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On the Euler Equations for Nonhomogeneous Fluids (II)

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1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the motion of a nonhomogeneous ideal incompressible fluid in a bounded connected open subset Ω of $\mathbb{R}^3$. We assume that the boundary Γ is a compact manifold of dimension 2, without boundary, and that Ω is locally situated on one side of Γ. Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ such that Γ_j $(j = 1, \ldots, m)$ are inside of Γ_0 and outside of one another. In Sections 2 and 3 we assume that Ω is simply connected; in Section 4 we drop this condition. We denote by $v(t, x)$ the velocity field, by $\rho(t, x)$ the mass density, and by $\pi(t, x)$ the pressure. The Euler equations of the motion are (see for instance Sédov [14, Chap. IV, Sect. 1, p. 164])

$$
\begin{aligned}
\rho\left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b\right] &= -\nabla\pi && \text{in} \quad Q_{T_0} \equiv [0, T_0] \times \bar{\Omega},\\
\operatorname{div} v &= 0 && \text{in} \quad Q_{T_0},\\
v \cdot n &= 0 && \text{on} \quad [0, T_0] \times \Gamma,\\
\frac{\partial \rho}{\partial t} + v \cdot \nabla\rho &= 0 && \text{in} \quad Q_{T_0},\\
\rho\,|_{t=0} &= \rho_0 && \text{in} \quad \bar{\Omega},\\
v\,|_{t=0} &= a && \text{in} \quad \bar{\Omega},
\end{aligned}
\tag{E}
$$

where $n = n(x)$ is the unit outward normal to the boundary Γ, $b = b(t, x)$ is the external force field, and $a = a(x)$, $\rho_0 = \rho_0(x)$ are the initial velocity field and the initial mass density, respectively.

Nonhomogeneous ideal incompressible fluids have been studied by several authors; see, for instance, Sédov [14], Zeytounian [18], Yih [17]. In some problems concerning oceanography (see, for instance, LeBlond and Mysak [10])

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or, more generally, rotating systems (see also Kazhikhov [9]), Eq. $(E)_1$ is replaced by

$$\rho\left[\frac{\partial v}{\partial t}+(v\cdot\nabla)v+2\omega\wedge v-b\right]=-\nabla\pi,$$

where ω is the angular velocity. The perturbation term $2\,\omega\wedge v$ does not give rise to any difficulty and our results and proofs hold again if one assumes that $\omega\in C^{0,1+\lambda}(Q_{T_0})$.

For the case in which the fluid is homogeneous, i.e., the density ρ_0 (and consequently ρ) is constant, Eqs. (E) have been studied by several authors. For the three-dimensional case see, for instance, Lichtenstein [11], Ebin and Marsden [6], Swann [15], Kato [8], Bourguignon and Brezis [5], Temam [16], Bardos and Frisch [2].

For nonhomogeneous fluids, Marsden [13] has proved (in the n-dimensional case) the existence of a local solution to problem (E), under the assumption that the external force field $b(t,x)$ is divergence free and tangential to the boundary, i.e., $\operatorname{div} b=0$ in Q_{T_0} and $b\cdot n=0$ on $[0,T_0]\times\Gamma$. The proof relies on techniques of Riemannian geometry on infinite-dimensional manifolds.[1] In a previous paper [4], we have proved, in the two-dimensional case, the existence of a local solution to problem (E) without any restriction on the external field $b(t,x)$. In this paper we prove the corresponding result for the three-dimensional case, i.e.,

THEOREM A. *Let Γ be of class $C^{3+\lambda}$, $0<\lambda<1$, and let $a\in C^{1+\lambda}(\bar\Omega)$ with $\operatorname{div} a=0$ in $\bar\Omega$ and $a\cdot n=0$ on Γ, $\rho_0\in C^{1+\lambda}(\bar\Omega)$ with $\rho_0(x)>0$ for each $x\in\bar\Omega$, and $b\in C^{0,1+\lambda}(Q_{T_0})$. Then there exists $T_1\in[0,T_0]$, $v\in C^{1,1+\lambda}(Q_{T_1})$, $\rho\in C^{1+\lambda,1+\lambda}(Q_{T_1})$ $\pi\in C^{0,2+\lambda}(Q_{T_1})$ such that (v,ρ,π) is a solution of* (E) *in Q_{T_1}.*

A uniqueness theorem for problem (E) is proved by Graffi [20]. See also [3]. For the study of nonhomogeneous *viscous* incompressible fluids see Kazhikhov [9], [21] Antoncev and Kazhikhov [1], Ladyzhenskaya and Solonnikov [22], Lions [12], and Simon [23].

2. PRELIMINARIES AND EXISTENCE OF A LOCAL SOLUTION OF THE AUXILIARY SYSTEM (A)[2]

In this section and in Section 3 we assume that Ω is simply connected. We use the notations introduced in [4]. We need only to define for a vector function φ the operator

$$\operatorname{curl}\varphi=\left(\frac{\partial\varphi_3}{\partial x_2}-\frac{\partial\varphi_2}{\partial x_3},\frac{\partial\varphi_1}{\partial x_3}-\frac{\partial\varphi_3}{\partial x_1},\frac{\partial\varphi_2}{\partial x_1}-\frac{\partial\varphi_1}{\partial x_2}\right).$$

[1] For the analytic case on compact manifolds without boundary see [19].

[2] See the end of this section.

In the following $\varphi(t, x) \in C^{0,\lambda}(Q_T)$, $T \in]0, T_0]$, will be a generic element of the sphere

$$\| \varphi \|_{0,\lambda} \leqslant A \tag{2.1}$$

(where the radius A is a positive constant which we will specify below) such that for each $t \in [0, T]$

$$\operatorname{div} \varphi = 0 \quad \text{in} \quad \mathscr{D}'(\Omega), \qquad \int_{\Gamma_i} \varphi \cdot n \, d\Gamma = 0 \qquad \forall i = 1,..., m. \tag{2.2}$$

This condition is equivalent to the existence of a vector function v_0 such that $\varphi = \operatorname{curl} v_0$; see for instance Foias and Temam [7, Proposition 1.3]. We denote by c, c_1, c_2,..., positive constants depending at most on λ and Ω.

Under our assumptions on Ω, the conditions on φ assure the existence of a unique solution $v \in C^{0,1+\lambda}(Q_T)$ of the elliptic system

$$\begin{aligned} \operatorname{curl} v &= \varphi \qquad \text{in } Q_T, \\ \operatorname{div} v &= 0 \qquad \text{in } Q_T, \\ v \cdot n &= 0 \qquad \text{in } [0, T] \times \Gamma. \end{aligned} \tag{2.3}$$

Moreover,

$$\| v \|_{0,1+\lambda} \leqslant c \| \varphi \|_{0,\lambda} \leqslant cA, \tag{2.4}$$

which corresponds to inequality (3.4) in [4].

As in [4] we construct the functions $U(s, t, x)$, $\rho(t, x)$, and $w(t, x)$ and we prove the corresponding Lemmas 3.2, 3.3, 4.1, 4.2.

We want now to study the equation

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + (v \cdot \nabla) \zeta = \beta + w \wedge \frac{\nabla \rho}{\rho^2} + (\zeta \cdot \nabla) v \qquad &\text{in} \quad Q_T, \\ \zeta |_{t=0} = \alpha \qquad &\text{in} \quad \bar{\Omega}, \end{aligned} \tag{2.5}$$

where $\alpha \equiv \operatorname{curl} a$ and $\beta \equiv \operatorname{curl} b$.

To solve (2.5) we use the well known method of characteristics. Consider in Q_T the C^1-change of variable $(t, x) \to (t, x')$ defined by

$$x' = U(0, t, x), \qquad \text{i.e.} \quad x = U(t, 0, x'), \quad \forall t \in [0, T]. \tag{2.6}$$

Set $\gamma \equiv \beta + w \wedge (\nabla \rho / \rho^2)$; system (2.5) becomes then

$$\begin{aligned} \frac{d\tilde{\zeta}}{dt}(t, x') &= \gamma(t, U(t, 0, x')) + Dv(t, U(t, 0, x')) \cdot \tilde{\zeta}(t, x'). \\ \tilde{\zeta}(0, x') &= \alpha(x'), \end{aligned} \tag{2.7}$$

where

$$\tilde{\zeta}(t, x') = \zeta(t, U(t, 0, x')), \tag{2.8}$$

i.e.,

$$\zeta(t, x) = \tilde{\zeta}(t, U(0, t, x)). \tag{2.9}$$

Dv is the matrix with $\partial v_i/\partial x_j$ in the ith row, jth column, and $Dv \cdot \tilde{\zeta}$ is the matrix product.

The linear ordinary system (2.7) has a unique solution for each $x' \in \bar{\Omega}$. Since $\alpha(x')$, $\gamma(t, U(t, 0, x'))$ and $Dv(t, U(t, 0, x'))$ are not differentiable with respect to x', $\tilde{\zeta}(t, x')$ is generally not differentiable with respect to this last variable; hence $\zeta(t, x)$, being not differentiable in x, is not a classical solution of (2.5). For this reason we must define $\zeta(t, x)$ by (2.9). We denote by $\bar{c}$, $\bar{c}_1$, $\bar{c}_2$,..., positive constants depending at most on λ, Ω, ρ_0, and b.

LEMMA 2.1. *The solution $\tilde{\zeta}$ of system (2.7) satisfies*

$$\begin{aligned}
\|\tilde{\zeta}\|_\infty &\leqslant (\|\alpha\|_\infty + T\|\gamma\|_\infty)\, e^{T\|Dv\|_\infty} \leqslant \|\alpha\|_\infty\, e^{cTA} + T\bar{c}(A, T),\\
[\tilde{\zeta}]_{0,\lambda} &\leqslant ([\alpha]_\lambda + Te^{\lambda T[v]_{0,\mathrm{lip}}}[\gamma]_{0,\lambda})\, e^{T\|Dv\|_\infty}\\
&\quad + (\|\alpha\|_\infty + T\|\gamma\|_\infty)\, T[Dv]_{0,\lambda}\, e^{T(2\|Dv\|_\infty + \lambda[v]_{0,\mathrm{lip}})}\\
&\leqslant [\alpha]_\lambda\, e^{cTA} + T\bar{c}(A, T)\,(1 + \|\alpha\|_\infty),\\
[\tilde{\zeta}]_{\lambda,0} &\leqslant [\|\gamma\|_\infty + (\|\alpha\|_\infty + T\|\gamma\|_\infty)\,\|Dv\|_\infty\, e^{T\|Dv\|_\infty}]\, T^{1-\lambda}\\
&\leqslant T^{1-\lambda}\bar{c}(A, T)\,(1 + \|\alpha\|_\infty),
\end{aligned} \tag{2.10}$$

where $\bar{c}(A, T)$ is nondecreasing in the variables A and T

Proof From (2.7) we have

$$\begin{aligned}
\frac{d\,|\tilde{\zeta}(t, x')|}{dt} &\leqslant \left|\frac{d\tilde{\zeta}(t, x')}{dt}\right| \leqslant \|\gamma\|_\infty + \|Dv\|_\infty\, |\tilde{\zeta}(t, x')|,\\
|\tilde{\zeta}(0, x')| &= |\alpha(x')|.
\end{aligned}$$

By comparison theorems and (2.4) one obtains $(2.10)_1$. Moreover we have

$$\begin{aligned}
\frac{d}{dt}\,|\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')| &\leqslant \left|\frac{d}{dt}\,[\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')]\right|\\
&\leqslant ([\gamma]_{0,\lambda}\,[U]^\lambda_{0,\mathrm{lip}} + \|\tilde{\zeta}\|_\infty\,[Dv]_{0,\lambda}\,[U]^\lambda_{0,\mathrm{lip}})\,|x' - x''|^\lambda\\
&\quad + \|Dv\|_\infty\,|\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')|,\\
|\tilde{\zeta}(0, x') - \tilde{\zeta}(0, x'')| &\leqslant [\alpha]_\lambda\,|x' - x''|^\lambda.
\end{aligned}$$

From $(2.10)_1$ and estimate $(3.7)_1$ of [4] we obtain

$$\frac{d}{dt}\,|\tilde{\zeta}(t,x') - \tilde{\zeta}(t,x'')| \leqslant [[\gamma]_{0,\lambda} + (\|\alpha\|_\infty + T\|\gamma\|_\infty)\,[Dv]_{0,\lambda}\,e^{T\|Dv\|_\infty}]\,e^{\lambda T[v]_{0,\mathrm{lip}}} \times |x' - x''|^\lambda + \|Dv\|_\infty\,|\tilde{\zeta}(t,x') - \tilde{\zeta}(t,x'')|\,.$$

By comparison theorems we have $(2.10)_2$.

Finally, from (2.7), $(2.10)_1$, and

$$|\tilde{\zeta}(t,x') - \tilde{\zeta}(s,x')| = \left|\int_s^t \frac{d}{d\tau}\,\tilde{\zeta}(\tau,x')\,d\tau\right|$$

one easily gets $(2.10)_3$. ∎

From this lemma, (2.9), and estimate $(3.7)_1$, $(3.7)_2$ of [4], one easily obtains

LEMMA 2.2. *The function $\zeta(t,x)$ defined in (2.8) is such that $\zeta \in C^{\lambda,\lambda}(Q_T)$ and*

$$\begin{aligned}
&\|\zeta\|_\infty = \|\tilde{\zeta}\|_\infty \leqslant \|\alpha\|_\infty\,e^{cTA} + T\bar{c}(A,T),\\
&[\zeta]_{0,\lambda} \leqslant [\tilde{\zeta}]_{0,\lambda}\,[U]^\lambda_{0,\mathrm{lip}} \leqslant [\alpha]_\lambda\,e^{cTA} + T\bar{c}(A,T)\,(1 + \|\alpha\|_\infty),\\
&[\zeta]_{\lambda,0} \leqslant [\tilde{\zeta}]_{\lambda,0} + [\tilde{\zeta}]_{0,\lambda}\,[U]^\lambda_{\mathrm{lip},0} \leqslant c_1 A^\lambda[\alpha]_\lambda\,e^{cTA} + T^{1-\lambda}\bar{c}(A,T)\,(1 + \|\alpha\|_\infty).
\end{aligned} \tag{2.11}$$

Now we want to prove that for each $t \in [0,T]$ $\operatorname{div}\zeta = 0$ in $\mathscr{D}'(\Omega)$ and $\int_{\Gamma_i} \zeta\cdot n\,d\Gamma = 0$ for each $i = 1,\ldots,m$. First of all we observe that

$$\gamma = \operatorname{curl} g, \qquad g \in C^{0,1+\lambda}(Q_T)$$

since $w \wedge \nabla\rho/\rho^2 = \operatorname{curl} w/\rho$, as one easily sees.

LEMMA 2.3. *Let $\zeta(t,x)$ be defined by (2.9). Then*

$$\operatorname{div}\zeta = 0 \qquad \textit{in} \qquad \mathscr{D}'(\Omega)$$

and (2.12)

$$\int_{\Gamma_i} \zeta\cdot n\,d\Gamma = 0 \qquad \forall i = 1,\ldots,m, \quad \forall t \in [0,T].$$

Proof. Suppose that $a \in C^2(\bar{\Omega})$, $g \in C^{0,2}(Q_T)$, $v \in C^{0,2}(Q_T)$,

$$\operatorname{div} v = 0 \text{ in } Q_T, \qquad \text{and} \qquad v\cdot n = 0 \text{ on } [0,T]\times\Gamma.$$

Then the solution $\tilde{\zeta}$ of (2.7) belongs to $C^1(Q_T)$, and consequently $\zeta \in C^1(Q_T)$ is a classical solution of (2.5).

Since

$$(v\cdot\nabla)\zeta - (\zeta\cdot\nabla)v = v\operatorname{div}\zeta - \zeta\operatorname{div} v - \operatorname{curl}(v\wedge\zeta), \tag{2.13}$$

we obtain that ζ is the solution of

$$\frac{\partial \zeta}{\partial t} + v \operatorname{div} \zeta = \operatorname{curl}(v \wedge \zeta) + \gamma \qquad \text{in} \qquad Q_T ,$$

$$\zeta\,|_{t=0} = \alpha \qquad \text{in} \qquad \bar{\Omega}.$$

Let $\theta \in C^{1,2}(Q_T)$, $\theta = 0$ on $[0, T] \times \Gamma$, $\theta(T, x) = 0$ for each $x \in \bar{\Omega}$. We obtain

$$\int_{Q_T} \frac{\partial \zeta}{\partial t} \cdot \nabla \theta \, dx \, dt + \int_{Q_T} (\operatorname{div} \zeta) v \cdot \nabla \theta \, dx \, dt = 0,$$

since curl grad $= 0$ and $\nabla \theta \wedge n = 0$ on $[0, T] \times \Gamma$. By integrating by parts

$$- \int_{Q_T} \zeta \cdot \nabla \frac{\partial \theta}{\partial t} \, dx \, dt + \int_{Q_T} (\operatorname{div} \zeta) v \cdot \nabla \theta \, dx \, dt = 0,$$

since $\theta\,|_{t=T} = 0$, $\operatorname{div} \zeta\,|_{t=0} = \operatorname{div} \alpha = 0$ and $\theta = 0$ on $[0, T] \times \Gamma$.

Moreover

$$- \int_{Q_T} \zeta \cdot \nabla \frac{\partial \theta}{\partial t} \, dx \, dt = \int_{Q_T} \operatorname{div} \zeta \, \frac{\partial \theta}{\partial t} \, dx \, dt$$

since $\theta\,|_{[0,T] \times \Gamma} = 0$.

Hence we have

$$\int_{Q_T} \operatorname{div} \zeta \left(\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta \right) dx \, dt = 0,$$

and consequently

$$\int_{Q_T} (\operatorname{div} \zeta) \, \psi \, dx \, dt = 0 \qquad \forall \psi \in \mathscr{D}(Q_T),$$

since the solution $\theta(t, x)$ of

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = \psi \qquad \text{in} \qquad Q_T ,$$

$$\theta\,|_{t=T} = 0 \qquad \text{in} \qquad \bar{\Omega}$$

is in $C^{1,2}(Q_T)$ and satisfies $\theta\,|_{[0,T] \times \Gamma} = 0$.

In conclusion we have $\operatorname{div} \zeta = 0$ in Q_T. Moreover

$$\frac{d}{dt} \int_{\Gamma_i} \zeta \cdot n \, d\Gamma = \int_{\Gamma_i} \frac{\partial \zeta}{\partial t} \cdot n \, d\Gamma = \int_{\Gamma_i} \gamma \cdot n \, d\Gamma + \int_{\Gamma_i} [(\zeta \cdot \nabla) v - (v \cdot \nabla) \zeta] \cdot n \, d\Gamma$$

$$= 0$$

by using (2.13). Hence for each $i = 1, \dots, m$

$$\int_{\Gamma_i} \zeta \cdot n \, d\Gamma = \int_{\Gamma_i} \alpha \cdot n \, d\Gamma = 0 \qquad \forall t \in [0, T].$$

If a, g, and v are not regular, we can approximate them in the following way. By using the Friedrichs mollifiers we can find

$$a^m \in C^{2+\lambda}(\bar{\Omega}), \; a^m \to a \text{ in } C^{1+\lambda/2}(\bar{\Omega}); \qquad g^m \in C^{0,2+\lambda}(Q_T),$$
$$g^m \to g \text{ in } C^{0,1+\lambda/2}(Q_T); \; \tilde{v}^m \in C^{0,2+\lambda}(Q_T), \; \tilde{v}^m \to v \text{ in } C^{0,1+\lambda/2}(Q_T).$$

Hence we have that

$$\begin{aligned} \alpha^m &= \operatorname{curl} a^m \to \alpha && \text{in } C^{\lambda/2}(\bar{\Omega}), \\ \gamma^m &= \operatorname{curl} g^m \to \gamma && \text{in } C^{0,\lambda/2}(Q_T), \\ \varphi^m &= \operatorname{curl} \tilde{v}^m \to \varphi && \text{in } C^{0,\lambda/2}(Q_T). \end{aligned}$$

From this last result we see that the solutions v^m of

$$\begin{aligned} \operatorname{curl} v^m &= \varphi^m && \text{in } Q_T, \\ \operatorname{div} v^m &= 0 && \text{in } Q_T, \\ v^m \cdot n &= 0 && \text{on } [0, T] \times \Gamma \end{aligned}$$

are such that $v^m \in C^{0,2+\lambda}(Q_T)$, $v^m \to v$ in $C^{0,1+\lambda/2}(Q_T)$. Define now the vector function ζ^m by using α^m, γ^m, and v^m; by the first part of the proof it follows that $\operatorname{div} \zeta^m = 0$ in Q_T and $\int_{\Gamma_i} \zeta^m \cdot n \, d\Gamma = 0$ for each $t \in [0, T]$. Moreover, by using (2.7), we easily see that $\zeta^m \to \zeta$ in $C^0(Q_T)$; hence the lemma is proved. ∎

The function ζ defined in (2.9) trivially satisfies $(2.5)_2$; moreover ζ is a solution of $(2.5)_1$ in the following weak sense:

LEMMA 2.4. *For each $\Phi \in C^1(\bar{\Omega})$ one has*

$$\frac{d}{dt}(\zeta, \Phi) = (\gamma, \Phi) + ((\zeta \cdot \nabla) v, \Phi) + ((v \cdot \nabla) \Phi, \zeta), \tag{2.14}$$

where $(\, , \,)$ is the scalar product in $L^2(\Omega)$.

Proof. We have

$$\int_\Omega \zeta(t, x) \cdot \Phi(x)\, dx = \int_\Omega \tilde{\zeta}(t, x') \cdot \Phi(U(t, 0, x'))\, dx';$$

hence by $(2.7)_1$

$$\frac{d}{dt}\int_\Omega \zeta(t, x) \cdot \Phi(x)\, dx$$

$$= \sum_{i=1}^{3} \int_\Omega \left[\frac{d\tilde{\zeta}_i}{dt}(t, x')\, \Phi_i(U(t, 0, x'))\right.$$

$$\left. + \sum_{j=1}^{3} \tilde{\zeta}_i(t, x') \frac{\partial \Phi_i}{\partial x_j}(U(t, 0, x')) \cdot v_j(t, U(t, 0, x'))\right] dx'$$

$$= \sum_{i=1}^{3} \int_\Omega \left\{ \left[\gamma_i(t, x) + \sum_{j=1}^{3} \frac{\partial v_i}{\partial x_j}(t, x)\, \zeta_j(t, x)\right] \Phi_i(x) + \sum_{j=1}^{3} \zeta_i(t, x) \frac{\partial \Phi_i}{\partial x_j}(x)\, v_j(t, x)\right\} dx.$$

∎

Now we define the map F as follows. The domain of F consists of the functions φ of the sphere defined by (2.1) with A satisfying

$$A > \| \alpha \|_\lambda , \tag{2.15}$$

and such that (2.2) holds.

Finally we put $\zeta = F[\varphi]$.

It follows from estimates (2.11) and from Lemma 2.3 that there exists $T_1 \in]0, T_0]$ such that the set

$$S = \{\varphi \in C^{\lambda,\lambda}(Q_{T_1}) \mid \| \varphi \|_{0,\lambda} \leqslant A,\ [\varphi]_{\lambda,0} \leqslant c_1 A^{1+\lambda},\ \varphi \text{ satisfies } (2.2)\}$$

satisfies $F[S] \subset S$, where F, the norms, and the seminorms correspond to the interval $[0, T_1]$.

S is a convex set and by the Ascoli–Arzelà theorem it follows that S is compact in $C^0(Q_{T_1})$.

Moreover, as in [4], we obtain

Lemma 2.5. *The map F: $S \to S$ has a fixed point.*

Hence we have construct a solution ζ, v, ρ, w of the auxiliary system

$$
\begin{aligned}
&\frac{\partial \zeta}{\partial t} + (v \cdot \nabla)\,\zeta = \beta + w \wedge \frac{\nabla \rho}{\rho^2} + (\zeta \cdot \nabla)\, v && \text{in} \quad Q_{T_1}, \\
&\operatorname{curl} v = \zeta && \text{in} \quad Q_{T_1}, \\
&\operatorname{div} v = 0 && \text{in} \quad Q_{T_1}, \\
&v \cdot n = 0 && \text{on} \quad [0, T_1] \times \Gamma, \\
&\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0 && \text{in} \quad Q_{T_1}, \\
&\rho\,|_{t=0} = \rho_0 && \text{in} \quad \bar{\Omega}, \\
&\operatorname{curl} w = 0 && \text{in} \quad Q_{T_1}, \\
&\operatorname{div} w = \frac{\nabla \rho}{\rho} \cdot w + \rho \sum_{i,j} (D_i v_j)(D_j v_i) - \rho \operatorname{div} b && \text{in} \quad Q_{T_1}, \\
&w \cdot n = -\rho \sum_{i,j} (D_i n_j)\, v_i v_j - \rho b \cdot n && \text{on} \quad [0, T_1] \times \Gamma, \\
&\zeta\,|_{t=0} = \alpha && \text{in} \quad \bar{\Omega},
\end{aligned}
\tag{A}
$$

where equation $(A)_1$ is satisfied in the sense described in Lemma 2.4.

3. Existence of a Solution of System (E)

First we prove that $D_t v$ exists in the classical sense and belongs to $C^{0,\lambda}(Q_{T_1})$. We need two lemmas:

Lemma 3.1. *If* $v \in C^1(\bar{\Omega})$, $\operatorname{div} v = 0$ *in* Ω, *and* $v \cdot n = 0$ *on* Γ, *then*

$$
\begin{aligned}
&\operatorname{div}[(v \cdot \nabla)\, v] = \sum_{i,j} (D_i v_j)(D_j v_i) && \text{in} \quad \Omega, \\
&[(v \cdot \nabla)\, v] \cdot n = -\sum_{i,j} (D_i n_j)\, v_i v_j && \text{on} \quad \Gamma,
\end{aligned}
\tag{3.1}
$$

where the operator div *is in the sense of distributions in* Ω.

See Bourguignon and Brezis [5, Sect. 3] or Temam [16, Lemma 1.1].

Lemma 3.2. *If* $v \in C^1(\bar{\Omega})$, $\zeta = \operatorname{curl} v$, *we have*

$$
((v \cdot \nabla)v, \operatorname{curl} \Phi) = -((v \cdot \nabla)\Phi, \zeta) - ((\zeta \cdot \nabla)v, \Phi) \qquad \forall \Phi \in C_0^\infty(\Omega). \tag{3.2}
$$

Proof. If $v \in C^2(\bar{\Omega})$, by a direct computation we have

$$\operatorname{curl}[(v \cdot \nabla)v] = (v \cdot \nabla)\zeta - (\zeta \cdot \nabla)v + (\operatorname{div} v)\zeta,$$

and this leads easily to (3.2).

If $v \in C^1(\bar{\Omega})$, we approximate it with $v_n \in C^2(\bar{\Omega})$. ∎

Now we can prove the existence of $D_t v$.

LEMMA 3.3. *We have*

$$\frac{\partial v}{\partial t} = b + \frac{w}{\rho} - (v \cdot \nabla)\, v \qquad \text{in} \qquad Q_{T_1}; \tag{3.3}$$

hence $\partial v/\partial t \in C^{0,\lambda}(Q_{T_1})$.

Proof. Let $\Phi \in C_0^\infty(\Omega)$. We have

$$D_t(v, \operatorname{curl} \Phi) = D_t(\zeta, \Phi)$$

since $\operatorname{curl} v = \varphi = \zeta$. Moreover from (2.14), (3.2), and the equation $\gamma = \operatorname{curl}(b + w/\rho)$ we obtain

$$D_t(v, \operatorname{curl} \Phi) = (\gamma, \Phi) + ((\zeta \cdot \nabla)\, v, \Phi) + ((v \cdot \nabla)\, \Phi, \zeta)$$

$$= (\gamma, \Phi) - ((v \cdot \nabla)\, v, \operatorname{curl} \Phi) = \left(b + \frac{w}{\rho} - (v \cdot \nabla)\, v, \operatorname{curl} \Phi\right).$$

Hence

$$(v, \operatorname{curl} \Phi) = \Big(v(0, \cdot), \operatorname{curl} \Phi\Big) + \int_0^t \left(b + \frac{w}{\rho} - (v \cdot \nabla)\, v, \operatorname{curl} \Phi\right) d\tau$$

$$= \left(v(0, \cdot) + \int_0^t \left[b + \frac{w}{\rho} - (v \cdot \nabla)\, v\right](\tau, \cdot)\, d\tau, \operatorname{curl} \Phi\right),$$

and consequently for each $t \in [0, T_1]$

$$v(t, x) - v(0, x) - \int_0^t \left[b + \frac{w}{\rho} - (v \cdot \nabla)\, v\right](\tau, x)\, d\tau = \nabla \Xi(t, x),$$

where $\Xi \in C^{1,\lambda}(\bar{\Omega})$, $\forall t \in [0, T_1]$. From $(2.3)_2$, $(2.3)_3$, $(\mathrm{A})_8$, $(\mathrm{A})_9$, and (3.1) we conclude that $\operatorname{div} \nabla \Xi = 0$ in the distributions sense, and $\nabla \Xi \cdot n = 0$ on $[0, T_1] \times \Gamma$, hence (3.3). ∎

From (3.3) and $(\mathrm{A})_7$ it follows that

$$\rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v - b\right] = -\nabla \pi \qquad \text{in} \qquad Q_{T_1},$$

i.e., $(E)_1$ holds, with $\pi \in C^{0,2+\lambda}(\bar{Q}_{T_1})$ (see Lemmas 4.1 and 4.2 in [4]). Furthermore

$$\begin{aligned} \operatorname{curl}(v|_{t=0} - a) = \zeta|_{t=0} - \alpha &= 0 \quad \text{in } \bar{\Omega}, \\ \operatorname{div}(v|_{t=0} - a) &= 0 \quad \text{in } \bar{\Omega}, \\ (v|_{t=0} - a) \cdot n &= 0 \quad \text{on } \Gamma, \end{aligned}$$

and consequently $(E)_6$ holds.

Finally, as in Remark 5.5 in [4], we prove that $\rho \in C^{1+\lambda,1+\lambda}(\bar{Q}_{T_1})$, and the proof of Theorem A is complete.

4. The Case Ω not Simply Connected

By the hypotheses on the domain Ω (see Section 1), it is clear that if Ω is not simply connected, one can make it so by means of a finite number of regular cuts. The number N of these cuts is the dimension of the first cohomology space $H_c(\Omega)$ of Ω, i.e., the quotient of the space of closed differential forms by the space of exact differential forms.

Moreover one can construct N functions $q_1, q_2, \ldots, q_N$ such that $v^{(k)} = \operatorname{grad} q_k$ are linearly independent and satisfy $v^{(k)} \in C^{1+\lambda}(\bar{\Omega})$, $\operatorname{div} v^{(k)} = 0$, $\operatorname{curl} v^{(k)} = 0$, $v^{(k)} \cdot n = 0$ on Γ. These $v^{(k)}$ are a basis of the space $H_c(\Omega)$.

Finally, one sees that a function w is a gradient if and only if $\operatorname{curl} w = 0$ and $(w, v^{(k)}) = 0$ for each $k = 1,\ldots, N$ (for these results see Foias and Temam [7, Remark 1.2, Lemma 1.3, and Proposition 1.1]).

We can orthonomalize the $v^{(k)}$; if we denote the orthonomal system thus obtained by $u^{(k)}$, we have constructed a system of vectors which has the properties of that introduced in [8, Sect. 1].

The difference between two solutions v_1 and v_2 of (2.3) is given by

$$v_1(t, x) - v_2(t, x) = \sum_k \theta_k(t)\, u^{(k)}(x),$$

where the $\theta_k(t) \in C^0([0, T])$ are arbitrary.

We denote by $v(t, x)$ the solution of (2.3) such that $(v, u^{(k)}) = 0$ for each $k = 1,\ldots, N$. Such a solution is obviously unique, and we have

$$\| v \|_{0,1+\lambda} \leqslant c \| \varphi \|_{0,\lambda} .$$

Moreover each solution $\bar{v}$ of (2.3) can be written in the form

$$\bar{v}(t, x) = v(t, x) + \sum_k \theta_k(t)\, u^{(k)}(x).$$

Hence, arguing as is [4], we obtain a solution $\bar{v}$, $\bar{\rho}$, $\bar{w}$ of system (6.1)–(6.5) and we prove Lemmas 7.2 and 7.3 and Remark 7.4 of [4]. Hence, by proceeding as before, we construct a function $\bar{\zeta}$ which satisfies the usual properties and we find a fixed point $\varphi = \bar{\zeta}$ (see Section 2 of this paper). The regularity of $D_t\bar{v}$ is proved as in Lemma 3.3 of this paper, by also using the fact that

$$D_t(\bar{v}, u^{(k)}) = \left(\frac{\bar{w}}{\bar{\rho}} - (\bar{v}\cdot\nabla)\,\bar{v} + b, u^{(k)}\right), \qquad \forall t \in [0, T_1], \quad \forall k = 1,\dots, N;$$

finally one has

$$\begin{aligned} \operatorname{curl}(\bar{v}\,|_{t=0} - a) &= \bar{\zeta}\,|_{t=0} - \alpha = 0 && \text{in } \bar{\Omega}, \\ (\bar{v}\,|_{t=0} - a, u^{(k)}) &= 0, && \forall k = 1,\dots, N, \\ \operatorname{div}(\bar{v}\,|_{t=0} - a) &= 0 && \text{in } \bar{\Omega}, \\ (\bar{v}\,|_{t=0} - a)\cdot n &= 0 && \text{on } \Gamma, \end{aligned}$$

that is, $\bar{v}\,|_{t=0} = a$ in $\bar{\Omega}$.

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