HBV-26

Rend. Sem. Mat. Univ. Padova, Vol. 59 (1978)

On the Motion of a Non-Homogeneous Ideal Incompressible Fluid in an External Force Field.

Hugo Beirão da Veiga - Alberto Valli (*)

1. Introduction and main results.

In this paper we consider the motion of a non-homogeneous ideal incompressible fluid in a bounded connected open subset Ω of $\mathbb{R}^2$.

We denote in the sequel by $v(t, x)$ the velocity field, by $\varrho(t, x)$ the mass density and by $\pi(t, x)$ the pressure. The Euler equations of the motion are

$$
\text{(E)}\quad \begin{cases}
\varrho\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = -\nabla\pi & \text{in } Q_T \equiv [0, T]\times\bar{\Omega}\,,\\
\operatorname{div} v = 0 & \text{in } Q_T\,,\\
\dfrac{\partial\varrho}{\partial t} + v\cdot\nabla\varrho = 0 & \text{in } Q_T\,,\\
v\cdot n = 0 & \text{on } [0, T]\times\Gamma\,,\\
v|_{t=0} = a & \text{in } \bar{\Omega}\,,\\
\varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}\,,
\end{cases}
$$

where $n = n(x)$ is the unit outward normal vector to the boundary Γ of Ω, $b = b(t, x)$ is the external force field and $a = a(x)$, $\varrho_0 = \varrho_0(x)$

(*) Indirizzo degli A.: Università di Trento, Dipartimento di Fisica e Matematica - 38050 Povo (Trento), Italy.

are the initial velocity field and the initial mass density respectively.

When the fluid is homogeneous, i.e. the density ϱ_0 (and consequently ϱ), is constant, equations (E) have been studied by several authors. As regards the two-dimensional case, we recall the papers of Wolibner [13], Leray [6], Hölder [3], Schaeffer [10], Yudovich [14], [15], Golovkin [2], Kato [5], Mc Grath [9] and Bardos [1]; for the case of a variable boundary see Valli [12]. For the n-dimensional case we recall the papers of Lichtenstein, Ebin and Marsden, Swann, Kato, Bourguignon and Brezis, Temam, Bardos and Frisch.

For non-homogeneous fluids, Marsden [8] has proved the existence of a local solution to problem (E), under the assumption that the external force field $b(t, x)$ is divergence free and tangential to the boundary, i.e. $\operatorname{div} b = 0$ in Q_T and $b \cdot n = 0$ on $[0, T] \times \Gamma$. The proof relies on techniques of Riemannian geometry on infinite dimensional manifolds. See also the reference [16].

In this paper we prove the existence of a local solution of problem (E) without any restriction on the external force field $b(t, x)$ but we need condition (A) on the initial mass density $\varrho_0(x)$ (1).

Our techniques are based on the method of characteristics and on Schauder's fixed point theorem, and in this sense related to the methods of Kato [5] and Mc Grath [9].

We prove the following results (2).

THEOREM A. *Let Ω be of class $C^{3+\lambda}$, $0 < \lambda < 1$, and let $a \in C^{2+\lambda}(\bar{\Omega})$ with $\operatorname{div} a = 0$ in $\bar{\Omega}$ and $a \cdot n = 0$ on Γ, $\varrho_0 \in C^{2+\lambda}(\bar{\Omega})$ with $\varrho_0(x) > 0$ for each $x \in \bar{\Omega}$, and $b \in C^{0,1+\lambda}(Q_T) \cap C^{\lambda,0}(Q_T)$ with $\operatorname{rot} b \in C^{0,1+\lambda}(Q_T) \cap C^{\lambda,0}(Q_T)$.*

Moreover we assume that (1)

$$\text{(A)} \qquad \left\| \frac{D\varrho_0}{\varrho_0} \right\|_\infty < \begin{cases} \dfrac{1}{K_1} & \textit{if } \Omega \textit{ is simply connected}, \\[2ex] \dfrac{1}{K_1(1 + K_3 K_4)} & \textit{otherwise}. \end{cases}$$

(1) *Added in proofs.* In the authors' papers « On the Euler equations for non-homogeneous fluids » (I), (II) (to appear) condition (A) is dropped and the three dimensional case is proved.

(2) The definition of K_1 is given in (3.4); those of K_3 and K_4 in (7.21), (7.11) and (7.24).

Then there exist

$$T_1 \in]0, T], \qquad v \in C^{1,2+\lambda}(Q_{T_1}) \cap C^{1+\lambda,0}(Q_{T_1}),$$
$$\varrho \in C^{2+\lambda,2+\lambda}(Q_{T_1}), \qquad \pi \in C^{0,2+\lambda}(Q_{T_1}) \cap C^{\lambda,1}(Q_{T_1}),$$

such that (v, ϱ, π) *is a solution of* (E) *in* Q_{T_1}.

THEOREM B. *Assume that* ϱ_0 *and* $\nabla\varrho_0$ *belong to* $L^\infty(\Omega)$, $\min \varrho_0 > 0$ *and that* b *belongs to* $L^1(0, T; L^\infty(\Omega))$. *Then problem* (E) *has at most a solution* (v, ϱ, π) *in the class of vector functions* $v \in L^\infty(Q_T)$ *such that* $\partial v/\partial t$, $\partial v/\partial x_1$ *and* $\partial v/\partial x_2$ *are in* $L^1(0, T; L^\infty(\Omega))$. *The pressure is unique up to an arbitrary function of* t *which may be added to it. This result holds in dimension* $n \geq 2$.

For other uniqueness theorems see also Serrin [11].

The paper consists of two parts. In Part I we prove Theorem A for a simply connected domain Ω, and Theorem B. In Part II we prove Theorem A in the general case, i.e. we assume that Γ consists of $m+1$ simple closed curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, where Γ_j $(j = 1, \ldots, m)$ are inside of Γ_0 and outside of one another.

PART I

2. Notations.

Let Ω be a bounded simply connected open subset of $\mathbb{R}^2$.

We denote by $C^{k+\lambda}(\bar{\Omega})$, k non negative integer, $0 < \lambda < 1$, the space of k-times continuously differentiable functions in $\bar{\Omega}$ with λ-Hölder continuous derivatives of order k; by $C^0(Q_T)$ the space of continuous functions in Q_T; by $C^1(Q_T)$ the space of continuously differentiable functions in Q_T.

We set

$$D_i\varphi \equiv \frac{\partial\varphi}{\partial x_i}, \qquad D^\alpha D_t^j \varphi \equiv \frac{\partial^{|\alpha|+j}\varphi}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\partial t^j},$$

and

$$C^{k,h}(Q_T) \equiv \{\varphi \in C^0(Q_T) | D^\alpha D_t^j \varphi \in C^0(Q_T) \text{ if } 0 \leq j \leq k,$$
$$|\alpha| \leq h \text{ and } j + |\alpha| \leq \max(k, h)\},$$

$$C^{\lambda,0}(Q_T) \equiv \{\varphi \in C^0(Q_T) | \varphi \text{ is } \lambda\text{-Hölder continuous in } t, \text{ uniformly with respect to } x\},$$

$$C^{0,\lambda}(Q_T) \equiv \{\varphi \in C_0(Q_T) | \varphi \text{ is } \lambda\text{-Hölder continuous in } x, \text{ uniformly with respect to } t\},$$

$$C^{k+\lambda,h}(Q_T) \equiv \{\varphi \in C^{k,h}(Q_T) | D^\alpha D_t^j \varphi \in C^{\lambda,0}(Q_T) \text{ if } j + |\alpha| = \max(k, h) \text{ or if } j = k\},$$

$$C^{k,h+\lambda}(Q_T) \equiv \{\varphi \in C^{k,h}(Q_T) | D^\alpha D_t^j \varphi \in C^{0,\lambda}(Q_T) \text{ if } j + |\alpha| = \max(k, h) \text{ or if } |\alpha| = h\},$$

$$C^{k+\lambda,h+\lambda}(Q_T) \equiv C^{k+\lambda,h}(Q_T) \cap C^{k,h+\lambda}(Q_T).$$

We denote by $\|\cdot\|_\infty$ the supremum norm, both in $\bar{\Omega}$ or in Q_T, and by $[\cdot]_\lambda$ the usual λ-Hölder seminorm in $\bar{\Omega}$. Furthermore we define

$$[\varphi]_{\lambda,0} \equiv \sup_{\substack{t,s\in[0,T]\\ t\neq s\\ x\in\bar{\Omega}}} \frac{|\varphi(t,x) - \varphi(s,x)|}{|t-s|^\lambda},$$

$$[\varphi]_{0,\lambda} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y\\ t\in[0,T]}} \frac{|\varphi(t,x) - \varphi(t,y)|}{|x-y|^\lambda},$$

$$[\varphi]_{\mathrm{lip},0} \equiv \sup_{\substack{t,s\in[0,T]\\ t\neq s\\ x\in\bar{\Omega}}} \frac{|\varphi(t,x) - \varphi(s,x)|}{|t-s|},$$

$$[\varphi]_{0,\mathrm{lip}} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y\\ t\in[0,T]}} \frac{|\varphi(t,x) - \varphi(t,y)|}{|x-y|},$$

Finally, we set

$$\|D\varphi\|_\infty \equiv \sum_{|\alpha|=1} \|D^\alpha\varphi\|_\infty,$$

$$\|D^2\varphi\|_\infty \equiv \sum_{|\alpha|=2} \|D^\alpha\varphi\|_\infty,$$

and analogously for the seminorms $[\cdot]_{\lambda,0}$ and $[\cdot]_{0,\lambda}$.

If $u = (u_1, u_2)$ is a vector field defined in Q_T, we write $u \in C^{\lambda,0}(Q_T)$ if $u_1, u_2 \in C^{\lambda,0}(Q_T)$, and we set $[u]_{\lambda,0} \equiv [u_1]_{\lambda,0} + [u_2]_{\lambda,0}$; the same convention is used for the other vector spaces and norms (in $\bar{\Omega}$ or in Q_T).

We put

$$\operatorname{Rot}\varphi \equiv \left(\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right),$$

$$\operatorname{rot} u \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

where φ is a scalar function and $u = (u_1, u_2)$ is a vector function.

3. Preliminaries.

Let $\varphi \in C^{1,1+\lambda}(Q_T)$ with $D_t\varphi \in C^{\lambda,0}(Q_T)$; we assume that

$$(3.1) \qquad \begin{cases} \|\varphi\|_{0,1+\lambda} \equiv \|\varphi\|_\infty + \|D\varphi\|_\infty + [D\varphi]_{0,\lambda} \leqslant A, \\ \|D_t\varphi\|_\infty \leqslant B, \\ \|D_t\varphi\|_{0,\lambda} \equiv \|D_t\varphi\|_\infty + [D_t\varphi]_{0,\lambda} \leqslant C, \\ [D_t\varphi]_{\lambda,0} \leqslant D, \end{cases}$$

where A, B, C, D are positive constants that we will specify in the following (see (4.9)).

Let ψ be the solution of

$$(3.2) \qquad \begin{cases} -\Delta\psi(t, x) = \varphi(t, x) & \text{in } \Omega, \\ \psi|_\Gamma = 0, \end{cases}$$

for each $t \in [0, T]$, i.e.

$$(3.2)' \qquad \psi(t, x) = \int_\Omega G(x, y)\varphi(t, y)\, dy$$

where $G(x, y)$ is the Green function for the operator $-\Delta$ with zero boundary condition.

Put $\|\chi\|_{k+\lambda} \equiv \sum_{|\alpha|\leqslant k} \|D^\alpha\chi\|_\infty + \sum_{|\alpha|=k} [D^\alpha\chi]_\lambda$. It is well known that there exist constants $c = c(\lambda, \Omega)$ such that

$$(3.3) \qquad \begin{aligned} &\|\chi\|_{3+\lambda} \leqslant c\|\Delta\chi\|_{1+\lambda}, \\ &\|\chi\|_{2+\lambda} \leqslant c\|\Delta\chi\|_\lambda, \\ &[D\chi]_\lambda \leqslant c\|\Delta\chi\|_\infty, \end{aligned}$$

for each $\chi \in C^\infty(\bar\Omega)$ vanishing on Γ.

Moreover there exists $K_1 = K_1(\Omega)$ such that

$$\sup_{x\in\bar\Omega}\int_\Omega |\nabla_x G(x,y)|\,dy \leqslant \frac{1}{2}K_1\,. \tag{3.4}$$

It is sufficient to choose $K_1 = 4\pi K \operatorname{diam}\Omega$, where K is such that

$$|\nabla_x G(x,y)| \leqslant \frac{K}{|x-y|}, \qquad \forall x, y \in \bar\Omega\,,\ x\neq y\,, \tag{3.4$'$}$$

(see for instance Lichtenstein [7], pag. 248).

We obtain

LEMMA 3.1. *Let* $\varphi \in C^{1,1+\lambda}(Q_T)$ *with* $D_t\varphi \in C^{\lambda,0}(Q_T)$ *and let* ψ *be defined in* (3.2). *Put*

$$v \equiv \operatorname{Rot}\psi\,; \tag{3.5}$$

then $v\in C^{1,2+\lambda}(Q_T)$, $D_t v \in C^{\lambda,0}(Q_T)$ *and*

$$\begin{aligned}
&\|v\|_{0,2+\lambda} \leqslant c\|\varphi\|_{0,1+\lambda}\,,\\
&\|D_t v\|_\infty \leqslant \tfrac{1}{2}K_1\|D_t\varphi\|_\infty\,,\\
&\|D_t v\|_{0,1+\lambda} \leqslant c\|D_t\varphi\|_{0,\lambda}\,,\\
&[D_t v]_{\lambda,0} \leqslant \tfrac{1}{2}K_1[D_t\varphi]_{\lambda,0}\,,\\
&[D_t v]_{0,\lambda} \leqslant c\|D_t\varphi\|_\infty\,.
\end{aligned} \tag{3.6}$$

Moreover $\operatorname{div} v = 0$ *and* $\operatorname{rot} v = \varphi$ *in* Q_T, $v\cdot n = 0$ *on* $[0,T]\times\Gamma$.

PROOF. Since $\varphi\in C^{0,1+\lambda}(Q_T)\subset C^0([0,T];C^{1+\lambda'}(\bar\Omega))$ for each $\lambda'<\lambda$ (see for instance Kato [5], Lemma 1.2), it follows from Schauder's estimates that $v\in C^0([0,T];C^{2+\lambda'}(\bar\Omega))$; hence $v, Dv, D^2v \in C^0(Q_T)$. Moreover estimate $(3.6)_1$ follows directly from $(3.3)_1$, i.e. $v\in C^{0,2+\lambda}(Q_T)$. Differentiating (3.2)$'$ with respect to t, we have

$$D_t\psi(t,x) = \int_\Omega G(x,y)\,D_t\varphi(t,y)\,dy\,; \tag{3.7}$$

since $D_t\varphi\in C^{0,\lambda}(Q_T)$, arguing as above it follows that $D_t v\in C^{0,1+\lambda}(Q_T)$ and $(3.6)_3$ holds.

Applying the operator Rot to (3.7) we have

$$D_t v(t,x) = \int_\Omega \operatorname{Rot}_x G(x,y)\,D_t\varphi(t,y)\,dy$$

and (3.4) yields $(3.6)_2$ and $(3.6)_4$.

Estimate $(3.6)_5$ follows directly from (3.5) and $(3.3)_3$. Finally remark that rot Rot $= -\Delta$ and that $v\cdot n$ is a tangential derivative of ψ at the boundary. □

By using (3.1) one has

$$\begin{aligned}
&\|v\|_{0,2+\lambda} \leqslant cA\,,\\
&\|D_t v\|_\infty \leqslant \tfrac{1}{2}K_1 B\,,\\
&\|D_t v\|_{0,1+\lambda} \leqslant cC\,,\\
&[D_t v]_{\lambda,0} \leqslant \tfrac{1}{2}K_1 D\,,\\
&[D_t v]_{0,\lambda} \leqslant cB\,.
\end{aligned} \tag{3.8}$$

Now we construct the stream lines of the vector field $v(t,x)$. We denote by $c, c_1, c_2, \ldots,$ constants depending at most on λ and Ω.

We put $U(\sigma,t,x)\equiv y(\sigma)$, $\sigma, t\in[0,T]$, $x\in\bar{\Omega}$, where $y(\sigma)$ is the solution of the ordinary differential equation

$$\begin{cases} \dfrac{dy}{d\sigma} = v(\sigma, y(\sigma)) \qquad \text{in } [0,T]\,,\\ y(t) = x\,. \end{cases} \tag{3.9}$$

Such a solution is global since $v\cdot n = 0$ on $[0,T]\times\Gamma$; from $v\in C^{1,2}(Q_T)$ one has $U\in C^2([0,T]\times Q_T)$.

We denote by $\|DU\|_\infty \equiv \sup\limits_{\sigma\in[0,T]}\|DU(\sigma,\cdot,\cdot)\|_\infty$ and analogously for each norm and seminorm involving U and its derivatives.

We have:

LEMMA 3.2. *The vector function $U(\sigma,t,x)$ satisfies the following estimates:*

$$\begin{aligned}
&\|DU\|_\infty \leqslant 2\exp[cTA]\,,\\
&\|D^2U\|_\infty \leqslant cTA\exp[cTA]\,,\\
&[D^2U]_{0,\lambda} \leqslant cTA(1+TA)\exp[cTA]\,,\\
&[U]_{\mathrm{lip},0} \leqslant cA\exp[cTA]\,,\\
&[DU]_{\lambda,0} \leqslant cT^{1-\lambda}A(1+T^\lambda A^\lambda)\exp[cTA]\,,\\
&[D^2U]_{\lambda,0} \leqslant cT^{1-\lambda}A(1+T^\lambda A^\lambda)(1+TA)\exp[cTA]\,.
\end{aligned} \tag{3.10}$$

PROOF. One obtains these estimates by direct computation of the resolutive formula

$$U(\sigma,t,x) = x + \int_t^\sigma v(\tau, U(\tau,t,x))\,d\tau \,. \tag{3.11}$$

We give only the explicity proof of $(3.10)_3$. From (3.11) one gets

$$D_i U_j(\sigma,t,x) = \delta_{ij} + \int_t^\sigma \sum_h (D_h v_j)(\tau, U(\tau,t,x))\, D_i U_h(\tau,t,x)\,d\tau \tag{3.12}$$

and

$$\begin{aligned} D^2_{ik} U_j(\sigma,t,x) = \\ = \int_t^\sigma \Big[\sum_{r,h} (D^2_{rh} v_j)(\tau, U(\tau,t,x))\, D_k U_r(\tau,t,x) D_i U_h(\tau,t,x) + \\ + \sum_h (D_h v_j)(\tau, U(\tau,t,x))\, D^2_{ik} U_h(\tau,t,x)\Big]\,d\tau \,. \end{aligned} \tag{3.13}$$

Hence one obtains

$$\begin{aligned} \sum_{i,k,j} |D^2_{ik} U_j(\sigma,t,x) - D^2_{ik} U_j(\sigma,t,y)| \leqslant T|x-y|^\lambda \{[D^2 v]_{0,\lambda} [U]^\lambda_{0,\mathrm{lip}} \|DU\|^2_\infty + \\ + 2\|D^2 v\|_\infty \|DU\|_\infty [DU]_{0,\lambda} + \|D^2 U\|_\infty [Dv]_{0,\lambda} [U]^\lambda_{0,\mathrm{lip}}\} + \\ + \|Dv\|_\infty \left| \int_t^\sigma \sum_{i,k,h} |D^2_{ik} U_h(\tau,t,x) - D^2_{ik} U_h(\tau,t,y)|\,d\tau \right| \end{aligned}$$

and from Gronwall's lemma

$$\begin{aligned} \sum_{i,j,k} |D^2_{ik} U_j(\sigma,t,x) - D^2_{ik} U_j(\sigma,t,y)| \leqslant T|x-y|^\lambda \cdot \\ \cdot \{[D^2 v]_{0,\lambda} [U]^\lambda_{0,\mathrm{lip}} \|DU\|^2_\infty + 2\|D^2 v\|_\infty \|DU\|_\infty [DU]_{0,\lambda} + \\ + \|D^2 U\|_\infty [Dv]_{0,\lambda} [U]^\lambda_{0,\mathrm{lip}}\} \exp\,[T\|Dv\|_\infty] \,. \end{aligned}$$

From $(3.10)_1$, $(3.10)_2$ and $(3.8)_1$ one obtains $(3.10)_3$.

On proving $(3.10)_4$ and (3.10_5), recall that

$$\frac{\partial U(\sigma,t,x)}{\partial t} = -\sum_h \frac{\partial U(\sigma,t,x)}{\partial x_h} v_h(t,x) \,. \tag{3.14}$$

□

We now study the equation

$$\left\{\begin{array}{lll}\dfrac{\partial\varrho}{\partial t}+v\cdot\nabla\varrho=0 & & \text{in } Q_T\,,\\[2ex] \varrho|_{t=0} & =\varrho_0 & \text{in } \bar{\Omega}\,.\end{array}\right. \tag{3.15}$$

LEMMA 3.3. *Let $\varrho_0\in C^{2+\lambda}(\bar{\Omega})$ and $\varrho_0(x)>0$ for each $x\in\bar{\Omega}$. Then the solution of* (3.15) *is given by*

$$\varrho(t,x)=\varrho_0\big(U(0,t,x)\big)\,. \tag{3.16}$$

Moreover $\varrho\in C^{2+\lambda,2+\lambda}(Q_T)$ and

$$\begin{aligned}
&\left\|\frac{D\varrho}{\varrho}\right\|_\infty\leqslant 2\left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty\exp[cTA]\,,\\
&\left\|\frac{D^2\varrho}{\varrho}\right\|_\infty\leqslant c\left(TA\left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty+\left\|\frac{D^2\varrho_0}{\varrho_0}\right\|_\infty\right)\exp[cTA]\,,\\
&\left[\frac{D\varrho}{\varrho}\right]_{0,\lambda}\leqslant c\left(TA\left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty+\left[\frac{D\varrho_0}{\varrho_0}\right]_\lambda\right)\exp[cTA]\,,\\
&\left[\frac{D^2\varrho}{\varrho}\right]_{0,\lambda}\leqslant c\left\{TA(1+TA)\left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty+TA\left[\frac{D\varrho_0}{\varrho_0}\right]_\lambda+\right.\\
&\qquad\qquad\left.+TA\left\|\frac{D^2\varrho_0}{\varrho_0}\right\|_\infty+\left[\frac{D^2\varrho_0}{\varrho_0}\right]_\lambda\right\}\exp[cTA]\,,\\
&\left[\frac{D\varrho}{\varrho}\right]_{\lambda,0}\leqslant c\left\{T^{1-\lambda}A(1+T^\lambda A^\lambda)\left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty+A^\lambda\left[\frac{D\varrho_0}{\varrho_0}\right]_\lambda\right\}\exp[cTA]\,.
\end{aligned} \tag{3.17}$$

PROOF. One easily obtains (3.16) by using the method of characteristics. From (3.16) one has

$$\begin{aligned}
\frac{D_i\varrho}{\varrho}(t,x)&=\sum_h\frac{D_h\varrho_0}{\varrho_0}\big(U(0,t,x)\big)D_iU_h(0,t,x)\,,\\
\frac{D^2_{ik}\varrho}{\varrho}(t,x)&=\sum_{r,h}\frac{D^2_{rh}\varrho_0}{\varrho_0}\big(U(0,t,x)\big)D_kU_r(0,t,x)D_iU_h(0,t,x)+\\
&\qquad+\sum_h\frac{D_h\varrho_0}{\varrho_0}\big(U(0,t,x)\big)D^2_{ik}U_h(0,t,x)\,.
\end{aligned}$$

By using (3.10), we obtain easily estimates (3.17). □

4. The vorticity equation.

In this number we study the auxiliary equation

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} + v \cdot \nabla \zeta = \gamma & \text{in } Q_T, \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}, \end{cases} \tag{4.1}$$

where $\alpha(x) \equiv \operatorname{rot} a(x)$, $\beta(t, x) \equiv \operatorname{rot} b(t, x)$, and $\gamma(t, x)$ is defined in Q_T by

$$\gamma \equiv \beta + \frac{\operatorname{Rot} \varrho}{\varrho} \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right], \tag{4.2}$$

where a and b are as in Theorem A.

One integrates (4.1) by the method of characteristics and one obtains

$$\zeta(t, x) = \alpha(U(0, t, x)) + \int_0^t \gamma(\tau, U(\tau, t, x))\, d\tau . \tag{4.3}$$

We denote by $\bar{c}, \bar{c}_1, \bar{c}_2, \ldots,$ constants that depend at most on λ, Ω, $\|D\varrho_0/\varrho_0\|_\lambda$, $\|D^2\varrho_0/\varrho_0\|_\lambda$, $\|b\|_{0,1+\lambda}$, $\|b\|_{\lambda,0}$, $\|\beta\|_{0,1+\lambda}$ and $\|\beta\|_{\lambda,0}$.

LEMMA 4.1. *Under the above conditions the following estimates hold*:

$$\begin{aligned} &\|\gamma\|_\infty \leqslant \bar{c}(A^2 + 1) \exp[cTA] + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_\infty K_1 B \exp[cTA], \\ &[\gamma]_{0,\lambda} \leqslant \bar{c}(1 + TA)(A^2 + B + 1) \exp[cTA], \\ &\|D\gamma\|_\infty \leqslant \bar{c}\{(1 + TA)(A^2 + B + 1) + C\} \exp[cTA], \\ &[D\gamma]_{0,\lambda} \leqslant \bar{c}\{(1 + T^2A^2)(A^2 + B + 1) + (1 + TA)C\} \exp[cTA], \\ &[\gamma]_{\lambda,0} \leqslant \bar{c}\{T^{1-\lambda}AC + 1 + A^\lambda(A^2 + B + 1)(1 + TA)\} \cdot \\ &\qquad \cdot \exp[cTA] + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_\infty K_1 D \exp[cTA]. \end{aligned} \tag{4.4}$$

PROOF. It follows by direct computations, using (4.2), (3.8) and (3.17). □

Finally we have

LEMMA 4.2. *The solution $\zeta(t, x)$ of (4.1) satisfies:*

$$\|\zeta\|_{0,1+\lambda} \leq 2\|\alpha\|_{1+\lambda} \exp[cTA] + \bar{c}T\{A\|D\alpha\|_\infty + (1 + T^2A^2)(A^2 + B + 1) + (1 + TA)C\} \exp[cTA],$$

$$\|D_t\zeta\|_\infty \leq c_1 A\|D\alpha\|_\infty \exp[cTA] + \bar{c}_1(A^2 + 1)\exp[cTA] + \bar{c}TA[(1 + TA)(A^2 + B + 1) + C]\exp[cTA] + \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_1 B \exp[cTA],$$

$$(4.5)\qquad [D_t\zeta]_{0,\lambda} \leq c_2 A(\|D\zeta\|_\infty + [D\zeta]_{0,\lambda}) + \bar{c}_2(1 + TA)(A^2 + B + 1)\exp[cTA],$$

$$[D_t\zeta]_{\lambda,0} \leq c_4 A[D\zeta]_{\lambda,0} + cT^{1-\lambda}B\|D\zeta\|_\infty + \bar{c}_3 A^\lambda(A^2 + B + 1)(1 + TA)\exp[cTA] + \bar{c}_3(T^{1-\lambda}AC + 1)\exp[cTA] + \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_1 D \exp[cTA],$$

where $\|D\zeta\|_\infty$, $[D\zeta]_{0,\lambda}$ and $[D\zeta]_{\lambda,0}$ are bounded respectively by (4.6), (4.7) *and* (4.8).

PROOF. From (4.3), (3.10) and Lemma 4.1 it follows easily that

$$\|\zeta\|_\infty \leq \|\alpha\|_\infty + \bar{c}T(A^2 + B + 1)\exp[cTA],$$

$$(4.6)\qquad \|D\zeta\|_\infty \leq 2\|D\alpha\|_\infty \exp[cTA] + \bar{c}T[(1 + TA)(A^2 + B + 1) + C]\exp[cTA],$$

$$(4.7)\qquad [D\zeta]_{0,\lambda} \leq 2[D\alpha]_\lambda \exp[cTA] + \bar{c}T[A\|D\alpha\|_\infty + (1 + T^2A^2)(A^2 + B + 1) + (1 + TA)C]\exp[cTA],$$

hence $(4.5)_1$ holds.

From (4.1), one has $D_t\zeta = -v\cdot\nabla\zeta + \gamma$, and by direct computation one obtains $(4.5)_2$, $(4.5)_3$ and $(4.5)_4$.

Finally, from (4.3) it follows that:

$$(4.8)\qquad [D\zeta]_{\lambda,0} \leq c_3 A^\lambda[D\alpha]_\lambda \exp[cTA] + cT^{1-\lambda}A(1 + T^\lambda A^\lambda)\|D\alpha\|_\infty \exp[cTA] + \bar{c}T^{1-\lambda}[(1 + TA)(A^2 + B + 1) + C](1 + T^{1+\lambda}A^{1+\lambda})\exp[cTA]. \qquad \square$$

We assume in the sequel that condition (A) of Theorem A holds, and we choose the constants A, B, C, D such that

$$(4.9)\quad \begin{cases} A > 2\|\alpha\|_{1+\lambda}\,, \\ B > \left\|\dfrac{D\varrho_0}{\varrho_0}\right\|_\infty K_1 B + c_1 A\|D\alpha\|_\infty + \bar{c}_1(A^2+1)\,, \\ C > c_1 A\|D\alpha\|_\infty + 2c_2 A\|D\alpha\|_\lambda + (\bar{c}_1+\bar{c}_2)(A^2+1) + \\ \qquad\qquad + \left[\left\|\dfrac{D\varrho_0}{\varrho_0}\right\|_\infty K_1 + \bar{c}_2\right] B\,, \\ D > \left\|\dfrac{D\varrho_0}{\varrho_0}\right\|_\infty K_1 D + c_3 c_4 A^{1+\lambda}[D\alpha]_\lambda + \bar{c}_3(A^2+B+1)A^\lambda + \bar{c}_3\,. \end{cases}$$

From (4.5), (4.6), (4.7) and (4.8) it follows that there exists $T_1 \in]0, T]$ such that

$$(4.10)\quad \begin{aligned} &\|\zeta\|_{0,1+\lambda} \leqslant A\,, \\ &\|D_t\zeta\|_\infty \leqslant B\,, \\ &\|D_t\zeta\|_{0,\lambda} \leqslant C\,, \\ &[D_t\zeta[_{\lambda,0} \leqslant D\,, \end{aligned}$$

where the norms are taken on the cylinder $Q_{T_1} \equiv [0, T_1]\times\bar{\Omega}$. The set

$$(4.11)\quad S \equiv \{\varphi \in C^1(Q_{T_1}) \,|\, \|\varphi\|_{0,1+\lambda} \leqslant A,\ \|D_t\varphi\|_\infty \leqslant B\,,\ \|D_t\varphi\|_{0,\lambda} \leqslant C,\ [D_t\varphi]_{\lambda,0} \leqslant D\}$$

is a convex, bounded and closed subset of $C^1(Q_{T_1})$.

Moreover the map $F: \varphi \mapsto \zeta$ defined by (3.2), (3.5), (3.9), (3.15) and (4.3) satisfies

$$(4.12)\quad F(S) \subset S\,,$$

and, from (4.8),

$$(4.13)\quad [D\zeta]_{\lambda,0} \leqslant \text{const}\,, \qquad \forall \varphi \in S\,.$$

By the Ascoli-Arzelà theorem and (4.11), (4.13) it follows that $F(S)$ is relatively compact in $C^1(Q_{T_1})$.

Finally, we shall see that F is continuous in the $C^1(Q_{T_1})$ topology, hence, by the Schauder fixed point theorem, one has

LEMMA 4.3. *$F: S \to S$ has a fixed point.*

PROOF. It is sufficient to prove that F is continuous from $C^1(Q_{T_1})$ in $C^0(Q_{T_1})$, since $F(S)$ is relatively compact in $C^1(Q_{T_1})$.

Let $\varphi_n \in S$, $\varphi_n \to \varphi$ in $C^1(Q_{T_1})$. From (3.2) and (3.5), one has

$$\begin{aligned} v^n &\to v && \text{in } C^0(Q_{T_1})\,, \\ Dv^n &\to Dv && \text{in } C^0(Q_{T_1})\,. \end{aligned} \tag{4.14}$$

Moreover, from (3.7) and (3.4′)

$$\frac{\partial v^n}{\partial t} \to \frac{\partial v}{\partial t} \qquad \text{in } C^0(Q_{T_1})\,. \tag{4.15}$$

On the other hand

$$\begin{aligned} |U^n(\sigma, t, x) - U(\sigma, t, x)| &\leqslant \Big| \int_t^\sigma [\,|v^n(\tau, U^n(\tau, t, x)) - v(\tau, U^n(\tau, t, x))| + \\ &\quad + |v(\tau, U^n(\tau, t, x)) - v(\tau, U(\tau, t, x))|\,]\, d\tau \Big| \leqslant \\ &\leqslant T_1 \|v_n - v\|_\infty + [v]_{0,\mathrm{lip}} \Big| \int_t^\sigma |U^n(\tau, t, x) - U(\tau, t, x)|\, d\tau \Big|\,, \end{aligned}$$

and from Gronwall's lemma

$$|U^n(\sigma, t, x) - U(\sigma, t, x)| \leqslant T_1 \|v_n - v\|_\infty \exp\,[T_1 [v]_{0,\mathrm{lip}}]\,,$$

hence $U^n \to U$ uniformly in $[0, T_1] \times Q_{T_1}$.

Analogously, one evaluates $|D_i U_j^n(\sigma, t, x) - D_i U_j(\sigma, t, x)|$ by using (3.12), and this gives

$$\begin{aligned} \|DU^n - DU\|_\infty \leqslant T_1 \big([Dv^n]_{0,\mathrm{lip}} \|DU^n\|_\infty \|U^n - U\|_\infty + \|DU\|_\infty \|Dv^n - Dv\|_\infty \big) \cdot \\ \cdot \exp\,[T_1 \|Dv^n\|_\infty]\,. \end{aligned}$$

Hence $DU^n \to DU$ uniformly in $[0, T_1] \times Q_{T_1}$. Consequently

$$\frac{\operatorname{Rot} \varrho_n}{\varrho_n} \to \frac{\operatorname{Rot} \varrho}{\varrho} \qquad \text{in } C^0(Q_{T_1}) \tag{4.16}$$

and

$$\gamma_n(\sigma, U^n(\sigma, t, x)) \to \gamma(\sigma, U(\sigma, t, x)) \quad \text{uniformly in } [0, T_1]\times Q_{T_1}.$$

From (4.3) the thesis follows. □

This fixed point $\zeta = \varphi = F[\varphi]$, together with the corresponding v and ϱ, is a solution of the system

$$(4.17) \quad \begin{cases} \dfrac{\partial \zeta}{\partial t} + v\cdot\nabla\zeta = \beta + \dfrac{\operatorname{Rot}\varrho}{\varrho}\cdot\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] & \text{in } Q_{T_1}, \\ \zeta = \operatorname{rot} v & \text{in } Q_{T_1}, \\ \operatorname{div} v = 0 & \text{in } Q_{T_1}, \\ \dfrac{\partial \varrho}{\partial t} + v\cdot\nabla\varrho = 0 & \text{in } Q_{T_1}, \\ v\cdot n = 0 & \text{on } [0, T_1]\times\Gamma, \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}, \\ \varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}. \end{cases}$$

5. Existence of a solution of system (E) when Ω is simply connected.

Since

$$\operatorname{rot}[(v\cdot\nabla)v] = (\operatorname{div} v)\operatorname{rot} v + v\cdot\nabla(\operatorname{rot} v)$$

one has from $(4.17)_1$, $(4.17)_2$ and $(4.17)_4$

$$\varrho \operatorname{rot}\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = \operatorname{Rot}\varrho\cdot\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right].$$

We recall the general identity

$$\operatorname{rot}(\varrho w) = \varrho \operatorname{rot} w - (\operatorname{Rot}\varrho)\cdot w,$$

where ϱ is an arbitrary scalar and w an arbitrary vector, and applying it we obtain

$$(5.1) \qquad \operatorname{rot}\left\{\varrho\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right]\right\} = 0 \quad \text{in } Q_{T_1}.$$

When Ω is simply connected, it is well known that there exists a scalar function $\pi \in C^{0,1}(Q_{T_1})$ such that $(E)_1$ holds in Q_{T_1}.

Moreover $\pi \in C^{\lambda,1}(Q_{T_1}) \cap C^{0,2+\lambda}(Q_{T_1})$: in fact $\pi(t, x)$ is determined as the integral of $\nabla\pi \cdot ds$ from a fixed point x_0 to x, along a path independent of t. Since $\nabla\pi \in C^{\lambda,0}(Q_{T_1})$, it follows that $\pi \in C^{\lambda,0}(Q_{T_1})$. The other statement follows directly from $(E)_1$. Furthermore

$$\begin{cases} \operatorname{rot}\,(v|_{t=0} - a) = 0 & \text{in } \bar{\Omega}\,, \\ \operatorname{div}\,(v|_{t=0} - a) = 0 & \text{in } \bar{\Omega}\,, \\ (v|_{t=0} - a)\cdot n = 0 & \text{on } \Gamma\,, \end{cases}$$

and consequently $(E)_5$ holds.

Hence we have found a solution (v, π, ϱ) to problem (E) in Q_{T_1}. This solution verifies the regularity conditions stated in Theorem A, as follows from Lemmas 3.1 and 3.3.

6. Uniqueness of the solution of system (E).

Let (v, π, ϱ) and $(\tilde{v}, \tilde{\pi}, \tilde{\varrho})$ be two solutions of (E) in $[0, T] \times \bar{\Omega}$, under the conditions of Theorem B. We set $u \equiv \tilde{v} - v$, $\sigma = \tilde{\pi} - \pi$, $\eta = \tilde{\varrho} - \varrho$. On subtracting the two equations $(E)_1$, we obtain

$$\tilde{\varrho}\left[\frac{\partial u}{\partial t} + (\tilde{v}\cdot\nabla)u + (u\cdot\nabla)v\right] = -\nabla\sigma - \eta\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right]. \tag{6.1}$$

On the other hand from $(E)_3$ one gets

$$\left(\tilde{\varrho}\frac{\partial u}{\partial t}, u\right) = \frac{1}{2}\frac{d}{dt}(\tilde{\varrho}u, u) + \frac{1}{2}((\tilde{v}\cdot\nabla\tilde{\varrho})u, u)\,,$$

where (,) denotes the scalar product in $L^2(\Omega)$ or in $[L^2(\Omega)]^2$. Taking the scalar product of (6.1) with u it follows

$$\frac{1}{2}\frac{d}{dt}(\tilde{\varrho}u, u) = -(\tilde{\varrho}(u\cdot\nabla)v, u) - \left(\eta\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right], u\right), \tag{6.2}$$

since

$$\left(\tilde{\varrho}(\tilde{v}\cdot\nabla)u,\, u\right) + \tfrac{1}{2}\left((\tilde{v}\cdot\nabla\tilde{\varrho})u,\, u\right) = 0 \ ;$$

recall that $\operatorname{div}\tilde{v} = 0$ and $\tilde{v}\cdot n = 0$.

Moreover, on subtracting the two equations $(\mathrm{E})_3$, we obtain

$$\frac{\partial\eta}{\partial t} + v\cdot\nabla\eta = -\, u\cdot\nabla\tilde{\varrho} \tag{6.3}$$

and taking the scalar product of (6.3) with η it follows

$$\frac{1}{2}\,\frac{d}{dt}(\eta, \eta) = -\,(u\cdot\nabla\tilde{\varrho}, \eta)\,, \tag{6.4}$$

since $(v\cdot\nabla\eta, \eta) = 0$.

From (6.2) and (6.4) one obtains

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\int_\Omega(\tilde{\varrho}|u|^2 + \eta^2)\,dx = -\int_\Omega\tilde{\varrho}[(u\cdot\nabla)v]\cdot u\,dx\,- \\ -\int_\Omega\eta\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b + \nabla\tilde{\varrho}\right]\cdot u\,dx\,.\end{aligned} \tag{6.5}$$

Set

$$f(t) \equiv \tfrac{1}{2}\int_\Omega\left(\tilde{\varrho}|u|^2 + \eta^2\right)dx\,.$$

Obviously $f(0) = 0$; moreover from (3.16) and (3.11)

$$\|\nabla\tilde{\varrho}\|_\infty \leqslant 2\|\nabla\varrho_0\|_\infty \exp\left[\|Dv\|_{L^1(0,T;L^\infty(\Omega))}\right]$$

and consequently from (6.5)

$$f'(t) \leqslant c(t)\,f(t)$$

where $c(t)\in L^1(0, T)$. By Gronwall's lemma $f(t)$ vanishes identically in $[0, T]$, i.e. $\tilde{v} = v$ and $\tilde{\varrho} = \varrho$ in Q_T.

Finally, from $(\mathrm{E})_1$ it follows that $\nabla\pi = \nabla\tilde{\pi}$ in Q_T, i.e. $\pi = \tilde{\pi}$ up to an arbitrary function of t.

PART II

7. Existence of a solution of system (E) when Ω is not simply connected.

Let Ω be a bounded connected open subset of $\mathbb{R}^2$. We assume that Γ consists of $m+1$ simple closed curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, where Γ_j $(j = 1, \ldots, m)$ are inside of Γ_0 and outside of one another.

We denote by v the vector field defined in (3.5) and by $u^{(k)}$, $k = 1, \ldots, m$, the vector fields introduced at the end of § 1 in [4]. We have $u^{(k)} \in C^{2+\lambda}(\bar{\Omega})$, $\operatorname{rot} u^{(k)} = 0$, $\operatorname{div} u^{(k)} = 0$ in $\bar{\Omega}$ and $u^{(k)} \cdot n = 0$ on Γ. We put

$$\bar{v}(t, x) \equiv v(t, x) + \sum_{k=1}^{m} \theta_k(t) u^{(k)}(x) \equiv v(t, x) + v'(t, x) , \tag{7.1}$$

and consequently we have $\operatorname{div} \bar{v} = 0$ and $\operatorname{rot} \bar{v} = \varphi$ in Q_T, $\bar{v} \cdot n = 0$ on $[0, T] \times \Gamma$.

We define $\bar{\varrho}(t, x)$ to be the solution of

$$\begin{cases} \dfrac{\partial \bar{\varrho}}{\partial t} + \bar{v} \cdot \nabla \bar{\varrho} = 0 & \text{in } Q_T , \\ \bar{\varrho}|_{t=0} = \varrho_0 & \text{in } \bar{\Omega} . \end{cases} \tag{7.2}$$

Now we prove that there exist $\theta_k(t) \in C^{1+\lambda}([0, T])$ such that

$$\left(\bar{\varrho} \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right) = 0 \qquad \forall t \in [0, T] , \tag{7.3}$$

$$(\bar{v}|_{t=0} - a, u^{(k)}) = 0 \tag{7.4}$$

for each $k = 1, \ldots, m$. We are going to use the Schauder fixed point theorem.

We consider the map $\bar{\theta}_k \mapsto \bar{v}$ from $C^0([0, T])$ in $C^{0,2+\lambda}(Q_T)$ defined by (7.1), the map $\bar{v} \mapsto \bar{\varrho}$ from $C^{0,2+\lambda}(Q_T)$ in $C^{\lambda,0}(Q_T)$ defined by (7.2) and finally the map $(\bar{v}, \bar{\varrho}) \mapsto \theta_k$ defined by (7.3), (7.4), i.e.

$$\sum_{s=1}^{m} \mu_{ks}(t) \frac{d\theta_s(t)}{dt} + \sum_{s,h=1}^{m} \mu_{ksh}(t) \theta_s(t) \theta_h(t) + \sum_{s=1}^{m} [\nu_{ks}(t) + \eta_{ks}(t)] \theta_s(t) + \\ + \mu_k(t) + \nu_k(t) + \eta_k(t) = 0 \qquad \text{in } [0, T] , \tag{7.3$'$}$$

$$\theta_k(0) = (a, u^{(k)}) , \tag{7.4$'$}$$

for each $k = 1, \ldots, m$. We have defined

$$\begin{aligned}
&\mu_{ks}(t) \equiv (\bar\varrho u^{(s)}, u^{(k)})\,, &&\mu_{ksh}(t) \equiv (\bar\varrho(u^{(s)}\cdot\nabla)u^{(h)}, u^{(k)})\,,\\
&\nu_{ks}(t) \equiv (\bar\varrho(v\cdot\nabla)u^{(s)}, u^{(k)})\,, &&\eta_{ks}(t) \equiv (\bar\varrho(u^{(s)}\cdot\nabla)v, u^{(k)})\,,\\
(7.5)\qquad &\mu_k(t) \equiv \left(\bar\varrho\,\frac{\partial v}{\partial t}, u^{(k)}\right), &&\nu_k(t) \equiv (\bar\varrho(v\cdot\nabla)v, u^{(k)})\,,\\
& &&\eta_k(t) \equiv -(\bar\varrho b, u^{(k)})\,.
\end{aligned}$$

Since $u^{(k)} \in C^{2+\lambda}(\bar\Omega)$, $v \in C^{1,2+\lambda}(Q_T) \cap C^{1+\lambda,0}(Q_T)$ and $\bar\varrho \in C^{\lambda,0}(Q_T)$, all these coefficients belong to $C^{\lambda}([0, T])$.

The notation $\tilde c, \tilde c_1, \tilde c_2, \ldots,$ will be used for constants depending at most on λ, Ω, a, b, ϱ_0, m, $u^{(k)}$.

Assume that estimates (3.1) hold and moreover

$$(7.6)\qquad \sup_{t\in[0,T]}\Big[\sum_{k=1}^{m}\bar\theta_i(t)^2\Big]^{\frac12} \equiv \|\bar\theta\|_\infty < E\,,$$

where $\bar\theta \equiv (\bar\theta_1, \ldots, \bar\theta_m)$ and E is a constant that will be fixed in the following.

One has

$$\begin{aligned}
&\|\bar v\|_\infty \;\; < \|v\|_\infty + \sum_k \|\bar\theta_k\|_\infty \|u^{(k)}\|_\infty < \tilde c(A + E)\,,\\
(7.7)\qquad &\|D\bar v\|_\infty < \|Dv\|_\infty + \sum_k \|\bar\theta_k\|_\infty \|Du^{(k)}\|_\infty < \tilde c(A + E)\,.
\end{aligned}$$

Define $\bar U(\sigma, t, x)$ to be the solution of

$$\begin{cases} \dfrac{d\bar U}{d\sigma}(\sigma, t, x) = \bar v(\sigma, \bar U(\sigma, t, x))\,,\\ \bar U(t, t, x) \quad = x\,; \end{cases}$$

one has, as in $(3.10)_1$:

$$(7.8)\qquad [\bar U]_{\mathrm{lip},0} = \left\|\frac{\partial\bar U}{\partial t}\right\|_\infty = \|D\bar U\|_\infty \|\bar v\|_\infty < 2\|\bar v\|_\infty \exp\left[T\|D\bar v\|_\infty\right].$$

It follows from (7.7) and (7.8) that

$$(7.9)\qquad [\bar U]_{\mathrm{lip},0} < \tilde c(A + E)\exp\left[\tilde c T(A + E)\right].$$

From (7.2) one has $\bar{\varrho}(t,x)=\varrho_0(\bar{U}(0,t,x))$, hence

$$\begin{aligned}&\|\bar{\varrho}\|_\infty\leqslant\|\varrho_0\|_\infty\,,\\ &[\bar{\varrho}]_{\lambda,0}\leqslant[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\lambda,0}\leqslant T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0}\,.\end{aligned}\tag{7.10}$$

Define

$$K_2\equiv\sup_k\|u^{(k)}\|_{L^2(\Omega)}\,.\tag{7.11}$$

We have from (7.5)

$$\begin{aligned}&\|\mu_{ks}\|_\infty\leqslant K_2^2\|\varrho_0\|_\infty\,, &&[\mu_{ks}]_\lambda\leqslant\tilde{c}T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0}\,,\\ &\|\mu_{ksh}\|_\infty\leqslant\tilde{c}\|\varrho_0\|_\infty\,, &&[\mu_{ksh}]_\lambda\leqslant\tilde{c}T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0}\,,\\ &\|\nu_{ks}\|_\infty+\|\eta_{ks}\|_\infty\leqslant\tilde{c}A\|\varrho_0\|_\infty\,,\\ &&&[\nu_{ks}]_\lambda+[\eta_{ks}]_\lambda\leqslant\tilde{c}A(\|\varrho_0\|_\infty+T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0})\,,\\ &\|\mu_k\|_\infty\leqslant K_1K_2B|\Omega|^{\frac12}\|\varrho_0\|_\infty\,,\\ &&&[\mu_k]_\lambda\leqslant\tilde{c}(D\|\varrho_0\|_\infty+T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0}B)\,,\\ &\|\nu_k\|_\infty\leqslant\tilde{c}A^2\|\varrho_0\|_\infty\,, &&[\nu_k]_\lambda\leqslant\tilde{c}A^2(\|\varrho_0\|_\infty+T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0})\,,\\ &\|\eta_k\|_\infty\leqslant\tilde{c}\|\varrho_0\|_\infty\,, &&[\eta_k]_\lambda\leqslant\tilde{c}(\|\varrho_0\|_\infty+T^{1-\lambda}[\varrho_0]_{\mathrm{lip}}[\bar{U}]_{\mathrm{lip},0})\,,\end{aligned}\tag{7.12}$$

where $|\Omega|\equiv\mathrm{meas}\,\Omega$.

Let $M(t)$ be the $(m\times m)$-symmetric matrix $\{\mu_{ks}(t)\}$. One sees easily that $|\xi|^2\min_{\bar{\Omega}}\varrho_0\leqslant M(t)\xi\cdot\xi\leqslant\max_{\bar{\Omega}}\varrho_0|\xi|^2$ for each $\xi\in\mathbb{R}^m$, and

$$0<\left(\min_{\bar{\Omega}}\varrho_0\right)^m\leqslant\det M(t)\leqslant\left(\max_{\bar{\Omega}}\varrho_0\right)^m\qquad\forall t\in[0,T]\,.\tag{7.13}$$

The element $\bar{\mu}_{ks}(t)$ of $[M(t)]^{-1}$ has the form

$$\bar{\mu}_{ks}(t)=\frac{(-1)^{k+s}M_{ks}(t)}{\det M(t)}\,,\tag{7.14}$$

where $M_{ks}(t)$ is the minor of the matrix $M(t)$ corresponding to the (k,s)-element of $M(t)$.

Hence

$$\begin{aligned}&\|\bar{\mu}_{ks}\|_\infty\leqslant(m-1)!\,K_2^{2(m-1)}\frac{\|\varrho_0\|^{m-1}}{(\min_{\bar{\Omega}}\varrho_0)^m}\,,\\ &[\bar{\mu}_{ks}]_\lambda\leqslant\sum_{i,j,h,r}\left(\|\mu_{ij}\|_\infty^{m-2}\frac{1}{(\min_{\bar{\Omega}}\varrho_0)^m}+\|\mu_{ij}\|_\infty^{2(m-1)}\frac{1}{(\min_{\bar{\Omega}}\varrho_0)^{2m}}\right)[\mu_{hr}]_\lambda\,,\end{aligned}\tag{7.15}$$

and by using (7.12)

$$[\bar{\mu}_{ks}]_\lambda \leqslant \tilde{c}T^{1-\lambda}[\varrho_0]_{\text{lip}}[\overline{U}]_{\text{lip},0}\left[\frac{\|\varrho_0\|_\infty^{m-2}}{(\min\limits_{\bar{\Omega}} \varrho_0)^m} + \frac{\|\varrho_0\|_\infty^{2(m-1)}}{(\min\limits_{\bar{\Omega}} \varrho_0)^{2m}}\right]. \tag{7.16}$$

Applying $[M(t)]^{-1}$ to (7.3)′, one obtains

$$\frac{d\theta_k(t)}{dt} = \sum_{s,h=1}^m \bar{\mu}_{ksh}(t)\theta_s(t)\theta_h(t) + \sum_{s=1}^m [\bar{\nu}_{ks}(t) + \bar{\eta}_{ks}(t)]\theta_s(t) + \\ + \bar{\mu}_k(t) + \bar{\nu}_k(t) + \bar{\eta}_k(t) \qquad \text{in } [0, T], \tag{7.17}$$

where

$$\bar{\mu}_k(t) \equiv -\sum_{i=1}^m \bar{\mu}_{ki}(t)\mu_i(t) \tag{7.18}$$

and analogously for the other coefficients.

Obviously the system (7.17), (7.4)′ has an unique local solution $\theta_k(t)$, $k = 1, \dots, m$.

Moreover, taking the scalar product of (7.17) with $\theta(t)$, one has

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}|\theta(t)|^2 \leqslant \tilde{c}\dfrac{\|\varrho_0\|_\infty^m}{(\min\limits_{\bar{\Omega}} \varrho_0)^m} \cdot \\ \qquad\qquad \cdot\left\{|\theta(t)|^3 + A|\theta(t)|^2 + (A^2 + B + 1)|\theta(t)|\right\}, \\ |\theta(0)|^2 \leqslant \tilde{c}_5^2\|a\|_\infty^2 . \end{cases} \tag{7.19}$$

Hence, if we choose $E > \tilde{c}_5\|a\|_\infty$ in (7.6), we see that there exists $T^* \in]0, T]$ such that

$$|\theta(t)| \leqslant E \qquad \text{in } [0, T^*].$$

If we put

$$S_1 \equiv \left\{\bar{\theta}(t) \in C^0([0, T^*]) \mid \|\bar{\theta}\|_\infty \leqslant E\right\}$$

and we denote by F_1 the map $\bar{\theta} \mapsto \theta$ defined by (7.1), (7.2), (7.17) and (7.4)′, we have $F_1(S_1) \subset S_1$.

Moreover from (7.17) and the Ascoli-Arzelà theorem, it follows that $F_1(S_1)$ is relatively compact in $C^0([0, T^*])$.

Finally, we see easily that $F_1 \colon S_1 \to S_1$ is continuous, consequently F_1 has a fixed point in S_1.

Hence equation (7.3), (7.4) has a local solution $\theta(t) \in C^{1+\lambda}$. We want to prove that $\theta(t)$ is a global solution.

From (7.3) we have

$$0 = \left(\bar{\varrho}\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - b\right], v'\right) = \left(\bar{\varrho}\,\frac{\partial v'}{\partial t}, v'\right) + \left(\bar{\varrho}\,\frac{\partial v}{\partial t}, v'\right) + \\ + \left(\bar{\varrho}(\bar{v}\cdot\nabla)v', v'\right) + \left(\bar{\varrho}(\bar{v}\cdot\nabla)v, v'\right) - (\bar{\varrho}b, v')\,.$$

Moreover

$$\left(\bar{\varrho}(\bar{v}\cdot\nabla)v', v'\right) = -\tfrac{1}{2}\left((\bar{v}\cdot\nabla\bar{\varrho})v', v'\right),$$

and from (7.2)

$$\frac{1}{2}\,\frac{d}{dt}\,(\bar{\varrho}v', v') = \left(\bar{\varrho}\,\frac{\partial v'}{\partial t}, v'\right) - \frac{1}{2}\left((\bar{v}\cdot\nabla\bar{\varrho})v', v'\right).$$

Hence

$$0 = \frac{1}{2}\,\frac{d}{dt}\,(\bar{\varrho}v', v') + \left(\bar{\varrho}\,\frac{\partial v}{\partial t}, v'\right) + \left(\bar{\varrho}(v'\cdot\nabla)v, v'\right) + \left(\bar{\varrho}(v\cdot\nabla)v, v'\right) - (\bar{\varrho}b, v')\,,$$

i.e.

$$\frac{1}{2}\,\frac{d}{dt}\sum_{k,s}\mu_{ks}(t)\theta_k(t)\theta_s(t) = \\ = -\sum_k \mu_k(t)\theta_k(t) - \sum_{k,s}\eta_{ks}(t)\theta_k(t)\theta_s(t) - \sum_k \nu_k(t)\theta_k(t) - \sum_k \eta_k(t)\theta_k(t)\,.$$

Consequently

$$\frac{1}{2}\,\frac{d}{dt}\,[M(t)\theta(t)\cdot\theta(t)] \leqslant \tilde{c}_6\|\varrho_0\|_\infty[A|\theta(t)|^2 + (A^2+B+1)|\theta(t)|] \leqslant \\ \leqslant \tilde{c}_6\|\varrho_0\|_\infty\left[\frac{A}{\min \varrho_0}\,M(t)\theta(t)\cdot\theta(t) + \frac{(A^2+B+1)}{\sqrt{\min_{\bar{\Omega}} \varrho_0}}\sqrt{M(t)\theta(t)\cdot\theta(t)}\right],$$

$$M(0)\theta(0)\cdot\theta(0) \leqslant \tilde{c}_7\|\varrho_0\|_\infty\|a\|_\infty^2\,.$$

Set

$$\alpha_1 \equiv 2\tilde{c}_6\,\frac{\|\varrho_0\|_\infty}{\sqrt{\min_{\bar{\Omega}} \varrho_0}}\,(A^2+B+1)\,, \qquad \alpha_2 \equiv 2\tilde{c}_6\,\frac{\|\varrho_0\|_\infty}{\min_{\bar{\Omega}} \varrho_0}\,A\,;$$

the solution $y(t)$ of

$$\begin{cases} y'(t) = \alpha_1\sqrt{y(t)} + \alpha_2 y(t)\,, \\ y(0) = \tilde{c}_7\|\varrho_0\|_\infty\,\|a\|_\infty^2\,, \end{cases}$$

satisfies

$$\alpha_1 + \alpha_2\sqrt{y(t)} = [\alpha_1 + \alpha_2\sqrt{y(0)}]\exp[(\alpha_2/2)t]\,,$$

Hence by comparison theorems

$$|\theta(t)| \leqslant \frac{A^2+B+1}{A}\left(\exp\left[\frac{\alpha_2}{2}t\right]-1\right) + \sqrt{\tilde{c}_7\frac{\|\varrho_0\|_\infty}{\min\limits_{\bar{\Omega}}\varrho_0}}\,\|a\|_\infty\exp\left[\frac{\alpha_2}{2}t\right] \qquad \forall t\in[0,T]\,,$$

i.e. $\theta(t)$ is a global solution in $[0, T]$ and

$$\text{(7.20)}\qquad \|\theta\|_\infty \leqslant \frac{A^2+B+1}{A}(\exp[\tilde{c}AT]-1) + \tilde{c}_8\exp[\tilde{c}AT] \leqslant \leqslant \tilde{c}(A^2+B+1)T\exp[\tilde{c}AT] + \tilde{c}_8\exp[\tilde{c}AT]\,.$$

Define

$$\text{(7.21)}\qquad K_3 \equiv m!\,\frac{\|\varrho_0\|_\infty^m}{(\min\limits_{\bar{\Omega}}\varrho_0)^m}\,K_2^{2m-1}\,|\Omega|^{\frac{1}{2}}\,.$$

From (7.12), (7.15) and (7.18) one has

$$\|\tilde{\mu}_k\|_\infty \leqslant K_3K_1B\,.$$

Consequently, from (7.17) and (7.15), (7.16)

$$\text{(7.22)}\qquad \left\|\frac{d\theta}{dt}\right\|_\infty \leqslant \tilde{c}_9(\|\theta\|_\infty^2 + A^2 + 1) + K_3K_1B\,,$$

$$\text{(7.23)}\qquad \left[\frac{d\theta}{dt}\right]_\lambda \leqslant \tilde{c}_{10}(A^2 + A\|\theta\|_\infty + 1) + K_3K_1D + + \tilde{c}T^{1-\lambda}([\bar{U}]_{\mathrm{lip},0} + \|\theta\|_\infty + A)(\|\theta\|_\infty^2 + A^2 + B + 1)\,.$$

Finally, define

$$\text{(7.24)}\qquad K_4 \equiv \sqrt{2}\left[\sum_k \|u^{(k)}\|_\infty^2\right]^{\frac{1}{2}};$$

from (7.1) and (3.8) one has,

$$
(7.25)\qquad
\begin{aligned}
\|\bar v\|_{0,2+\lambda} &\leqslant \tilde c(A + \|\theta\|_\infty)\,,\\
\|D_t\bar v\|_\infty &\leqslant K_1 B + K_4\left\|\frac{d\theta}{dt}\right\|_\infty,\\
\|D_t\bar v\|_{0,1+\lambda} &\leqslant \tilde c\left(C + \left\|\frac{d\theta}{dt}\right\|_\infty\right),\\
[D_t\bar v]_{\lambda,0} &\leqslant K_1 D + K_4\left[\frac{d\theta}{dt}\right]_\lambda,\\
[D_t\bar v]_{0,\lambda} &\leqslant \tilde c\left(B + \left\|\frac{d\theta}{dt}\right\|_\infty\right),
\end{aligned}
$$

which replace estimates (3.8).

Set

$$
\bar\gamma \equiv \beta + \frac{\operatorname{Rot}\bar\varrho}{\bar\varrho}\cdot\left[\frac{\partial\bar v}{\partial t} + (\bar v\cdot\nabla)\bar v - b\right] \qquad \text{in } Q_T\,;
$$

by replacing v, U, ϱ, γ with $\bar v$, $\bar U$, $\bar\varrho$, $\bar\gamma$ in the proofs of Lemmas 3.2, 3.3, 4.1 and by using (7.25) one obtains:

LEMMA 7.1. *Let $\bar\xi(t,x)$ be the solution of*

$$
\begin{cases}
\dfrac{\partial\bar\xi}{\partial t} + \bar v\cdot\nabla\bar\xi = \bar\gamma & \text{in } Q_T\,,\\
\bar\xi|_{t=0} = \alpha & \text{in } \bar\Omega\,.
\end{cases}
$$

Then Lemma 4.2 is true if we substitute in (4.5), (4.6), (4.7) and (4.8) ξ with $\bar\xi$, A with $A + \|\theta\|_\infty$, $K_1 B$ with $K_1 B + K_4\|d\theta/dt\|_\infty$, B with $B + \|d\theta/dt\|_\infty$, C with $C + \|d\theta/dt\|_\infty$, $K_1 D$ with $K_1 D + K_4[d\theta/dt]_\lambda$. Constants c and $\bar c$, c_i and $\bar c_i$ must be replaced respectively by $\tilde c$, $\tilde c_i$. We will denote these new estimates by (4.5)′, (4.6)′, (4.7)′ and (4.8)′.

Hence the existence of a solution of system (4.17) will be a consequence of the existence of a fixed point for the map $\bar F\colon \varphi \mapsto \bar\xi$.

First of all, we prove that there exists $T_1 \in\,]0, T]$ such that $\bar F(S) \subset S$, provided that A, B, C, D are chosen in a suitable way in (4.11).

By using (7.20), (7.22) and (7.23), we obtain from Lemma 7.1

that in Q_T one has

$$(7.36)\qquad \begin{aligned} &\|\bar{\zeta}\|_{0,1+\lambda} \leqslant f_1(T, A, B, C)\,, \\ &\|D_t\bar{\zeta}\|_\infty \leqslant f_2(T, A, B, C)\,, \\ &\|D_t\bar{\zeta}\|_{0,\lambda} \leqslant f_3(T, A, B, C)\,, \\ &[D_t\bar{\zeta}]_{\lambda,0} \leqslant f_4(T, A, B, C, D)\,, \end{aligned}$$

where the functions f_i are continuous, non-negative, and non-decreasing with respect to each variable. Hence, if we fix A, B, C, D such that

$$(7.27)\qquad \begin{aligned} &f_1(0, A, B, C) < A\,, \\ &f_2(0, A, B, C) < B\,, \\ &f_3(0, A, B, C) < C\,, \\ &f_4(0, A, B, C, D) < D\,, \end{aligned}$$

there exists $T_1 \in\,]0, T]$ for which

$$(7.28)\qquad \begin{aligned} &\|\bar{\zeta}\|_{0,1+\lambda} \leqslant f_1(T_1, A, B, C) \leqslant A\,, \\ &\|D_t\bar{\zeta}\|_\infty \leqslant f_2(T_1, A, B, C) \leqslant B\,, \\ &\|D_t\bar{\zeta}\|_{0,\lambda} \leqslant f_3(T_1, A, B, C) \leqslant C\,, \\ &[D_t\bar{\zeta}]_{\lambda,0} \leqslant f_4(T_1, A, B, C, D) \leqslant D\,, \end{aligned}$$

in Q_{T_1}.

It is easy to verify that (7.27) has a solution, provided that condition (A) is satisfied. For example, one can choose successively

$$(7.29)\qquad \begin{aligned} &A > 2\|\alpha\|_{1+\lambda}\,, \\ &B > \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_1(1 + K_3K_4)B + \tilde{c}_1(A + \tilde{c}_8)\|D\alpha\|_\infty + \\ &\qquad + \tilde{c}_1[(A + c_8)^2 + 1] + \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_4\tilde{c}_9(\tilde{c}_8^2 + A^2 + 1)\,, \\ &C > B + 2\tilde{c}_2(A + \tilde{c}_8)\|D\alpha\|_\lambda + \\ &\qquad + \tilde{c}_2\{(A + \tilde{c}_8)^2 + B + \tilde{c}_9(\tilde{c}_8^2 + A^2 + 1) + K_3K_1B + 1\}\,, \\ &D > \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_1(1 + K_3K_4)D + \tilde{c}_3\tilde{c}_4(A + \tilde{c}_8)^{1+\lambda}[D\alpha]_\lambda + \\ &\qquad + \tilde{c}_3[(A + \tilde{c}_8)^2 + B + \tilde{c}_9(\tilde{c}_8^3 + A^2 + 1) + K_3K_1B + 1]\cdot \\ &\qquad \cdot(A + \tilde{c}_8)^\lambda + \tilde{c}_3 + \left\|\frac{D\varrho_0}{\varrho_0}\right\|_\infty K_4\tilde{c}_{10}(A^2 + \tilde{c}_8A + 1)\,. \end{aligned}$$

Lemma 4.3 is proved as before, provided that $\overline{F}: S \mapsto S$ is continuous from $C^1(Q_{T_1})$ in $C^0(Q_{T_1})$.

Hence, we must prove that if $\varphi_n \to \varphi$ in $C^1(Q_{T_1})$, $\varphi_n \in S$, then $\overline{v}_n$ and $\overline{v}$ satisfy (4.14) and (4.15). Since v^n and v satisfy these last conditions, it is sufficient to prove that

$$\theta^n \to \theta \qquad \text{uniformly in } [0, T_1],$$
$$\frac{d\theta^n}{dt} \to \frac{d\theta}{dt} \qquad \text{uniformly in } [0, T_1];$$

we begin by recalling that $\overline{v}^n$ and ϱ_n satisfy

$$\text{(7.30)} \qquad \begin{cases} \dfrac{\partial \overline{\varrho}_n}{\partial t} + \overline{v}^n \cdot \nabla \overline{\varrho}_n = 0 & \text{in } Q_{T_1}, \\ \overline{\varrho}_n|_{t=0} = \varrho_0 & \text{in } \overline{\Omega}, \end{cases}$$

$$\text{(7.31)} \qquad \left(\overline{\varrho}_n \left[\frac{\partial \overline{v}^n}{\partial t} + (\overline{v}^n \cdot \nabla)\overline{v}^n - b\right], u^{(k)}\right) = 0 \qquad \forall t \in [0, T_1],$$

$$\text{(7.32)} \qquad (\overline{v}^n|_{t=0} - a, u^{(k)}) = 0,$$

for each $k = 1, \ldots, m$.

Set now $\eta \equiv \overline{\varrho}_n - \overline{\varrho}$, $u' \equiv v'^n - v'$, $u \equiv v^n - v$, $\overline{u} \equiv \overline{v}^n - \overline{v} = u + u'$; one obtains from (7.3)

$$\text{(7.33)} \qquad \left(\overline{\varrho}_n \left[\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\overline{v} - b\right], u^{(k)}\right) - \left(\eta \left[\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\overline{v} - b\right], u^{(k)}\right) = 0.$$

On substrating (7.33) from (7.31) one has

$$\left(\overline{\varrho}_n \left[\frac{\partial \overline{u}}{\partial t} + (\overline{v}^n \cdot \nabla)\overline{u} + (\overline{v} \cdot \nabla)\overline{v}\right], u^{(k)}\right) = -\left(\eta \left[\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\overline{v} - b\right], u^{(k)}\right),$$

and multipling by $\theta_k^n - \theta_k$

$$\text{(7.34)} \qquad \left(\overline{\varrho}_n \left[\frac{\partial u'}{\partial t} + (\overline{v}^n \cdot \nabla)u' + (u' \cdot \nabla)\overline{v}\right], u'\right) =$$
$$= -\left(\overline{\varrho}_n \left[\frac{\partial u}{\partial t} + (\overline{v}^n \cdot \nabla)u + (u \cdot \nabla)\overline{v}\right], u'\right) - \left(\eta \left[\frac{\partial \overline{v}}{\partial t} + (\overline{v} \cdot \nabla)\overline{v} - b\right], u'\right).$$

From $(7.30)_1$

$$\left(\bar{\varrho}_n \frac{\partial u'}{\partial t}, u'\right) = \frac{1}{2}\frac{d}{dt}(\bar{\varrho}_n u', u') + \frac{1}{2}\left((\bar{v}^n\cdot\nabla\bar{\varrho}_n)u', u'\right)$$

and moreover

$$(\bar{\varrho}_n(\bar{v}^n\cdot\nabla)u', u') = \frac{1}{2}\sum_{i,j}\int_\Omega \bar{\varrho}_n \bar{v}_i^n \frac{\partial u_j'^2}{\partial x_i}\,dx = -\frac{1}{2}\left((\bar{v}^n\cdot\nabla\bar{\varrho}_n)u', u'\right).$$

Hence (7.34) becomes

$$\frac{1}{2}\frac{d}{dt}(\bar{\varrho}_n u', u') + (\bar{\varrho}_n(u'\cdot\nabla)\bar{v}, u') = \tag{7.35}$$

$$= -\left(\bar{\varrho}_n\left[\frac{\partial u}{\partial t} + (\bar{v}^n\cdot\nabla)u + (u\cdot\nabla)\bar{v}\right], u'\right) - \left(\eta\left[\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - b\right], u'\right).$$

On other hand, from (7.30) and (7.2) one obtains

$$\frac{\partial\eta}{\partial t} + \bar{v}\cdot\nabla\eta = -u'\cdot\nabla\bar{\varrho}_n - u\cdot\nabla\bar{\varrho}_n\,,$$

and taking the scalar product with η

$$\frac{1}{2}\frac{d}{dt}(\eta,\eta) = -(\eta u', \nabla\bar{\varrho}_n) - (\eta u, \nabla\bar{\varrho}_n)\,. \tag{7.36}$$

Set $f(t) \equiv \frac{1}{2}(\bar{\varrho}_n u', u') + \frac{1}{2}(\eta,\eta)$: from (7.35) and (7.36) one has

$$\frac{d}{dt}f(t) \leqslant c(\bar{\varrho}_n u', u') + \left\|\frac{\partial v^n}{\partial t} - \frac{\partial v}{\partial t}\right\|_\infty \int_\Omega \bar{\varrho}_n |u'|\,dx +$$

$$+ c\|Dv^n - Dv\|_\infty \int_\Omega \bar{\varrho}_n|u'|\,dx + c\|v^n - v\|_\infty \int_\Omega \bar{\varrho}_n |u'|\,dx +$$

$$+ c\int_\Omega |\eta|\,|u'|\,dx + c\|v^n - v\|_\infty \int_\Omega |\eta|\,dx\,,$$

since $\|\bar{v}^n\|_\infty$ and $\|\nabla\bar{\varrho}_n\|_\infty$ are bounded, and $0 < \min_{\bar{\Omega}} \varrho_0 \leq \bar{\varrho}_n(t, x) \leq \|\varrho_0\|_\infty$. Hence

$$\begin{cases} f'(t) \leq cf(t) + c_n\sqrt{f(t)}\,, \\ f(0) = 0 \end{cases}$$

where $c_n \to 0$, and consequently by comparison theorems

$$f(t) \leq \left(\frac{c_n}{c}\right)^2 \left(\exp\left[\frac{ct}{2}\right] - 1\right)^2 \qquad \text{in } [0, T_1]\,. \tag{7.37}$$

Estimate (7.37) gives

$$\begin{aligned} &\sup_{t\in[0,T_1]} \|v'^n - v'\|_{L^2(\Omega)} \underset{n}{\to} 0\,, \\ &\sup_{t\in[0,T_1]} \|\bar{\varrho}_n - \bar{\varrho}\|_{L^2(\Omega)} \underset{n}{\to} 0\,, \end{aligned} \tag{7.38}$$

i.e.

$$\sup_{t\in[0,T_1]} |\theta^n(t) - \theta(t)| = \sup_{t\in[0,T_1]} \|v'^n - v'\|_{L^2(\Omega)} \underset{n}{\to} 0\,. \tag{7.39}$$

Moreover from (7.5) one has

$$\|\mu^n_{ks} - \mu_{ks}\|_\infty \leq c \sup_{t\in[0,T_1]} \|\bar{\varrho}_n - \bar{\varrho}\|_{L^2(\Omega)} \underset{n}{\to} 0\,,$$

and analogously for the other coefficients.

Consequently from (7.14) it follows that

$$\|\bar{\mu}^n_{ks} - \bar{\mu}_{ks}\|_\infty \underset{n}{\to} 0\,,$$

and the same is true for each coefficient in (7.17); hence we conclude that

$$\left\|\frac{d\theta^n}{dt} - \frac{d\theta}{dt}\right\|_\infty \underset{n}{\to} 0\,.$$

As in § 4, we have proved that

$$\operatorname{rot}\left\{\bar{\varrho}\left[\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - b\right]\right\} = 0 \qquad \text{in } Q_{T_1}\,,$$

and moreover

$$\left(\bar{\varrho}\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} - b\right], u^{(k)}\right) = 0 \qquad \forall t \in [0, T_1]\,,$$
$$(\bar{v}|_{t=0} - a, u^{(k)}) = 0\,,$$

for each $k = 1, \ldots, m$.

As in Kato [5], Lemma 1.6 (see also Hopf [4]), it follows that there exists a scalar function $\bar{\pi} \in C^{0,1}(Q_{T_1})$ such that $(\mathrm{E})_1$ holds in Q_{T_1}. The further regularity properties of $\bar{\pi}$ are proved as in § 4, since $\bar{v}$ has the same regularity of v.

Finally, by using (7.4) one obtains that $\bar{v}|_{t=0} = a$ in $\bar{\Omega}$, i.e. we have found a solution $(\bar{v}, \bar{\pi}, \bar{\varrho})$ of system (E) in Q_{T_1}.

REFERENCES

[1] C. Bardos, *Existence et unicité de la solution de l'équation d'Euler en dimension deux,* J. Math. Anal. Appl., **40** (1972), pp. 769-790.

[2] K. K. Golovkin, *Vanishing viscosity in Cauchy's problem for hydromechanics equations,* Proc. Steklov Inst. Math., **92** (1966), pp. 33-53 (previously in Trudy Mat. Inst. Steklov, **92** (1966), pp. 31-49 [russian]).

[3] E. Hölder, *Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit,* Math. Z., **37** (1933), pp. 727-738.

[4] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen,* Math. Nachr., **4** (1950-51), pp. 213-231.

[5] T. Kato, *On Classical Solutions of the Two-Dimensional Non-Stationary Euler Equation,* Arch. Rat. Mech. Anal., **25** (1967), pp. 188-200.

[6] J. Leray, *Sur les mouvements des liquides illimites,* C.R.A.S. Paris, **194** (1932), 1892-1894.

[7] L. Lichtenstein, *Neuere entwicklung der Potentialtheorie. Konforme Abbildung,* Encycl. Math. Wiss., II C **3** (1918), pp. 177-377.

[8] J. E. Marsden, *Well-posedness of the equations of a non-homogeneous perfect fluid,* Comm. Partial Diff. Eq., **1** (1976), pp. 215-230.

[9] F. J. Mc Grath, *Nonstationary Plane Flow of Viscous and Ideal Fluids,* Arch. Rat. Mech. Anal., **27** (1967), pp. 329-348.

[10] A. C. Schaeffer, *Existence theorem for the flow of an ideal incompressible fluid in two dimensions,* Trans. Amer. Math. Soc., **42** (1937), pp. 497-513.

[11] J. SERRIN, *On the uniqueness of compressible fluid motions*, Arch. Rat. Mech. Anal., **3** (1959), pp. 271-288.

[12] A. VALLI, *Soluzioni classiche dell'equazione di Eulero dei fluidi bidimensionali in domini con frontiera variabile*, Ricerche di Mat., **26** (1977), pp. 301-333.

[13] W. WOLIBNER, *Un theorème sur l'esistence du mouvement plan d'un fluide parfait, homogène. incompressible, pendant un temps infiniment long*, Math. Z., **37** (1933), pp. 698-726.

[14] V. I. YUDOVICH, *Non-stationary flows of ideal incompressible fluids*, Zhur. Vych. Mat. i Mat. Fiz., **3** (1963), pp. 1032-1066 [russian].

[15] V. I. YUDOVICH, *A two dimensional problem of unsteady flow of an ideal incompressible fluid across a given domain*, Amer. Math. Soc. Translations, **57** (1966), pp. 277-304 (previously in Mat. Sb., **64** (1964), pp. 562-588 [russian]).

Added in proof: the analytic case in compact manifolds without boundary was studied in

[16] M. S. BAOUENDI - C. GOULAOUIC, *Solutions analytiques de l'equation d'Euler d'un fluide incompressible*, Seminaire Goulaouic-Schwartz, 1976-77 (Paris).

Manoscritto pervenuto in redazione il 20 luglio 1978.