2)_1$ by u, by integrating over Ω and by doing some well known devices. □

Let now $1<p\leq 5/4$ and define $q=q(p)$ and $r=r(p)$ as follows:

$$\frac{1}{q}=\frac{1}{p}-\frac{1}{2}, \qquad \frac{1}{r}=\frac{1-(2/q)}{2}+\frac{2/q}{6}=\frac{1}{2}-\frac{2}{3q}.$$

Since $0<2/q<1$ it follows that:

$$|v|_r \leqq |v_2|^{1-2/q}|v|_6^{2/q} \leqq c|v|_2^{1-2/q}|\nabla v|_2^{2/q}, \tag{1.6}$$

where we used a well known Sobolev's theorem (*see* for instance [4], Chap. I, Lemma 3). Consequently:

$$\|v\|_{q,\,p,\,T} \leqq c(\|v\|_{\infty,\,2,\,T} + \|\nabla v\|_{2,\,T}). \tag{1.7}$$

We now prove the following result:

THEOREM 1.2. — *Under the hypothesis of Theorem* 1.1 *the solution u, ∇p of problem* (1.2) *verifies the estimate:*

$$\|u'\|_{p,\,T} + \|u\|_{L^p(0,\,T;\,\mathbb{W}_p^2)} + \|\nabla p\|_{p,\,T} \leqq B(\tilde{u}_0, \tilde{f}) + c_2(\|v\|_{\infty,\,2,\,T} + \|\nabla v\|_{2,\,T})\|\nabla u\|_{2,\,T}. \tag{1.8}$$

Proof. — Note that $L^2(Q_T) \subsetneq L^p(Q_T)$ and $V \subsetneq H \subsetneq \mathbb{W}_p^{2-2/p}$. Estimate (1.8) follows from Theorem 15, paragraph 17, p. 102 of [10], by taking in account the estimates:

$$\|(v\,.\,\nabla)u\|_{p,\,T} \leqq \|v\|_{q,\,T}\|\nabla u\|_{2,\,T},$$

and (1.7). Note that $\|v\|_{q,\,T}$ is bounded by c times the left hand side of (1.7), since $q \leqq 10/3 \leqq r$.

Now we prove the approximation theorem stated in the introduction.

Proof of Theorem A. — Let $\varepsilon>0$ be fixed. The compactness of $\mathbb{K}$ guarantees the existence of a finite number of elements $\tilde{v}_1, \ldots, \tilde{v}_N \in \mathbb{K}$, such that:

$$\mathbb{K} \subset \bigcup_{i=1}^{N} B(\tilde{v}_i, \varepsilon/4),$$

where $B(x, \delta)$ denotes the ball with center in x and radius δ in the space X. Since $\mathbb{Q}$ is dense in $\mathbb{K}$ there exists elements $v_i \in \mathbb{Q}$, $i=1, \ldots, N$, such that:

$$\mathbb{K} \subset \bigcup_{i=1}^{N} B(v_i, \varepsilon/2).$$

Le now $\mathbb{Q}_0$ be the convex hull generated by the elements $v_1, \ldots, v_N$ and let $P: \mathbb{K} \to \mathbb{Q}_0$ be a continuous map on $\mathbb{K}$, such that:

$$\|Pu-u\| < \varepsilon, \qquad \forall u \in \mathbb{K}. \tag{1.9}$$

If X is an Hilbert space [in the sequel we will utilize Theorem A with $X=L^2(Q_T)$] it suffices to define P as the projection on to $\mathbb{Q}_0$. In this case P is continuous, moreover:

$$\|Pu-u\| = \inf_{v \in \mathbb{Q}_0} \|v-u\| < \varepsilon/2.$$

In the general case it is not difficult to prove the existence of a continuous P verifying (1.9), and we leave the construction to the reader.

Consider now the restriction of the map PS to $\mathbb{Q}_0$, $PS:\mathbb{Q}_0 \to \mathbb{Q}_0$. This map is continuous from a finite dimensional bounded, closed and convex set on iteself. By Brower's fixed point theorem there exists $v_\varepsilon \in \mathbb{Q}_0$ such that $PS\, v_\varepsilon = v_\varepsilon$. Defining $u_\varepsilon = S\, v_\varepsilon$ it follows from (1.9) that:

$$\| v_\varepsilon - u_\varepsilon \| = \| P u_\varepsilon - u_\varepsilon \| < \varepsilon. \quad \square$$

For brevity we will use the notation $L^p(\mathscr{X}) = L^p(0, T; \mathscr{X})$, where $\mathscr{X}$ is a Banach space and:

$$W_p^1(\mathscr{X}) = \{ v \in L^p(\mathscr{X}) : v' \in L^p(\mathscr{X}) \}.$$

We define the Banach space:

$$Y \equiv L^\infty(H) \cap L^2(V) \cap W_p^1(\mathbb{L}^p),$$

normalized according to the definition. Let A, A_1 and B be non-negative real numbers and define $\mathbb{K} = \mathbb{K}(A, A_1, B)$ as:

$$\mathbb{K} \equiv \{ v \in Y : \| v \|_{\infty,\, 2,\, T} \leqq A, \| \nabla v \|_{2,\, T} \leqq A_1, \| v' \|_{p,\, T} \leqq B \}. \tag{1.10}$$

$\mathbb{K}$ is a convex compact subset of $X \equiv L^2(Q_T)$ (*see* [5], Chap. I, Theorem 5.1). One has the following result:

LEMMA 1.3. — *The convex set:*

$$\mathbb{Q} \equiv \Lambda \cap \mathbb{K},$$

is dense in $\mathbb{K}$ *in the* X *topology.*

Proof. — Let $v \in \mathbb{K}$. We assume, without loose of generality, that v verifies strictly the inequalities appearing in definition (1.10). Extend v by reflexion to $[-T, 0]$ and to $[T, 2T]$, and define in the usual way the mollifiers:

$$v_n(t) = \int j_n(t-\tau)\, v(\tau)\, d\tau, \qquad n = 1, 2, \ldots \tag{1.11}$$

Since $v_n \in \mathbb{K}$ and $v_n \to v$ in X when $n \to +\infty$, we don't loose generality by also assuming that $v \in C^1(0, T; V)$.

Let now $\varepsilon > 0$ be fixed and choose $\omega = \omega(\varepsilon)$ such that:

$$| v(t) - v(s) |_2 < \varepsilon, \ \| v(t) - v(s) \|_V < \varepsilon, \qquad | v'(t) - v'(s) |_p < \varepsilon,$$

whenever $s, t \in [0, T]$, $|s - t| < \omega$. Fix points t_i in $[0, T]$ verifying :

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

and:

$$| t_{i+1} - t_i | < \omega \qquad \text{for} \quad i = 0, 1, \ldots, n-1.$$

Finally fix $\varepsilon_1 = \varepsilon_1(\varepsilon)$ such that:

$$\varepsilon_1^p \sum_{i=0}^{n-1} (t_{i+1}-t_i)^{1-p} \leqq \varepsilon^p.$$

Define $v_i = v(t_i) \in V$, $i=0, \ldots, n$, and fix elements $w_i \in \mathscr{V}$ such that:

$$|w_i - v_i|_2 < \varepsilon, \qquad \|w_i - v_i\|_V < \varepsilon, \qquad |w_i - v_i|_p < \varepsilon_1, \qquad \text{for} \quad i=0, 1, \ldots, n.$$

Consider the function $w(t)$, which is linear on each interval $[t_i, t_{i+1}]$ and verifies :

$$w(t_i) = w_i \qquad \text{for} \quad i=0, 1, \ldots, n.$$

Arguing separately in each interval $[t_i, t_{i+1}]$ one easily shows that :

$$|w(t)-v(t)|_2 < 4\varepsilon, \qquad \|w(t)-v(t)\|_V < 4\varepsilon,$$

hence:

$$\text{(1.12)} \qquad \|w-v\|_{L^2(V)}^2 \leqq 16\varepsilon^2 T, \qquad \|w-v\|_{\infty, 2, T} \leqq 4\varepsilon.$$

On the other hand, for $t \in [t_i, t_{i+1}]$,

$$w'(t) = \frac{v_{i+1}-v_i}{t_{i+1}-t_i} + \frac{(w_{i+1}-v_{i+1})-(w_i-v_i)}{t_{i+1}-t_i},$$

consequently:

$$|w'(t)-v'(t)|_p \leqq \left| \frac{v(t_{i+1})-v(t_i)}{t_{i+1}-t_i} - v'(t) \right|_p + \frac{2\varepsilon_1}{t_{i+1}-t_i}.$$

By using the mean value theorem for functions with values in a Banach space, it follow that:

$$|w'(t)-v'(t)|_p \leqq \varepsilon + \frac{2\varepsilon_1}{t_{i+1}-t_i}.$$

for every $t \in [t_i, t_{i+1}]$. Taking in account the definition of ε_1, one easily shows that:

$$\text{(1.13)} \qquad \|w'-v'\|_{p, T}^p \leqq (2^{p-1} T + 2^{2p-1}) \varepsilon^p.$$

From (1.12), (1.13) it follows that $w \in \mathbb{K}$, for ε sufficiently small. Moreover, $w \to v$ in Y, hence in X, when $\varepsilon \to 0$.

The proof of lemma 1.3 is accomplished by approximating w by mollifiers w_n, as in (1.11). Recall that $w(t)$ is piecewise linear in [0, T], with values in $\mathscr{V}$. □

Let now $(\tilde{u}_0, \tilde{f}) \in V \times L^2(Q_T)$ be *fixed* and define a map:

$$S: \quad \Lambda \to X,$$

by $Sv = u$, where u is the solution of problem (1.2) corresponding to $v \in \Lambda$. We claim the following result:

LEMMA 1.4. — *The restriction of* S *to every finite dimensional subspace of* Λ *is continuous, with respect to the* X-*topology. In particular, the restriction of* S *to the convex hull of a finite number of elements of* $\mathbb{Q}$ *is continuous with respect to the* X-*topology.*

Proof. — Let Λ_0 be a finite dimensional subspace of Λ, and let v, $v_n \in \Lambda_0$, $\| v_n - v \| \to 0$ when $n \to +\infty$. Since in a finite dimensional vector space all norms are equivalent, one has in particular:

$$\lim_{n \to +\infty} \| v_n - v \|_{C^1(\bar{Q}_T)} = 0.$$

Let $u_n = S v_n$, $u = S v$ and consider the difference (side by side) between equation $(1.2)_1$ written for the couple u, v and written for the couple u_n, v_n. By taking the scalar product in $\mathbb{L}^2$ of $u_n - u$ with both sides of the equation just obtained, the thesis follows (without difficulty) by using well known devices. □

Finally, we prove the following lemma:

LEMMA 1.5. — *Let* $\tilde{u}_0$, $\tilde{f} \in V \times L^2(Q_T)$ *be given and fix* $\mathbb{K}$ *by choosing in definition* (1.10) *the values:*

$$(1.14) \quad \begin{cases} A = A(\tilde{u}_0, \tilde{f}), \\ A_1 = A_1(\tilde{u}_0, \tilde{f}), \\ B = B(\tilde{u}_0, \tilde{f}) + c_2 (A + A_1) A_1. \end{cases}$$

Then the map S, *defined via the system* (1.2), *verifies:*

$$(1.15) \quad S(\mathbb{Q}) \subset \mathbb{K}.$$

Proof. — The result follows from the estimates (1.5) and (1.8), since $v \in \mathbb{K}$. □

With the definitions given above, all the hypothesis of theorem A are verified. As a consequence, one obtains the following approximation theorem:

THEOREM 1.6. — *Let* $\tilde{u}_0 \in V$ *and* $\tilde{f} \in L^2(Q_T)$ *be given, and let* $1 < p \leqq 5/4$. *Then, in correspondence to every* $\varepsilon > 0$, *there exist* $v_\varepsilon \in \Lambda$, $u_\varepsilon \in Y$, $p_\varepsilon \in L^p(0, T; W_p^1)$ *verifying the system:*

$$(1.16)_\varepsilon \quad \begin{cases} u'_\varepsilon + (v_\varepsilon . \nabla) u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = \tilde{f} & \text{in } Q_T, \\ \nabla . u_\varepsilon = 0 & \text{in } Q_T, \\ u_\varepsilon = 0 & \text{on } \Sigma_T, \\ (u_\varepsilon)|_{t=0} = \tilde{u}_0 & \text{in } \Omega, \end{cases}$$

and for which:

$$(1.17) \quad \| u_\varepsilon - v_\varepsilon \|_{2, T} < \varepsilon.$$

Moreover, the following estimates hold:

$$(1.18)\quad \begin{cases} \|u_\varepsilon\|_{\infty,\,2,\,T} \leqq A(\tilde{u}_0, \tilde{f}), \\ \|\nabla u_\varepsilon\|_{2,\,T}^2 \leqq A_1^2(\tilde{u}_0, \tilde{f}), \\ \|u'_\varepsilon\|_{p,\,T} + \|u_\varepsilon\|_{L^p(\mathbb{W}_p^2)} + \|\nabla p_\varepsilon\|_{p,\,T} \\ \qquad \leqq B(\tilde{u}_0, \tilde{f}) + c_3 A_1^2(\tilde{u}_0, \tilde{f}). \end{cases}$$

Estimates $(1.18)_1$ *and* $(1.18)_2$ *hold also for* v_ε *and* ∇v_ε *respectively, and estimate* $(1.18)_3$ *holds for* v'_ε.

Remark. — If in definition (1.10) one add the condition $\|v\|_{L^p(0,\,T;\,\mathbb{W}_p^2)} \leqq B$, then $(1.18)_3$ holds also for $\|v_\varepsilon\|_{L^p(0,\,T;\,\mathbb{W}_p^2)}$.

However, this would not be useful.

2. The limit problem

Let $u_0 \in H \cap \mathbb{W}_p^{2-2/p}$ and $f \in L^1(0, T; \mathbb{L}^2) \cap L^p(Q_T)$ be given, ([3]) and consider sequences $u_0^{(n)} \in V$ and $f_n \in L^2(Q_T)$ such that:

$$(2.1)\quad \begin{cases} |u_0^{(n)} - u_0|_2 < n^{-1}, & \|u_0^{(n)} - u_0\|_{2-2/p,\,p} < n^{-1}, \\ \|f_n - f\|_{1,\,2,T} < n^{-1}, & \|f_n - f\|_{p,\,T} < n^{-1}. \end{cases}$$

Let now (u_n, v_n, p_n) be the solution of problem $(1.16)_\varepsilon$ for $\varepsilon = 1/n$ and with data $u_0 = u_0^{(n)}$, $\tilde{f} = f_n$. For the reader's convenience we rewrite theorem 1.6 for this case. One has:

THEOREM 2.1. — *Let* u_0, f, $u_0^{(n)}$, f_n, u_n, v_n *and* p_n *be as above.*
Then:

$$(2.2)\quad \begin{cases} u'_n + (v_n . \nabla) u_n - \Delta u_n + \nabla p_n = f_n & in\ Q_T, \\ \nabla . u_n = 0 & in\ Q_T, \\ u_n = 0 & on\ \Sigma_T, \\ (u_n)|_{t=0} = u_0^{(n)} & in\ \Omega, \end{cases}$$

and:

$$(2.3)\quad \|u_n - v_n\|_{2,\,T} \leqq n^{-1}.$$

([3]) We put u_0 and f directly in equation (2.2) and we drop $1/n$ in equation (2.4), if $u_0 \in V$ and $f \in L^2(Q_T)$.

Moreover:

$$(2.4)\qquad \left\{\begin{aligned} &\|u_n\|_{\infty,\,2,\,T}\leqq A\left(u_0+\frac{1}{n},f+\frac{1}{n}\right),\\ &\|\nabla u_n\|^2_{2,\,T}\leqq A_1^2\left(u_0+\frac{1}{n},f+\frac{1}{n}\right),\\ &\|u_n'\|_{p,\,T}+\|u_n\|_{L^p(\mathbb{W}_p^2)}+\|\nabla p_n\|_{p,\,T}\\ &\leqq B\left(u_0+\frac{1}{n},f+\frac{1}{n}\right)+c_3A_1^2\left(u_0+\frac{1}{n},f+\frac{1}{n}\right).\end{aligned}\right.$$

Finally, estimates $(2.4)_1$ *and* $(2.4)_2$ *hold for* v_n *and* ∇v_n, *respectively, and* $(2.4)_3$ *holds for* v_n'.

On the right hand sides of (2.4), the term $1/n$ is assumed to be added to the *norms* of u_0 and f, appearing in definition (1.3).

From the sequences u_n and v_n we can selct subsequences converging both to a solution u of problem (0.1). This is done by using well known devices, which we recall for the sake of completeness; see for instance [4], [5], [12], [1]. Note that the "usual" non-linear term $(u_n.\nabla)u_n$ is replaced by $(v_n.\nabla)u_n$. However, (2.3) guarantees that the sequences u_n and v_n have the same limit.

By using the estimates stated in Theorem 2.1 and the property (2.3), it follows the existence of subsequences u_ν, v_ν and p_ν and functions u, p such that:

$$(2.5)\qquad \begin{aligned} &u_\nu\rightharpoonup u \text{ weakly in } L^2(V), \text{ weakly in } L^p(\mathbb{W}_p^2), \text{ and weak-}* \text{ in } L^\infty(H).\\ &p_\nu\rightharpoonup p \quad \text{weakly in } L^p(\mathbb{W}_p^1).\\ &v_\nu\rightharpoonup u \quad \text{weakly in } L^2(V) \text{ and weak-}* \text{ in } L^\infty(H).\end{aligned}$$

Moreover, a well known compactness theorem ([5], Chap. 1, Theorem 5.1) guarantees that we can select subsequences (denoted by the same index ν) verifying:

$$(2.6)\qquad u_\nu\rightarrow u,\ v_\nu\rightarrow u,$$

strongly in $L^2(Q_T)$; strongly in $\mathbb{L}^2(\Omega)$ for almost all $t\in]0, T[$; and almost everywhere in Q_T.

On the other hand, from the embedding $L^\infty(\mathbb{L}^2)\cap L^2(\mathbb{L}^6)\subsetneq L^4(\mathbb{L}^3)$, it follows that the sequences $(v_\nu)_i\,(u_\nu)_j$, for which $i, j\in\{1, 2, 3\}$, are bounded in $L^2(\mathbb{L}^{3/2})$. A well known device (*see* [5], p. 76) gives:

$$(2.7)\qquad (v_\nu)_i(u_\nu)_j\rightharpoonup u_iu_j,\quad \text{weakly in } L^2(\mathbb{L}^{3/2}).$$

In particular $(v_\nu.\nabla)u_\nu\rightharpoonup(u.\nabla)u$ weakly in $L^2(\mathbb{W}_2^{-2})$. The convergence of the other terms in equation $(2.2)_1$ to the corresponding terms in equation $(0.1)_1$ is clear. Hence u, p is a solution of system (0.1). Finally, the estimates (0.4) follow from the corresponding estimates (2.4), by taking in account the lower semi-continuity of the norms with respect to the weak convergences.

Now we want to prove the local energy estimate (0.3), for every $\varphi \in C^2(\overline{Q_T})$, $\varphi \geqq 0$ on Q_T. By multiplying both sides of equation $(2.2)_1$ by φu_ν and by integrating over Q_t, one easily shows that:

$$(2.8)\quad \frac{1}{2}\int_\Omega |u_\nu|^2 \varphi + \iint_{Q_t} |\nabla u_\nu|^2 \varphi = \frac{1}{2}\int_{\Omega_0} |u_0^{(\nu)}|^2 \varphi + \frac{1}{2}\iint_{Q_t} |u_\nu|^2 (\varphi' + \Delta\varphi) + \frac{1}{2}\iint_{Q_t} |u_\nu|^2 v_\nu . \nabla\varphi + \iint_{Q_t} p_\nu u_\nu . \nabla\varphi + \iint_{Q_t} f_\nu . u_\nu \varphi.$$

Note that the limit functions u and p are not smooth enough to giustify a similar calculation starting from equation (0.1).

Now we pass to the limit in equation (2.8), when $\nu \to +\infty$. We start by proving (0.3) for the values t for which $u_\nu(t) \to u(t)$ strongly in $\mathbb{L}^2$. Later, we extend (0.3) to every $t \in [0, T]$.

It's clear that the integrals over Ω_0 and Ω_t in equation (2.8) converge to the corresponding integrals in (0.3). On the other hand, $D_i(u_\nu)_j \rightharpoonup D_i u_j$ weakly in $L^2(Q_T)$. Consequently $\sqrt{\varphi}\, D_i(u_\nu)_j \rightharpoonup \sqrt{\varphi}\, D_i u_j$ weakly in $L^2(Q_T)$, hence:

$$\iint_{Q_t} |\nabla u|^2 \varphi \leqq \liminf_{\nu \to +\infty} \iint_{Q_t} |\nabla u|^2 \varphi.$$

The convergence of the second and of the last term on the right hand side of (2.8) to the corresponding terms in (0.3) is obvious. Let us consider the two remaining terms.

From the embedding $L^\infty(\mathbb{L}^2) \cap L^2(\mathbb{L}^6) \subsetneq L^{10/3}(Q_T)$, one gets $\| |u_\nu|^2 v_\nu \|_{10/9,\, T} \leqq \text{Const}$. Taking in account the pointwise convergence in Q_T, one shows that $|u_\nu|^2 v_\nu$ converges weakly to $|u|^2 u$ in $L^{10/9}(Q_T)$; *see* [5], Chap. I, Lemma 1.3.

In particular:

$$\lim_{\nu \to +\infty} \iint_{Q_t} |u_\nu|^2 v_\nu . \nabla\varphi = \iint_{Q_t} |u|^2 v . \nabla\varphi.$$

Finally we consider the pressure term. From $(2.5)_2$ and from a well known Sobolev's embedding theorem it follows that:

$$(2.8)\qquad p_n \rightharpoonup p \quad \text{weakly in } L^p(0, T; L^{p^*}),$$

where $1/p^* = (1/p) - (1/3)$; we assume now that $10/9 < p \leqq 5/4$. Consequently the relation:

$$\lim_{\nu \to +\infty} \iint_{Q_t} p_\nu u_\nu . \nabla\varphi = \iint_{Q_t} pu . \nabla\varphi,$$

is proved if we show that:

$$(2.9)\qquad u_\nu \to u \text{ strongly in } L^{p'}(\mathbb{L}^{(p^*)'}),$$

where in general $1/\alpha' = 1 - (1/\alpha)$. There are not loose of generality on assuming that

$10/9 \equiv p_0 < p < 6/5$, since p' and $(p^*)'$ are decreasing functions of p. Estimate (1.7) for the value $q = 10$ show that:

$$\| u_\nu \|_{p'_0,\,(p_0^*)',\,T} \leqq \text{Const.}, \tag{2.10}$$

since $p'_0 = 10$ and $(p_0^*)' = 30/13$. On the other hand, for $p \in]10/9, 6/5[$ one has $p' \in]2, 10[$, $(p^*)' \in]2, 30/13[$. Hence, by fixing a value $\theta \in]0, 1[$ for which:

$$\frac{1}{p'} \geqq \frac{\theta}{p'_0} + \frac{1-\theta}{2}.$$

$$\frac{1}{(p^*)'} \geqq \frac{\theta}{(p_0^*)'} + \frac{1-\theta}{2},$$

one obtains:

$$\| u_\nu - u \|_{p',\,(p^*)',\,T} \leqq c \, \| u_\nu - u \|^{\theta}_{p'_0,\,(p_0^*)',\,T} \, \| u_\nu - u \|^{1-\theta}_{2,\,T}. \tag{2.11}$$

Statement (2.9) follows from (2.10) and (2.11), by recalling that $u_\nu \to u$ strongly in $L^2(Q_T)$.

To accomplish the proof of Theorem A it remains to show that (0.3) holds for every $t \in [0, T]$. Let t_n be a sequence of values for which (0.3) holds, and such that $t_n \to t$. Consider equation (0.3) for the values t_n, and take the lim inf when $n \to +\infty$. The function $u(t)\sqrt{\varphi(t)}$ is weakly continous in $[0, T]$ with values in $\mathbb{L}^2$, since the same property holds for $u(t)$. Consequently,

$$\int_{\Omega_t} |u|^2 \varphi \leqq \liminf_{t_n \to t} \int_{\Omega_{t_n}} |u|^2 \varphi.$$

The convergence of the integrals over Q_{t_n} to the corresponding integrals over Q_t is obvious. □

In order to prove theorem A in the time interval $[0, +\infty[$, we proceed as follows. We start by fixing an increasing sequence of positive values T_n converging to $+\infty$ [replace also in (2.1) the value T by T_n].

Then we apply our approximation argument to each fixed interval $[0, T_{n_0}]$, by starting each time from a subsequence of indices for which the convergence to (u, u, p) holds in $[0, T_{n_0-1}]$. We obtain a solution u, p in $Q_{+\infty}$, verifying all the requested properties, by selecting a diagonal subsequence.

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