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EXISTENCE OF C^∞ SOLUTIONS OF THE EULER EQUATIONS FOR NON-HOMOGENEOUS FLUIDS

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In this paper we prove the existence of C^∞ solutions for the system (see Sedov [11], chap. IV, § 1, pg. 164)

$$
(E)\quad \begin{cases}
\rho\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = -\nabla\pi & \text{in } \quad Q_{T_0} \equiv\,]0,T_0[\,\times\,\Omega\,, \\
\operatorname{div} v = 0 & \text{in } \quad Q_{T_0}\,, \\
v\cdot n = 0 & \text{on } \quad]0,T_0[\,\times\,\partial\Omega\,, \\
\dfrac{\partial\rho}{\partial t} + v\cdot\nabla\rho = 0 & \text{in } \quad Q_{T_0}\,, \\
\rho_{|t=0} = \rho_0 & \text{in } \quad \Omega\,, \\
v_{|t=0} = a & \text{in } \quad \Omega\,.
\end{cases}
$$

This problem has been studied by Marsden[10]and by us [3], [4], [5] (where one can find some references) from the point of view of (local in time) existence, uniqueness and regularity of the solution. Marsden also obtains a result for C^∞ solutions. The analytic case on compact manifolds without boundary was studied by Baouendi - Goulaouic [1]. These authors have proved analogous results also for manifolds with boundary (private communication).

Here we solve system (E), as in [4], [5], via the equivalent system (2.2) (with $\varphi = \zeta$), (2.4), (2.7), (2.11); and the essential tool is the use of elliptic system (2.7).

In proving the existence of a fixed point in Sobolev spaces (as in [2]), we give existence results in this context. Moreover, by generalizing the method of [6], we prove a C^∞ - regularity result, and we see that the interval of existence of the C^∞ solution is the maximal interval of existence of the solution in $L^\infty(\mathbb{R}^+ ; H^3(\Omega))$.

1. Main Results

Let Ω be a bounded connected open subset of $\mathbb{R}^3$. We assume that the boundary Γ is a compact manifold of dimension 2, without boundary, and that Ω is locally situated on one side of Γ. Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ such that $\Gamma_j (j = 1, \dots, m)$ are inside of Γ_0 and outside of one another.

We prove the following results

Theorem A. *Let* Γ *be of class* C^{k+3} *and let* $a \in H^{k+2}(\Omega)$, $k \geq 1$, *with* $\operatorname{div} a = 0$ *in* Ω *and* $a \cdot n = 0$ *on* Γ, $\rho_0 \in H^{k+2}(\Omega)$ *with* $\rho_0(x) > 0$ *for each* $x \in \bar{\Omega}$, *and* $b \in L^1(0, T_0 ; H^{k+2}(\Omega)) \cap L^p(0, T_0 ; H^{k+1}(\Omega))$, $p > 1$(1).

Then there exists $T_1 = T_1(k) \in]0, T_0]$, $v \in L^\infty(0, T_1 ; H^{k+2}(\Omega))$ *with* $\frac{\partial v}{\partial t} \in L^p(0, T_1 ; H^{k+1}(\Omega))$, $\rho \in L^\infty(0, T_1 ; H^{k+2}(\Omega))$ *with* $\frac{\partial \rho}{\partial t} \in L^\infty(0, T_1 ; H^{k+1}(\Omega))$, $\pi \in L^p(0, T_1 ; H^{k+2}(\Omega))$ *such that* (v, ρ, π) *is a solution of* (E) *in* Q_{T_1}.

(1) The condition $b \in L^p(0, T_0; H^{k+1}(\Omega))$ can be weakened. By using the same proofs we can choose for instance $b \in L^p(0, T_0; H^1(\Omega))$ and $X = C^0([0, T_1] ; L^2(\Omega))$ in Lemma 2.4.

Theorem B. *Let* Γ *be of class* C^∞, *and let* $a \in C^\infty(\overline{\Omega})$, $\rho_0 \in C^\infty(\overline{\Omega})$, $b \in C^\infty([0, +\infty[\times \overline{\Omega})$. *Then the solution* (v, ρ, π) *of* (E) *belongs to* $C^\infty(\overline{Q}_{T_1})$ *for each* $T_1 \in]0, T^*[$, *where* T^* *determines the maximal interval of existence of the solution* (v, ρ) *in* $L^\infty(\mathbb{R}^+; H^3(\Omega))$.

A uniqueness theorem for problem (E) is proved in [3] (see also Graffi [7]). The same results hold if $\Omega \subset \mathbb{R}^2$.

2. Proof of Theorem A

We suppose that Ω is simply-connected. Otherwise, we can prove the same results by proceeding as in [5], § 4 and [4], § 6.

Let $T \in]0, T_0]$ and let φ be a function in $L^\infty(0, T; H^{k+1}(\Omega)) \cap C^0([0, T]; H^k(\Omega))$ such that for each $t \in [0, T]$

$$\text{div}\,\varphi = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \int_{\Gamma_i} \varphi \cdot n \, d\Gamma = 0 \quad \forall i = 1, \dots, m \; . \tag{2.1}$$

Then there exists a unique solution v of the elliptic system

$$\begin{cases} \text{rot}\, v = \varphi & \text{in} \quad Q_T \, , \\ \text{div}\, v = 0 & \text{in} \quad Q_T \, , \\ v \cdot n = 0 & \text{on} \quad]0, T[\times \Gamma \; . \end{cases} \tag{2.2}$$

Moreover $v \in L^\infty(0, T; H^{k+2}(\Omega)) \cap C^0([0, T]; H^{k+1}(\Omega))$ with

$$\|v\|_{k+2,T} \leqslant c \, \|\varphi\|_{k+1,T} \leqslant cA \quad , \quad c = c(k, \Omega) \; , \tag{2.3}$$

where we have choosen φ such that $\|\varphi\|_{k+1,T} \leqslant A$ (which will be specified in (2.18)).

By Sobolev's theorems, we obtain $v \in L^\infty(0, T; C^1(\overline{\Omega})) \cap C^0(\overline{Q}_T)$, and consequently we can construct the solution ρ of

$$(2.4) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0 & \text{in} \quad Q_T \, , \\ \rho_{|t=0} = \rho_0 & \text{in} \quad \Omega \, , \end{cases}$$

by using the method of characteristics.

Moreover the following estimates hold:

Lemma 2.1 *Let* ρ *be the solution of (2.4). Then* $\rho \in L^\infty(0, T; H^{k+2}(\Omega))$, $\dfrac{\partial \rho}{\partial t} \in L^\infty(0, T; H^{k+1}(\Omega))$ *and*

$$(2.5) \qquad \|\rho\|_{k+2,T} \leqslant \|\rho_0\|_{k+2} \, e^{cAT} \, ,$$

$$(2.6) \qquad \left\|\frac{\partial \rho}{\partial t}\right\|_{k+1,T} \leqslant cA \, \|\rho_0\|_{k+2} \, e^{cAT} \, ,$$

where $c = c(k, \Omega)$.

Proof. Apply the operator D^γ to $(2.4)_1$, where γ is a multi-index with $|\gamma| \leqslant k+2$; multiply by $D^\gamma \rho$ and integrate over Ω. Recalling that

$$((v \cdot \nabla) D^\gamma \rho \, , \, D^\gamma \rho) = 0$$

since $\operatorname{div} v = 0$, $(v \cdot n)_{|\Gamma} = 0$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|D^\gamma \rho\|^2 \leqslant c \, \|D^\gamma \rho\| \sum_{0 \leqslant \sigma < \gamma} \|D^{\gamma-\sigma} v \cdot D^\sigma \nabla \rho\| \, .$$

By adding in γ, for $|\gamma| \leqslant k+2$, one gets

$$\frac{1}{2} \frac{d}{dt} \|\rho\|_{k+2}^2 \leqslant c \, \|\rho\|_{k+2}^2 \, \|v\|_{k+2} \, ,$$

since $H^{k+1}(\Omega)$ is an algebra for $k \geqslant 1$.

Hence

$$\frac{d}{dt} \|\rho\|_{k+2} \leqslant c \, \|v\|_{k+2} \, \|\rho\|_{k+2} \quad ;$$

then from Gronwall's lemma we have (2.5).

From equation $(2.4)_1$ we obtain

$$\left\| \frac{\partial \rho}{\partial t} \right\|_{k+1} \leqslant \|v\|_{k+1} \, \|\rho\|_{k+2} \quad ,$$

and consequently obtain (2.6). □

We now consider the elliptic system

$$(2.7) \qquad \begin{cases} \operatorname{rot} w = 0 & \text{in} \quad \Omega \, , \\ \operatorname{div} w - \dfrac{\nabla \rho}{\rho} \cdot w = \rho \sum\limits_{i,j} (D_i \, v_j)(D_j \, v_i) - \rho \operatorname{div} b & \text{in} \quad \Omega \, , \\ w \cdot n = - \rho \sum\limits_{i,j} (D_i \, n_j) \, v_i \, v_j - \rho \, b \cdot n & \text{on} \quad \Gamma \, , \end{cases}$$

which is equivalent to the Neumann problem

$$(2.8) \qquad \begin{cases} - \Delta \pi + \dfrac{\nabla \rho}{\rho} \cdot \nabla \pi = \rho \sum\limits_{i,j} (D_i \, v_j)(D_j \, v_i) - \rho \operatorname{div} b \equiv f & \text{in} \quad \Omega \, , \\ - \dfrac{\partial \pi}{\partial n} = - \rho \sum\limits_{i,j} (D_i \, n_j) \, v_i \, v_j - \rho \, b \cdot n \equiv g & \text{on} \quad \Gamma \, , \end{cases}$$

where $- \nabla \pi = w$.

We need some estimates for the solution of the elliptic problem (2.8). We shall see that

$$(2.9) \qquad \|\nabla \pi\|_{k+2} \leqslant c \left(k, \Omega, \left\| \frac{\nabla \rho}{\rho} \right\|_{k+1} \right) (\|f\|_{k+1} + \|g\|_{k+2}) \, , \quad \forall k \geqslant 1 \, ,$$

and

$$(2.10) \qquad \|\nabla \pi\|_{k+2} \leqslant c\,(k, \Omega, \rho_0, \|\rho\|_{k+1}, \|f\|_k, \|g\|_{k+1})(1 + \|f\|_{k+1} + \|g\|_{k+2} + \|\rho\|_{k+2}) \, , \quad \forall k \geqslant 2 \, .$$

We need this last estimate only for the C^∞ regularity result.

As in [4] one has the existence of a solution of (2.8) (unique up to an arbitrary constant) and the estimate

$$\|\nabla \pi\|_{C^{1+\alpha}} \leqslant c(\alpha, \Omega, \|\frac{\nabla\rho}{\rho}\|_{C^\alpha})(\|f\|_{C^\alpha} + \|g\|_{C^{1+\alpha}}) \quad , \quad 0<\alpha<1 \ ,$$

Letting $\alpha = 1/2$, it follows by Sobolev's embedding theorems that[2]

$$\|\nabla \pi\|_1 \leqslant c(\Omega, \|\frac{\nabla\rho}{\rho}\|_2)\ (\|f\|_2 + \|g\|_3) \ .$$

By a straightforward calculation one easily sees that this estimate holds also for $\|\nabla\pi\|_2$ and $\|\nabla\pi\|_3$, and by induction one gets

$$\|\nabla \pi\|_{k+2} \leqslant c(k, \Omega, \|\frac{\nabla\rho}{\rho}\|_k)\ \|\frac{\nabla\rho}{\rho}\|_{k+1}\ (\|f\|_k + \|g\|_{k+1}) +$$

$$+\ c(k,\Omega)\,(\|f\|_{k+1} + \|g\|_{k+2}) \quad , \quad \forall\, k \geqslant 2 \ .$$

Hence (2.9) and (2.10) hold.

From (2.9), (2.3) and (2.5) it follows that the unique solution w of (2.7) belongs to $L^1(0, T : H^{k+1}(\Omega))$; and moreover

$$\int_0^T \|w(t)\|_{k+1}\, dt \leqslant \overline{c}(A, T) \ ,$$

where $\overline{c}$ is a non-decreasing function in the variables A and T ($\overline{c}$ depends also on p, k, Ω, b and ρ_0). In addition $\lim_{T\to 0^+} \overline{c}(A, T) = 0.$

We want to study the equation

$$(2.11)\qquad \begin{cases} \dfrac{\partial\zeta}{\partial t} + (v\cdot\nabla)\zeta = \beta + w \wedge \dfrac{\nabla\rho}{\rho^2} + (\zeta\cdot\nabla)v & \text{in } Q_T \ , \\[2ex] \zeta_{|t=0} = \alpha & \text{in } \Omega \ , \end{cases}$$

where $\alpha \equiv \text{rot a}$ and $\beta \equiv \text{rot b}$.

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(2) One can also start from the more precise estimate (see Ladyženskaja - Ural'ceva [8], chap III, § 5 and 6)

$$\|\nabla\pi\|_1 \leqslant c(\Omega, \|\frac{\nabla\rho}{\rho}\|_{L^\infty})\,(\|f\| + \|g\|_1) \ .$$

As for equation (2.4), we can construct the solution ζ by using the method of characteristics (see also [5]). Moreover, one has the following estimates:

Lemma 2.2 *Let* ζ *be the solution of (2.11). Then* $\zeta \in L^\infty(0, T; H^{k+1}(\Omega))$,

$\frac{\partial \zeta}{\partial t} \in L^p(0, T; H^k(\Omega))$ *and*

$$\|\zeta\|_{k+1,T} \leqslant [\|\alpha\|_{k+1} + \overline{c}(A, T)] e^{cAT} , \tag{2.12}$$

$$\int_0^T \|\frac{\partial \zeta}{\partial t}(t)\|_k^p \, dt \leqslant \overline{c}_1(A, T) [\|\alpha\|_{k+1}^p + 1] , \tag{2.13}$$

where $\overline{c}$, $\overline{c}_1$ *are non-decreasing functions in the variables* A *and* T ($\overline{c}, \overline{c}_1$ *depend also on* p, k, Ω, b *and* ρ_0), *and* $\lim_{T \to 0^+} \overline{c}(A, T) = 0$.

Proof. Apply the operator D^γ to $(2.11)_1$, where γ is a multi-index with $|\gamma| \leqslant k+1$; multiply by $D^\gamma \zeta$ and integrate over Ω. Recalling that

$$((v \cdot \nabla) D^\gamma \zeta, D^\gamma \zeta) = 0 ,$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \|D^\gamma \zeta\|^2 \leqslant c \|D^\gamma \zeta\| \{ \|D^\gamma \beta\| + \sum_{0 \leqslant \sigma \leqslant \gamma} \|D^\sigma w \cdot D^{\gamma-\sigma} \nabla(\frac{1}{\rho})\| +$$

$$+ \sum_{0 \leqslant \sigma \leqslant \gamma} \|D^\sigma \zeta \cdot D^{\gamma-\sigma} Dv\| + \sum_{0 \leqslant \sigma < \gamma} \|D^{\gamma-\sigma} v \cdot D^\sigma D\zeta\| \} .$$

Adding in γ, for $|\gamma| \leqslant k+1$, one obtains

$$\frac{1}{2} \frac{d}{dt} \|\zeta\|_{k+1}^2 \leqslant c \|\zeta\|_{k+1} \{ \|b\|_{k+2} + \|w\|_{k+1} \|\frac{1}{\rho}\|_{k+2} + \|v\|_{k+2} \|\zeta\|_{k+1} \} ,$$

since $H^{k+1}(\Omega)$ is an algebra for $k \geqslant 1$.

Hence, by Gronwall's lemma

$$\|\zeta(t)\|_{k+1} \leqslant \left[\|\alpha\|_{k+1} + c \int_0^T (\|b(s)\|_{k+2} + \|w(s)\|_{k+1} \, \| \frac{1}{\rho}(s)\|_{k+2}) \, ds \right] \cdot$$

$$\cdot \exp\left(c \int_0^t \|v(s)\|_{k+2} \, ds \right) \leqslant [\|\alpha\|_{k+1} + \overline{c}(A,T)] \, e^{cAt} \, .$$

Finally, from equation $(2.11)_1$ one obtains easily (2.13). Recall that from (2.9) one gets

$$w \in L^p(0,T\,;\,H^{k+1}(\Omega)) \qquad , \qquad k \geqslant 1 \, .$$

If $k = 1$ we use instead of (2.9) a corresponding estimate obtained via the note (2). □

Lemma 2.3 *Let* ζ *be the solution of (2.11). Then, for each* $t \in [0,T]$.

(2.14) $$\operatorname{div} \zeta = 0 \qquad \text{a.e. in} \quad \Omega \, ,$$

(2.15) $$\int_{\Gamma_i} \zeta \cdot n \, d\Gamma = 0 \qquad \forall \, i = 1, \dots, m \, .$$

Proof. From the general formula

$$(v \cdot \nabla)\zeta - (\zeta \cdot \nabla)v = v \operatorname{div} \zeta - \zeta \operatorname{div} v - \operatorname{rot}(v \wedge \zeta)$$

it follows that

(2.16) $$\frac{\partial \zeta}{\partial t} + v \operatorname{div} \zeta = \operatorname{rot}(v \wedge \zeta) + \beta + w \wedge \frac{\nabla \rho}{\rho^2} \, .$$

On the other hand $\beta + w \wedge \frac{\nabla \rho}{\rho^2} = \operatorname{rot}\left(b + \frac{w}{\rho} \right)$. Hence applying the operator div to both sides of (2.16) one gets.

(2.17)
$$\begin{cases} \dfrac{\partial(\operatorname{div}\zeta)}{\partial t} + v\cdot\nabla(\operatorname{div}\zeta) = 0 & \text{in } Q_T, \\ (\operatorname{div}\zeta)_{|t=0} = \operatorname{div}\alpha = 0 & \text{in } \Omega, \end{cases}$$

since div rot = 0. Hence div $\zeta = 0$.

Finally, by using (2.16) we have

$$\frac{d}{dt}\int_{\Gamma_i}\zeta\cdot n\, d\Gamma = \int_{\Gamma_i}\frac{\partial\zeta}{\partial t}\cdot n\, d\Gamma = 0 \qquad \forall\, i = 1,\dots,m$$

since $\int_{\Gamma_i}$ rot $G\cdot n\, d\Gamma = 0$ for each G, and $(v\,.\,n)_{|\Gamma} = 0$. Hence, for each $t\in[0,T]$,

$$\int_{\Gamma_i}\zeta\cdot n\, d\Gamma = \int_{\Gamma_i}\alpha\cdot n\, d\Gamma = 0 \qquad \forall\, i = 1,\dots,m\,. \qquad \square$$

We can now construct a fixed point for the map $F:\varphi\to\zeta$. In fact, choose

(2.18)
$$A > \|\alpha\|_{k+1}\,.$$

Then from estimate (2.12) and from Lemma 2.3 one sees that there exists $T_1 \in\,]0, T_0]$ such that the set

$$S \equiv \{\varphi\in L^\infty(0,T_1;H^{k+1}(\Omega)) \cap C^0([0,T_1];H^k(\Omega)) \mid \|\varphi\|_{k+1,T_1} \leqslant A\,,$$
$$\varphi \text{ satisfies } (2.1)\}$$

satisifes $F[S]\subset S$, where F is related to the interval $]0,T_1[$. S is obviously convex and closed in $X\equiv C^0([0,T_1];H^k(\Omega))$.

Lemma 2.4. *The map* F *has a fixed point in* S.

Proof. We utilize the Schauder's fixed point theorem in the space X. From Lemma 2.2 one has

$$F[S] \subset \{\zeta \in S \mid \int_0^{T_1} \|\frac{\partial \zeta}{\partial t}(t)\|_k^p \, dt \leqslant \bar{c}_1(A, T_1)[\|\alpha\|_{k+1}^p + 1]\}.$$

In particular $F[S]$ is bounded in $C^{\alpha}([0, T_1]; H^k(\Omega)) \cap L^{\infty}(0, T_1; H^{k+1}(\Omega))$, $\alpha = (p-1)/p$, and from the Ascoli-Arzelà's theorem $F[S]$ is relatively compact in X.

Let now $\varphi, \varphi^n \in S$, $\varphi^n \to \varphi$ in X. Then the solutions v^n of the elliptic system (2.2) converge in $C^0([0, T_1]; H^{k+1}(\Omega))$ to v. Moreover for ρ_n and ρ one obtains

$$\frac{1}{2}\frac{d}{dt}\|\rho_n - \rho\|^2 \leqslant \|\rho_n - \rho\| \, \|\nabla\rho\|_{L^{\infty}(Q_{T_1})} \|v^n - v\|_{0,T_1},$$

and consequently $\rho_n \to \rho$ in $L^{\infty}(0, T_1; L^2(\Omega))$.

Hence by (2.5), (2.6) and a compactness argument it follows that $\rho_n \to \rho$ in $C^0([0, T_1]; H^2(\Omega))$. In particular

$$\frac{\nabla\rho_n}{\rho_n} \to \frac{\nabla\rho}{\rho} \text{ in } L^{\infty}(0, T_1; L^2(\Omega)), \quad \frac{\nabla\rho_n}{\rho_n^2} \to \frac{\nabla\rho}{\rho^2} \text{ in } L^{\infty}(0, T_1; L^2(\Omega)).$$

From the Neumann problem (2.8) one obtains with a straightforward calculation that $w^n \to w$ in $L^1(0, T_1; H^1(\Omega))$.

Finally, by evaluating $\frac{d}{dt}\|\zeta^n - \zeta\|^2$ in a standard way, from equations $(2.11)_1$ one easily gets $\zeta^n \to \zeta$ in $C^0([0, T_1]; L^2(\Omega))$. By the compactness of $\overline{F[S]}$, this implies that $\zeta^n \to \zeta$ in X. □

Let $\varphi = \zeta$ be a fixed point of F. Then the functions v, ρ and π determined in (2.2), (2.4) and (2.8) by this φ are the solutions of system (E). In fact, by differentiating in t system (2.2) we prove that $\frac{\partial v}{\partial t} \in L^p(0, T_1; H^{k+1}(\Omega))$. Then by $(2.11)_1$ and (2.7) we have

$$\begin{cases} \operatorname{rot}\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b - \dfrac{w}{\rho}\right] = \dfrac{\partial \zeta}{\partial t} + (v\cdot\nabla)\zeta - (\zeta\cdot\nabla)v - \beta - w\wedge\dfrac{\nabla\rho}{\rho^2} = 0 & \text{in } \Omega\,, \\ \operatorname{div}\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b - \dfrac{w}{\rho}\right] = \sum_{i,j}(D_i v_j)(D_j v_i) - \operatorname{div} b - \dfrac{1}{\rho}\operatorname{div} w + w\cdot\dfrac{\nabla\rho}{\rho^2} = 0 \text{ in } \Omega\,, \\ \left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b - \dfrac{w}{\rho}\right]\cdot n = -\sum_{i,j}(D_i n_j)\, v_i\, v_j - b\cdot n - \dfrac{1}{\rho}\, w\cdot n = 0 & \text{on } \Gamma\,. \end{cases}$$

Since $w = -\nabla\pi$, we have obtained equation $(E)_1$.

Moreover

$$\begin{cases} \operatorname{rot}(v_{|t=0} - a) = \zeta_{|t=0} - \alpha = 0 & \text{in } \Omega\,, \\ \operatorname{div}(v_{|t=0} - a) = 0 & \text{in } \Omega\,, \\ (v_{|t=0} - a)\cdot n = 0 & \text{on } \Gamma\,, \end{cases}$$

hence $v_{|t=0} = a$ in Ω.

3. Proof of theorem B.

We now prove that $v(t)$, $\rho(t)$ and $\pi(t)$ belong to $C^\infty(\overline{\Omega})$ for each $t \in [0, T_1]$, where $0 < T_1 < T^*$, and $[0, T^*[$ is the maximal interval of existence for the solution (v, ρ) in $L^\infty(\mathbb{R}^+ ; H^3(\Omega))$.

It is sufficient to prove that $T^*(k) = T^*(1)$ for each $k \geq 1$. Since it is clear that $T^*(k)$ is non-increasing in k, we want to prove that $T^*(k) \geq T^*(1)$.

Let $k \geq 2$. Applying the operator D^γ to $(E)_1$, where γ is a multi-index with $|\gamma| \leq k+2$, multiplying by $D^\gamma v$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D^\gamma v\|^2 \leq \|D^\gamma b\|\,\|D^\gamma v\| + c \sum_{0\leq\sigma<\gamma} \|D^{\gamma-\sigma} v \cdot D^\sigma Dv\|\,\|D^\gamma v\| + $$

$$+ \|D^\gamma(\frac{\nabla\pi}{\rho})\|\,\|D^\gamma v\|\,,$$

since $((v\cdot\nabla)D^\gamma v, D^\gamma v) = 0$.

By adding in γ for $|\gamma| \leqslant k+2$ we obtain (see also [6] , (1.7))

$$(3.1) \qquad \frac{d}{dt} \|v\|_{k+2} \leqslant c(k, \Omega) \{ \|b\|_{k+2} + \|v\|_{k+1} \|v\|_{k+2} \} + \| \frac{\nabla\pi}{\rho} \|_{k+2} .$$

From equation $(E)_4$ we have

$$(3.2) \qquad \frac{d}{dt} \|\rho\|_{k+2} \leqslant c(k, \Omega) \{ \|v\|_{k+1} \|\rho\|_{k+2} + \|\rho\|_{k+1} \|v\|_{k+2} \}.$$

On the othe hand from (2.10) and (3.2) one has

$$\| \frac{\nabla\pi}{\rho} \|_{k+2} \leqslant c(k, \Omega) \left[\|\nabla\pi\|_{k+1} \| \frac{1}{\rho} \|_{k+2} + \|\nabla\pi\|_{k+2} \| \frac{1}{\rho} \|_{k+1} \right] \leqslant$$

$$\leqslant c(k, \Omega, \rho_0, \|\rho\|_{k+1}, \|f\|_k, \|g\|_{k+1}) [1 + \|f\|_{k+1} + \|g\|_{k+2} + \|\rho\|_{k+2}] .$$

Recalling the definition of f and g, we obtain

$$(3.3) \qquad \| \frac{\nabla\pi}{\rho} \|_{k+2} \leqslant c(k, \Omega, \rho_0, b, \|\rho\|_{k+1}, \|v\|_{k+1}) [1 + \|v\|_{k+2} + \|\rho\|_{k+2}] .$$

Hence, from (3.1), (3.2) and (3.3)

$$(3.4) \qquad \frac{d}{dt} (\|v\|_{k+2} + \|\rho\|_{k+2}) \leqslant c(k, \Omega, \rho_0, b, \|\rho\|_{k+1}, \|v\|_{k+1}) [1 + \|v\|_{k+2} + \|\rho\|_{k+2}] .$$

Consequently, by induction on k we prove that $T^*(k) \geqslant T^*(1)$ for each $k \geqslant 1$.

The regularity in t is also proved by induction by verifying that if

$v^{(\ell)} \equiv \frac{d^\ell}{dt^\ell} v$, $\rho^{(\ell)} \equiv \frac{d^\ell}{dt^\ell} \rho$, $\ell \geqslant 0$, belong to $L^\infty(0, T_1; H^{k+2}(\Omega))$ for each $k \geqslant 1$, then the same holds for $v^{(\ell+1)}$ and $\rho^{(\ell+1)}$.

Formally, this can be done by differentiating in t equations $(E)_1$, $(E)_4$ and (2.8), recalling that this last equation gives

$$\begin{cases} -\Delta\pi^{(\ell)} + \dfrac{\nabla\rho}{\rho}\cdot\nabla\pi^{(\ell)} = f^{(\ell)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} \left(\dfrac{\nabla\rho}{\rho}\right)^{(\ell-j)} (\nabla\pi)^{(j)} \equiv F^{(\ell)} & \text{in } \Omega, \\ -\dfrac{\partial\pi^{(\ell)}}{\partial n} = g^{(\ell)} & \text{on } \Gamma. \end{cases}$$

Hence $\nabla\pi^{(\ell)}$ satisfies (2.9) with f and g replaced by $F^{(\ell)}$ and $g^{(\ell)}$ respectively.

For the complete proof we must use the well known method of differential quotients (see for instance Lions [9] , chap. V).

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