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On Bifurcation and Asymptotic Bifurcation for Nondifferentiable Potential Operators and for Systems of the Hammerstein Type

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The greatest positive bifurcation point (cf. Theorem 1.1) and the greatest positive asymptotic bifurcation point (cf. Theorems 1.2 and 1.3) for a class of nondifferentiable potential operators on Hilbert spaces are characterized. These results, announced in [3], are used to study bifurcation for systems of the Hammerstein type (cf. Section 4).

0. INTRODUCTION

Notation

Let H be a real Hilbert space with norm and inner product denoted by $\|\ \|$ and $(\ ,\ )$, respectively. The strong convergence in H is denoted by $\to$ and the weak convergence by $\rightharpoonup$.

In the following $\mathbb{K}$ is a closed convex cone in H with vertex at the origin, i.e., $\mathbb{K}$ is a closed subset of H such that $t\mathbb{K} \subset \mathbb{K}$ for all $t \geqslant 0$ and $\mathbb{K} + \mathbb{K} \subset \mathbb{K}$. We assume that $\mathbb{K}$ is nonvoid and different from $\{0\}$. We denote by $\operatorname{int} \mathbb{K}$ the interior of $\mathbb{K}$. If $\rho \geqslant 0$, we put

$$S_\rho = \{u \in H : \|u\| = \rho\}, \qquad V_\rho = \{u \in H : \|u\| \leqslant \rho\}, \quad \mathbb{K}_\rho = \mathbb{K} \cap V_\rho .$$

Some Definitions

A real number λ is said to be a *bifurcation point* for an operator $\Gamma : \mathbb{K} \to H$ if for every $\epsilon > 0$ there exist a pair $(\lambda_\epsilon, u_\epsilon) \in \mathbb{R} \times \mathbb{K}$ with $|\lambda - \lambda_\epsilon| < \epsilon$ and $0 < \|u_\epsilon\| < \epsilon$ such that $\Gamma(u_\epsilon) = \lambda_\epsilon u_\epsilon$.

Similarly $+\infty$ is said to be a *bifurcation point* for Γ if the preceding conditions hold with $\lambda_\epsilon > 1/\epsilon$ instead of $|\lambda - \lambda_\epsilon| < \epsilon$; a similar definition holds for $-\infty$.

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If in the preceding definition we replace the condition $0 < \|u_\epsilon\| < \epsilon$ by the condition $\|u_\epsilon\| > 1/\epsilon$, one says that λ (respectively, $+\infty$ or $-\infty$) is an *asymptotic bifurcation point* for the operator Γ.

If S is a nonempty subset of H and Φ is a real functional defined on S, one says that Φ is *weakly continuous on* S if $u_n \in S$ and $u_n \rightharpoonup u_0 \in S$ implies that $\Phi(u_0) = \lim_{n\to+\infty} \Phi(u_n)$. One says that Φ is *weakly upper semicontinuous on* S if under the same conditions one has $\Phi(u_0) \geqslant \lim_{n\to+\infty} \sup \Phi(u_n)$. The *weak lower semicontinuity on* S is defined similarly.

We denoted by $\overline{S}$ the *weak closure* of S in the following sense: $\overline{S} = \{u \in H : \exists u_n \in S, n = 1, 2, \ldots, u_n \rightharpoonup u\}$. Obviously $\overline{S} = S$ if S is a convex and (strongly) closed subset of H.

Let $\Phi: S \to \mathbb{R}$. We say that Φ is *weakly continuous on* $\overline{S}$ if there exists an extension $\overline{\Phi}$ of Φ to $\overline{S}$ that is weakly continuous on $\overline{S}$. One gives similar definitions for the *weak* (*upper or lower*) *semicontinuity of* Φ *on* $\overline{S}$.

In the following, different positive constants shall be denoted by the same symbol c.

The Classical Result

The following result is well known (cf. [6, §6, Theorem 2.1] and [10, §17, Theorem 17.6]):

THEOREM A. *Let $\Phi: H \to \mathbb{R}$, $\Phi(0) = 0$, be a weakly upper semicontinuous functional. Let Φ be differentiable and put $\Gamma \equiv \nabla\Phi$. Suppose that $\Gamma(0) = 0$ and let Γ be Fréchet differentiable at the origin with $D\Gamma(0) = B$ linear, continuous, self-adjoint, and compact. Then the largest eigenvalue λ_0 of B, if positive, is a bifurcation point for Γ.*

The aim of this chapter is to generalize Theorem A to a class of operators not necessarily differentiable at the origin[1]. The motivation was given by some elementary examples of potential operators on $\mathbb{R}^2$ to which this theorem does not apply. We give these examples now.

Examples with a Nondifferentiable Γ at the Origin

EXAMPLE 1. Consider the following set-up. Let $\phi: [0, +\infty[\to \mathbb{R}$ be a continuously differentiable function which does not vanish identically in any neighborhood of the origin and such that $\phi(0) = \phi'(0) = \phi''(0) = 0$. Let $\psi: [0, 2\pi] \to \mathbb{R}$ be a differentiable function such that $\psi(0) = \psi(2\pi)$, $\psi'(0) = \psi'(2\pi)$, and assume that ψ' is unbounded in $[0, 2\pi]$.

[1] We remark that there is a result of Turner [9, Theorem 2.6] on odd multiplicity bifurcation for nondifferentiable operators verifying $\|(\Gamma - B)(u)\| = O(\|u\|)$ at the origin with B linear and compact.

For any $y \in \mathbb{R}^2$ define a function Φ in polar coordinates (we write $y = (\xi, \eta) = [r, \theta]$, where $\xi = r\cos\theta$, $\eta = r\sin\theta$) by

$$\Phi(y) = \tfrac{1}{2}r^2 + \phi(r)\psi(\theta). \tag{0.1}$$

Then Φ is Fréchet differentiable (F-differentiable) on $\mathbb{R}^2$ and its gradient is given by $\nabla\Phi \equiv \Gamma = I + \omega$, where I is the identity operator on $\mathbb{R}^2$ and ω is given in each point $y = (\xi, \eta) = [r, \theta]$ by $\omega(y) = \omega_r(y) + \omega_t(y)$, where

$$\omega_r(y) = \phi'(r)\psi(\theta)y/r, \qquad \omega_t(y) = (\phi(r)/r)\psi'(\theta)(y^{\perp}/r), \tag{0.2}$$

and $y^{\perp} = (-\eta, \xi)$. Note that $\omega_r(y)$ is the radial component of $\omega(y)$ and $\omega_t(y)$ the tangential one.

Since $\Gamma(y) = y + \omega_r(y) + \omega_t(y)$, the equation $\Gamma(y) = \lambda y$ yields, by decomposition in the radial and the tangential components, $y + \omega_r(y) = \lambda y$ and $\omega_t(y) = 0$. Using (0.2) one obtains the system

$$[\phi'(r)/r] + 1 = \lambda, \qquad \psi'(\theta) = 0. \tag{0.3}$$

Obviously the second equation in (0.3) is solvable. Let θ_0 be a fixed solution of $\psi'(\theta_0) = 0$. The first equation in (0.3) admits, for each $r > 0$, the solution $\lambda_r = 1 + r^{-1}\phi'(r)\psi(\theta)$. Hence to each solution θ_0 of $\psi'(\theta_0) = 0$ corresponds a branch $y_r, \lambda_r \in \mathbb{R}^2 \times \mathbb{R}$ of solutions of $\Gamma(y_r) = \lambda_r y_r$, with $y_r = [r, \theta_0]$. Obviously $y_r \to 0$ as $r \to 0$; moreover $\lambda_r \to 1$ since $\phi''(0) = 0$. Thus 1 is a bifurcation point (the unique point) for the operator Γ. In order words the greatest bifurcation point for Γ (actually the unique one) is the greatest positive eigenvalue (the unique one) for the linear operator I. However, Theorem A does not apply since Γ is not F-differentiable at the origin. If it were, we would have $D\Gamma(0) = I$ (since this holds in the Gateaux sense) and the remainder $\omega(y)$ would verify

$$\lim_{|y| \to 0} |\omega(y)|/|y| = 0 \tag{0.4}$$

or equivalently, its radial and tangential components should verify

$$\lim_{|y| \to 0} |\omega_r(y)|/|y| = 0 \tag{0.5}$$

and

$$\lim_{|y| \to 0} |\omega_t(y)|/|y| = 0, \tag{0.6}$$

respectively. The first condition holds since $\phi''(0) = 0$, but the second one fails since

$$\sup_{|y| = r} |\omega_t(y)|/|y| = +\infty, \forall r > 0.$$

Our aim is to show that this is not a casual fact but a general one. More precisely, we shall prove in this chapter (see Theorem 1.1) that the thesis of Theorem A holds even if Γ is not F-differentiable at the origin provided that the radial component $\omega_r(u)$ of the remainder $\omega(u) \equiv \Gamma(u) - B(u)$ verifies

$$\lim_{\|u\| \to 0} \|\omega_r(u)\| / \|u\| = 0, \tag{0.7}$$

or, equivalently,

$$\lim_{\|u\| \to 0} \|(\omega(u), u)\| / \|u\|^2 = 0. \tag{0.8}$$

We emphasize that, as in the preceding example, we do not exclude the situation

$$\sup_{\|u\| = r} \|\omega(u)\| = +\infty, \qquad \forall r > 0.$$

EXAMPLE 2. We now give another example of a potential operator $\omega: \mathbb{R}^2 \to \mathbb{R}^2$ such that (0.5) holds but (0.6) fails. This function shall be utilized in Section 4 to show that there exist Hammerstein systems to which our results apply, but Theorem A does not apply; it is an easy exercise to construct other functions on $\mathbb{R}^2$ (or $\mathbb{R}^m$, $m > 2$) for which this holds. We turn now to our example. For the sake of clarity we put $e_1 = (1,0)$ and $e_2 = (0,1)$. Let $y = (\xi, \eta) \in \mathbb{R}^2$ and define a function $\Phi: \mathbb{R}^2 \to \mathbb{R}$ by

$$\Phi(y) = \phi(\xi/\eta^2)\xi\eta, \tag{0.9}$$

where $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(0) \neq 0$, is a continuously differentiable function with compact support on $\mathbb{R}$. Then Φ is continuously differentiable on $\mathbb{R}^2$ and its gradient $\nabla\Phi \equiv \omega$ has components given by

$$\begin{aligned} \omega_1(y) &\equiv \partial\Phi/\partial\xi\, e_1 = \{\phi(\xi/\eta^2)\eta + \phi'(\xi/\eta^2)\xi/\eta\}e_1, \\ \omega_2(y) &\equiv \partial\Phi/\partial\eta\, e_2 = \{\phi(\xi/\eta^2)\xi - 2\phi'(\xi/\eta^2)\xi^2/\eta^2\}e_2. \end{aligned} \tag{0.10}$$

By definition Φ and ω vanish when $\eta = 0$. We claim that ω verifies (0.8) but does not verify (0.4). First, one has

$$(\omega(y), y) = [2\phi(\xi/\eta^2) - \phi'(\xi/\eta^2)\xi/\eta^2]\xi\eta. \tag{0.11}$$

On the other hand, it follows from (0.9) that there exists a positive constant c such that

$$|\xi| \geqslant c\eta^2 \quad \Rightarrow \quad \omega(y) = 0 \tag{0.12}$$

since $\phi(\xi/\eta^2) = \phi'(\xi/\eta^2) = 0$ whenever $\xi/\eta^2 \notin \operatorname{supp}\phi$. Consequently one can assume that $|\xi| \leqslant c\eta^2$. Under this assumption (0.11) yields $|(\omega(y), y)| \leqslant$

$c|\xi|\,|\eta| \leqslant c|\eta|^3$ and consequently

$$|(\omega(y), y)|/|y|^2 \leqslant c|y| \to 0 \tag{0.13}$$

as $|y| \to 0$, as desired. However, if one considers the points $y_t = (0, t)$, $t \in \mathbb{R}$, one has

$$|\omega(y_t)|/|y_t| = |\phi(0)| \neq 0, \qquad \forall t \neq 0, \tag{0.14}$$

which contradicts (0.4). Thus our thesis is proved.

As a matter of fact we have proved only that 0 is not the F-derivative of ω at the origin. However, one can easily see that ω Fréchet differentiable at the origin implies $D\omega(0) = 0$.

We take the opportunity to state two estimates that shall be used in Section 4:

$$|\omega_1(y)| \leqslant c|\eta|, \qquad |\omega_2(y)| \leqslant c|\xi|. \tag{0.15}$$

These estimates follow from (0.10) and (0.12).

The two examples just presented show that (0.8) is the natural assumption to be made for the calculus of the largest positive bifurcation point.

Examples with a Nonlinear B

We shall give in this chapter another generalization of the F-differentiability condition requested in Theorem A, whose meaning is shown by elementary examples as, for instance, the function $(y^2/2)\operatorname{sgn} y$, defined in the real line, whose gradient is $|y|$.

Another elementary example (in the plane) is the following one. Using polar coordinates put

$$\Phi(y) = \tfrac{1}{2}r^2 \sin\theta. \tag{0.16}$$

This function is F-differentiable on $\mathbb{R}^2$ with $\nabla\Phi(y) \equiv B(y) = B_r(y) + B_t(y)$, where $B_r(y) = y\sin\theta$ and $B_t(y) = \frac{1}{2}y^{\perp}\cos\theta$ are, respectively, the radial and the tangential components of $B(y)$. Furthermore, the equation $B(y) = \lambda y$ is equivalent to the system (independent of r) $\lambda = \sin\theta$, $\cos\theta = 0$, which admits the solutions $\theta = \pi/2$, $\lambda = 1$ and $\theta = -\pi/2$, $\lambda = -1$. Hence -1 and 1 are the bifurcation points for B. However, Theorem A (as well as the generalization we have previously mentioned) does not apply since B is not decomposable as a sum $B = A + \omega$ with A linear and ω verifying (0.4) or (0.8). However, the bifurcation points for B, 1 and -1, coincide with

$$M(B) = \sup_{|x|=1} (By, y)/|y|^2 \qquad \text{and} \qquad m(B) = \inf_{|x|=1} (By, y)/|y|^2,$$

respectively. We claim that this is a general fact, provided B is a positively homogeneous operator [i.e., $B(ty) = tB(y)$, $\forall t \geqslant 0$].

In fact, we prove in this chapter that Theorem A holds if B is positively homogeneous and if λ_0 is replaced by $M(B)$. Note that $M(B)$ coincides with the largest eigenvalue λ_0 if B verifies the assumptions of Theorem A.

Operators Γ Defined Only on a Cone $\mathbb{K}$

The third direction in which we generalize Theorem A is to operators Γ defined only on a closed convex cone $\mathbb{K}$ with vertex at the origin. We assume in this case that a suitable geometrical condition holds (see Assumption 1.1). We emphasize that this geometrical condition holds if for each $u \in \mathbb{K}$ there exists $\epsilon = \epsilon(u) > 0$ such that $u + \epsilon\Gamma(u) \in \mathbb{K}$ (or equivalently if the intersection of $\mathbb{K}$ with the half-line $\{v : v = u + t\Gamma(u), t > 0\}$ is not empty) and consequently holds if $\Gamma\mathbb{K} \subset \mathbb{K}$. Note that there are elementary examples for which our condition holds but $\Gamma\mathbb{K} \subset \mathbb{K}$ fails. For instance, let $\mathbb{K}$ be the cone of positive functions (on a domain in $\mathbb{R}^n$) and consider $\Gamma u(x) = \pm \operatorname{sign} u(x)$ or $\Gamma u(x) = -cu(x)$, $c > 0$.

Examples Concerning the Asymptotic Bifurcation

Finally we shall prove also that our results hold for the asymptotic bifurcation case, as stated in Theorem 1.2 (to which we refer the reader). As for the ordinary bifurcation the gradient of function (0.16) gives us a clear motivation for considering in Theorem 1.2 a positively homogeneous B. Similarly, the two first examples of this section show, after slight alterations, that condition (1.2) is the natural one in Theorem 1.2. For the sake of completeness let us write the alterations: in the first example assume that $\lim_{t\to+\infty}[\phi'(t)/t] = \lim_{t\to+\infty}[\phi(t)/t^2] = 0$ instead of $\phi(0) = \phi'(0) = \phi''(0) = 0$. In the second example define

$$\Phi(y) = \phi(\xi^2/\eta)\xi\eta, \tag{0.17}$$

$\Phi(y) = 0$ if $\eta = 0$. This function is continuously differentiable on $\mathbb{R}^2$ and $\omega \equiv \nabla\Phi$ is given by its components

$$\begin{aligned} \omega_1(y) &= [\phi(\xi^2/\eta)\eta + 2\xi^2\phi'(\xi^2/\eta)]e_1, \\ \omega_2(y) &= [\phi(\xi^2/\eta)\xi - (\xi^3/\eta)\phi'(\xi^2/\eta)]e_2, \end{aligned} \tag{0.18}$$

being $\omega(y) = 0$ if $\eta = 0$. Since there exists a positive constant c such that $\omega(y) = 0$ whenever $\xi^2 \geqslant c|\eta|$ it follows easily that

$$|(\omega(y), y)|/|y|^2 \leqslant c/\sqrt{|y|}, \tag{0.19}$$

which implies (1.2). On the other hand, (0.14) holds again and consequently $|\omega(y)|/|y| \to 0$ as $|y| \to +\infty$ fails. Thus ω is not F-differentiable at infinity (note that $D\omega(\infty) = 0$ if ω were F-differentiable at infinity). Finally (0.15) holds again; this shall be used later.

Plan of the Chapter

The plan of the chapter is the following. In Section 1 we state the main results that shall be proved in Section 2. In Section 3 we discuss conditions (1.1) and (1.2) and their relationship with Gateaux differentiability (G-differentiability) and F-differentiability. In Section 4 we apply Theorems 1.1 and 1.2 to the study of bifurcation and asymptotic bifurcation for systems of the Hammerstein type and we establish sufficient conditions in order that (1.1) and (1.2) should hold for these systems. Finally in the Appendix we consider briefly the case $M(B) = 0$ and the case in which B is homogeneous of deg $\alpha > 0$.

1. Main Results

One says that $B: \mathbb{K} \to H$ is *positively homogeneous* if $B(tu) = tB(u)$ for any $u \in \mathbb{K}$ and any $t \geqslant 0$. We define

$$m_{\mathbb{K}}(B) = \inf_{u \in \mathbb{K} \cap S_1} (B(u), u), \qquad M_{\mathbb{K}}(B) = \sup_{u \in \mathbb{K} \cap S_1} (B(u), u).$$

One verifies easily that the following two propositions hold.

PROPOSITION 1.1. *Let $\Gamma = B + \omega$ with $B: \mathbb{K} \to \mathbb{R}$ positively homogeneous and $\omega: \mathbb{K} \to \mathbb{R}$ verifying*

$$\lim_{\substack{u \to 0 \\ 0 \neq u \in \mathbb{K}}} (\omega(u), u)/\|u\|^2 = 0. \tag{1.1}$$

Then the bifurcation points for Γ are contained in the closed interval $[m_{\mathbb{K}}(B), M_{\mathbb{K}}(B)]$.

PROPOSITION 1.2. *Let $\Gamma = B + \omega$ with $B: \mathbb{K} \to \mathbb{R}$ positively homogeneous and $\omega: \mathbb{K} \to \mathbb{R}$ verifying*

$$\lim_{\substack{\|u\| \to +\infty \\ u \in \mathbb{K}}} (\omega(u), u)/\|u\|^2 = 0. \tag{1.2}$$

Then the asymptotic bifurcation points for Γ are contained in the closed interval $[m_{\mathbb{K}}(B), M_{\mathbb{K}}(B)]$.

It is natural to study under what conditions the extreme points of the interval $[m_{\mathbb{K}}(B), M_{\mathbb{K}}(B)]$ are effectively bifurcation points for Γ. First we state some definitions and remarks.

Remark 1.1. Since λ is a bifurcation point (at the origin or asymptotic) for an operator Γ if and only if $-\lambda$ is a bifurcation point for $-\Gamma$, the study concerning the value $m_{\mathbb{K}}(B)$ can be reduced to the study concerning $M_{\mathbb{K}}(B)$. Hence, without loss of generality, we shall consider in the following only this last case.

If Φ is a real functional defined on $\mathbb{K}$, one says that Φ is *Gateaux differentiable* at a point $u_0 \in \mathbb{K}$ and that its Gateaux gradient at u_0 is $v \in H$ if

$$\Phi(u_0 + th) - \Phi(u_0) = (v, th) + \omega(u_0, th) \tag{1.3}$$

with $\lim_{t\to 0}[\omega(u_0, th)/t] = 0$, for all h such that $u_0 + h \in \mathbb{K}$ and all $t \in]0,1]$. We then write $v = \nabla\Phi(u_0)$. If $\Phi: \mathbb{K} - \mathbb{K}_{\rho_0} \to \mathbb{R}$ and if $u_0 \in \mathbb{K} - \mathbb{K}_{\rho_0}$, we consider in (1.3) only the increments h such that $\|h\| < \|u_0\| - \rho_0$.

We shall use the following assumption in the Theorem 1.1.

ASSUMPTION 1.1. *For every $u \in \mathbb{K}$ such that $(\Gamma(u), u) > 0$ and $\Gamma(u) \neq \lambda u$, $\forall \lambda \in \mathbb{R}$, there exists a $v \in \mathbb{K}_\rho$ ($\rho = \|u\|$) such that $v \neq u$ and $(\Gamma(u), v) \geqslant (\Gamma(u), u)$.*

The following assumption will be used in Theorem 1.2.

ASSUMPTION 1.2. *For every $u \in \mathbb{K}$ with $\|u\| > \rho_0$ such that $(\Gamma(u), u) > 0$ and $\Gamma(u) \neq \lambda u$, $\forall \lambda \in \mathbb{R}$, there exists a $v \in \mathbb{K}_\rho$ ($\rho = \|u\|$) such that $v \neq u$ and $(\Gamma(u), v) \geqslant (\Gamma(u), u)$.*

It is easily verified that for points $u \in \operatorname{int} \mathbb{K}$ Assumptions 1.1 and 1.2 hold with $(\Gamma(u), v) > (\Gamma(u), u)$. Moreover we have the following proposition.

PROPOSITION 1.3. *Assumptions 1.1 and 1.2 hold in a point u (with $(\Gamma(u), v) > (\Gamma(u), u)$) if there exists $\epsilon = \epsilon(u) > 0$ such that $u + \epsilon\Gamma(u) \in \mathbb{K}$.*

In particular, the following corollaries hold.

COROLLARY 1.1. *Assumption 1.1 holds if $\Gamma\mathbb{K} \subset \mathbb{K}$ or if $\mathbb{K} = H$.*

COROLLARY 1.2. *Assumption 1.2 holds if $\Gamma(\mathbb{K} - \mathbb{K}_{\rho_0}) \subset \mathbb{K}$ or if $\mathbb{K} = H$.*

Note that Propositions 1.1 and 1.2 have a local character. Similarly, in the following theorem it is sufficient that Φ and Γ be defined only in a $\mathbb{K}_{\rho_0}$ with $\rho_0 > 0$ arbitrary small.

THEOREM 1.1. *Let $\Phi: \mathbb{K} \to \mathbb{R}$, $\Phi(0) = 0$, be a weakly upper semicontinuous and Gateaux differentiable functional. Let $\Gamma = \nabla\Phi$ verify Assumption 1.1 and $\Gamma = B + \omega$ with B positively homogeneous in $\mathbb{K}$ and ω satisfying (1.1). Then if $M_{\mathbb{K}}(B)$ is positive, $M_{\mathbb{K}}(B)$ is a bifurcation point for Γ in $\mathbb{K}$.*

In the next theorem condition (i) is obviously fulfilled if Φ is defined and weakly upper semicontinuous on all of $\mathbb{K}$.

THEOREM 1.2. *Let $\Phi: \mathbb{K} - \mathbb{K}_{\rho_0} \to \mathbb{R}$ be a Gateaux differentiable functional on $\mathbb{K} - \mathbb{K}_{\rho_0}$ and let $\Gamma = \nabla\Phi$ verify Assumption 1.2. Assume that $\Gamma = B + \omega$ with B positively homogeneous in $\mathbb{K}$ and ω satisfying* (1.2). *Finally let $\Phi(u) - \frac{1}{2}(B(u), u)$ be bounded on each $\mathbb{K} \cap S_\rho$ for $\rho > \rho_0$, and assume that one of the following conditions holds*:

(i) *Φ is weakly upper semicontinuous on $\overline{\mathbb{K} - \mathbb{K}_{\rho_0}}$, or*

(ii) *Φ is weakly upper semicontinuous on $\mathbb{K} - \mathbb{K}_{\rho_0}$ and $(B(u), u)$ is weakly upper semicontinuous on $\mathbb{K}$.*

Then if $M_{\mathbb{K}}(B)$ is positive, $M_{\mathbb{K}}(B)$ is an asymptotic bifurcation point for Γ on $\mathbb{K}$.

As a corollary to Theorem 1.2 one has

THEOREM 1.3. *Let $\Phi: \mathbb{K} \to \mathbb{R}$ be a weakly upper semicontinuous and Gateaux differentiable functional. Let $\Gamma = \nabla\Phi$ verify Assumption 1.2 and $\Gamma = B + \omega$ with B positively homogeneous in $\mathbb{K}$ and ω satisfying* (1.2). *Assume that $\Phi(u)$ and $(B(u), u)$ are bounded on bounded sets. Then if $M_{\mathbb{K}}(B)$ is positive, $M_{\mathbb{K}}(B)$ is an asymptotic bifurcation point for Γ on $\mathbb{K}$.*

Remark 1.2. From hypothesis (1.2) it follows that, for fixed $K > 0$, there exists $\rho_1 > \rho_0$ such that $|(\omega(u), u)| \leqslant K\|u\|^2$, $\forall u \in \mathbb{K}$ with $\|u\| \geqslant \rho_1$. Since

$$\Phi(\xi u) = \Phi(u) + \int_1^\xi (\Gamma(tu), u)\, dt, \qquad \xi > 1,$$

it easily follows that if $\Phi(u) - \frac{1}{2}(B(u), u)$ is bounded on $\mathbb{K} \cap S_{\rho_1}$, then it is bounded on each $\mathbb{K} \cap S_\rho$ for all $p \geqslant \rho_1$. Hence, if in Theorem 1.2 one chooses ρ_0 sufficiently large ($\rho_0 \geqslant \rho_1$), it is sufficient to assume that $\Phi(u) - \frac{1}{2}(B(u), u)$ is bounded only on $S_{\rho_0} \cap \mathbb{K}$.

2. PROOF OF THE MAIN RESULTS

In this section we prove the results described in the preceding one. In order to prove Theorem 1.1 we state some lemmas.

LEMMA 2.1. *Let $\Phi: \mathbb{K} \to \mathbb{R}$, $\Phi(0) = 0$, be radially continuous at the origin.*[2] *Assume that Φ is Gateaux differentiable on $\mathbb{K} - \{0\}$ and put $\Gamma = \nabla\phi$. Let*

[2] That is, for any $u \in \mathbb{K}$, the function $t \to \Phi(tu)$, $t \geqslant 0$, is continuous for $t = 0$.

$\Gamma = B + \omega$ with B positively homogeneous in $\mathbb{K}$ and ω verifying (1.1). *Then*

$$\lim_{\substack{u \to 0 \\ 0 \neq u \in \mathbb{K}}} \left|\Phi(u) - \tfrac{1}{2}(B(u), u)\right| / \|u\|^2 = 0. \tag{2.1}$$

In particular, Φ is Gateaux differentiable at the origin (Frechet differentiable if $m_{\mathbb{K}}(B)$ and $M_{\mathbb{K}}(B)$ are finite) with $\Gamma(0) = 0$.

Proof. Let $u \neq 0$, $u \in \mathbb{K}$, and consider the function $f(t) = \Phi(tu)$, $t \in [0, 1]$. Since $f(t)$ is differentiable on $]0, 1]$, with bounded derivative $f'(t) = (\Gamma(tu), u)$, and continuous at $t = 0$, it follows that

$$\Phi(u) = \tfrac{1}{2}(B(u), u) + \int_0^1 (\omega(tu), u)\, dt.$$

Denoting by $\delta(\epsilon)$ a positive number such that $|(\omega(u), u)| \leqslant \epsilon\|u\|^2$ if $u \in \mathbb{K}$, $\|u\| \leqslant \delta(\epsilon)$, it follows that

$$\left|\Phi(u) - \tfrac{1}{2}(B(u), u)\right| \leqslant \tfrac{1}{2}\epsilon\|u\|^2 \tag{2.2}$$

if $u \in \mathbb{K}$, $\|u\| \leqslant \delta(\epsilon)$.

LEMMA 2.2. *Suppose that $u \in \mathbb{K}$, $\|u\| = \rho$, and $(\Gamma(u), u) > 0$. If there exists a $v \in \mathbb{K}_\rho$, $v \neq u$, such that $(\Gamma(u), v) \geqslant (\Gamma(u), u)$, then there exists a $v' \in \mathbb{K}_\rho$ such that $(\Gamma(u), v') > (\Gamma(u), u)$.*

Proof. Suppose that $(\Gamma(u), v) = (\Gamma(u), u)$ and put $v_t = tu + (1 - t)v, t \in]0, 1[$. Since we use only values of t close to 1 we assume, without loss of generality, that $v_t \neq 0$. Define $v_t' = \rho v_t / \|v_t\|$. Then $v_t' \in \mathbb{K}_\rho$ and $(\Gamma(u), v_t' - u) = (\Gamma(u), v_t' - v_t)$ since $v_t - u = (1 - t)(v - u)$. Suppose now that the lemma is false. It follows that $(\Gamma(u), v_t' - v_t) \leqslant 0$ for all t in a neighborhood of 1. Since $v_t' - v_t = \alpha v_t$ with $\alpha > 0$ it follows that $(\Gamma(u), v_t) \leqslant 0$; letting $t \to 1$ we conclude that $(\Gamma(u), u) \leqslant 0$ which is in contradiction to our assumptions.

LEMMA 2.3. *Let $u \in \mathbb{K}$, $u \neq 0$, and $\Gamma(u) \neq \lambda u$ for all $\lambda \in \mathbb{R}$. Assume that there exists $\epsilon = \epsilon(u) > 0$ such that $u + \epsilon\Gamma(u) \in \mathbb{K}$. Then, putting $\rho = \|u\|$, there exists $v' \in \mathbb{K}_\rho$ such that $(\Gamma(u), v') > (\Gamma(u), u)$.*

Proof. Put

$$v' = \frac{\rho}{\|u + \epsilon\Gamma(u)\|}(u + \epsilon\Gamma(u)).$$

One has $v' \in \mathbb{K}_\rho$. Moreover $v' \neq u$ since $v' = u$ implies that $\Gamma(u) = \lambda u$. Put $w = u + \epsilon\Gamma(u)$. One has $\|w\|(\Gamma(u), v' - u) = \epsilon^{-1}(w - u, \|u\|w - \|w\|u)$ and consequently

$$(\Gamma(u), v' - u) = \frac{\|u\| + \|w\|}{\epsilon\|w\|}(\|u\|\,\|w\| - (u, w)) > 0$$

as desired.

Proof of Proposition 1.3. The proof follows directly from Lemma 2.2.

Remark 2.1. Note that in Proposition 1.3 and Lemmas 2.2 and 2.3, $\Gamma(u)$ is an arbitrary element of H.

Proof of Theorem 1.1. Given $\epsilon \in]0, M[, M = M_{\mathbb{K}}(B)$, let $\rho \in]0, \delta(\epsilon/3)]$ be fixed. If $u \in \mathbb{K}$, $\|u\| \leqslant \rho$, one has

$$|(\omega(u), u)| \leqslant \tfrac{1}{3}\epsilon\|u\|^2 \tag{2.3}$$

and

$$|\Phi(u) - \tfrac{1}{2}(B(u), u)| \leqslant \tfrac{1}{6}\epsilon\|u\|^2. \tag{2.4}$$

From the definition of M follows the existence of a sequence $u_n \in \mathbb{K}$ such that $\|u_n\| = \rho$ and $(B(u_n), u_n) > M\rho^2 - (1/n)$. Hence we get from (2.4) that

$$\begin{aligned}\Phi(u_n) &\geqslant \tfrac{1}{2}(B(u_n), u_n) - |\Phi(u_n) - \tfrac{1}{2}(B(u_n), u_n)| \\ &\geqslant \tfrac{1}{2}M\rho^2 - (1/2n) - \tfrac{1}{6}\epsilon\rho^2;\end{aligned}$$

consequently

$$\sup_{u \in \mathbb{K}_\rho} \Phi(u) \geqslant (\tfrac{1}{2}M - \tfrac{1}{6}\epsilon)\rho^2. \tag{2.5}$$

On the other hand, Φ attains its maximum in $\mathbb{K}_\rho$ at (at least) one point u_0 since it is weakly upper semicontinuous in $\mathbb{K}$, and (2.5) yields

$$\Phi(u_0) \geqslant (\tfrac{1}{2}M - \tfrac{1}{6}\epsilon)\rho^2.$$

For convenience put $\|u_0\| = \rho'$. One has $\tfrac{1}{2}(B(u_0), u_0) + \tfrac{1}{2}(\omega(u_0), u_0) \geqslant (\tfrac{1}{2}M - \tfrac{1}{6}\epsilon)\rho^2 - \tfrac{1}{6}\epsilon\rho'^2 - \tfrac{1}{6}\epsilon\rho'^2$, consequently

$$\tfrac{1}{2}(\Gamma(u_0), u_0) \geqslant (\tfrac{1}{2}M - \tfrac{1}{2}\epsilon)\rho^2 > 0; \tag{2.6}$$

and in particular $\Gamma(u_0) \neq 0$. We claim that

$$\Gamma(u_0) = \lambda u_0 \tag{2.7}$$

for a real number λ. Notice first that $\|u_0\| = \rho$. If not, then putting $h = [(\rho - \rho')/\rho']u_0$ in (1.3) we obtain

$$\Phi(u_0 + th) = \Phi(u_0) + t[(\rho - \rho')/\rho](\Gamma(u_0), u_0) + \omega(u_0, th)$$

and this yields $\Phi(u_0 + th) > \Phi(u_0)$ for $t > 0$ sufficiently small which is false since $\Phi(u_0)$ is a maximum on $\mathbb{K}_\rho$; hence $\|u_0\| = \rho$.

Suppose now, by contradiction, that (2.7) does not hold. Then from Assumption 1.1 and Lemma 2.2 the existence follows of a $v' \in \mathbb{K}_\rho$ such that $(\Gamma(u_0), v' - u_0) > 0$. Hence by (1.3) one has $\Phi(u_0 + t(v' - u_0)) > \Phi(u_0)$ for $t > 0$ sufficiently small, which is not possible since $\Phi(u_0)$ is a maximum on $\mathbb{K}_\rho$. Thus (2.7) holds.

Finally $(B(u_0), u_0) + (\omega(u_0), u_0) = \lambda\rho^2$ consequently $\lambda\rho^2 \leqslant M\rho^2 + \frac{1}{3}\epsilon\rho^2$, i.e.,

$$\lambda \leqslant M + \tfrac{1}{3}\epsilon. \tag{2.8}$$

On the other hand, from (2.6) and (2.7) it follows easily that

$$\lambda \geqslant M - \epsilon. \tag{2.9}$$

We remark that we have proved that for any $\epsilon \in]0, M[$ and any $\rho \in]0, \delta(\epsilon/3)]$ there exists $u_0 \in \mathbb{K}_\rho$ and $\lambda \in [M - \epsilon, M + \frac{1}{3}\epsilon]$ such that $\Gamma(u_0) = \lambda u_0$. Furthermore $\Phi(u_0) = \max \Phi(u)$ on $\mathbb{K}_\rho$.

Remark 2.2. If the condition described in Proposition 1.3 (or in Corollary 1.1) holds, then Lemma 2.2 is unnecessary since the existence of a point $v' \in \mathbb{K}_\rho$ such that $(\Gamma(u_0), v' - u_0) > 0$ follows directly from Lemma 2.3.

We now proceed to prove Theorem 1.2. We begin by stating the following lemma.

LEMMA 2.4. *Let* $\Phi: \mathbb{K} - \mathbb{K}_{\rho_0} \to \mathbb{R}$ *be a Gateaux differentiable functional and put* $\Gamma = \nabla\Phi$. *Let* $\Gamma = B + \omega$ *with* B *positively homogeneous in* $\mathbb{K}$ *and* ω *satisfying* (1.2) *and suppose that* $|\Phi(u) - \frac{1}{2}(B(u), u)|$ *is bounded in* $\mathbb{K} \cap S_\rho$ *for each* $\rho > \rho_0$. *Then*

$$\lim_{\|u\| \to +\infty} |\Phi(u) - \tfrac{1}{2}(B(u), u)|/\|u\|^2 = 0. \tag{2.10}$$

Proof. For every $\epsilon > 0$ there exists $R_\epsilon > \rho_0$ such that $|(\omega(u), u)| \leqslant \epsilon\|u\|^2$ if $u \in K$, $\|u\| \geqslant R_\epsilon$. Let $u \in K$, $\|u\| > R_\epsilon$. One has

$$\Phi(u) - \tfrac{1}{2}(B(u), u) = \Phi\left(\frac{R_\epsilon}{\|u\|}u\right) - \frac{1}{2}\left(B\left(\frac{R_\epsilon}{\|u\|}u\right), \frac{R_\epsilon}{\|u\|}u\right) + \int_{R_\epsilon/\|u\|}^{1} (\omega(tu), u)\, dt. \tag{2.11}$$

On the other hand,

$$\left|\int_{R_\epsilon/\|u\|}^{1} (\omega(tu), u)\, dt\right| < \tfrac{1}{2}\epsilon\|u\|^2. \tag{2.12}$$

On putting

$$S_\epsilon = \sup_{\substack{\|u\|=R_\epsilon \\ u\in\mathbb{K}}} \left|\Phi(u) - \tfrac{1}{2}(B(u),u)\right|,$$

it follows from (2.11) and (2.12) that

$$\left|\Phi(u) - \tfrac{1}{2}(B(u),u)\right| \leqslant \epsilon\|u\|^2, \qquad \forall u\in\mathbb{K}, \quad \|u\| \geqslant N_\epsilon, \tag{2.13}$$

where

$$N_\epsilon = \max(R_\epsilon, (2S_\epsilon/\epsilon)^{1/2}).$$

Proof of Theorem 1.2. First we verify that $M = M_{\mathbb{K}}(B) < +\infty$. Let $\rho > \rho_0$ and suppose, by contradiction, that $M_{\mathbb{K}}(B) = +\infty$. Then there exists a sequence $u_n \in \mathbb{K}\cap S_\rho$ such that $(B(u_n),u_n) \to +\infty$. If hypothesis (i) of Theorem 1.2 holds, one has $\Phi(u_n) \to +\infty$ and (at least for a subsequence) $u_n \rightharpoonup u_0 \in \overline{\mathbb{K} - \mathbb{K}_{\rho_0}}$. But this contradicts (i). If hypothesis (ii) holds, then (at least for a subsequence) $u_n \rightharpoonup u_0 \in \mathbb{K}$, which contradicts the weak upper semicontinuity of $(B(u),u)$ on $\mathbb{K}$; thus $M < +\infty$.

Given $\epsilon\in\,]0,M[$, let ρ_1, $\rho \in [N(\epsilon/6), +\infty[$, with $\rho_0 \leqslant \rho_1 < \rho$ fixed. If hypothesis (i) holds, we take ρ sufficiently large so that

$$\tfrac{1}{3}M\rho^2 \geqslant \sup_{u\in\mathbb{K}_{\rho_1}\cap\overline{\mathbb{K}-\mathbb{K}_{\rho_1}}} \bar{\Phi}(u). \tag{2.14i}$$

If (ii) holds, we take ρ such that

$$\rho^2 \geqslant 3\rho_1{}^2. \tag{2.14ii}$$

From the definition of M the existence follows of a sequence $u_n \in S_\rho \cap \mathbb{K}$ such that $(B(u_n),u_n) > M\rho^2 - (1/n)$. By using estimate (2.13) one obtains

$$\sup_{\mathbb{K}_\rho - \mathbb{K}_{\rho_1}} \Phi(u) \geqslant (\tfrac{1}{2}M - \tfrac{1}{6}\epsilon)\rho^2.$$

Now let $v_n \in \mathbb{K}_\rho - \mathbb{K}_{\rho_1}$ be a maximizing sequence for Φ such that $v_n \to u_0$. We claim that $u_0 \in \mathbb{K}_\rho - \mathbb{K}_{\rho_1}$. Suppose that condition (i) holds and suppose, by contradiction, that $u_0 \in \mathbb{K}_{\rho_1}$. Since $u_0 \in \overline{\mathbb{K}_\rho - \mathbb{K}_{\rho_1}}$ it follows from (2.14i) that $\bar{\Phi}(u_0) \leqslant \frac{1}{3}M\rho^2 < \lim \Phi(v_n)$, which contradicts (i). If condition (ii) holds, one has $\frac{1}{2}(B(v_n),v_n) \geqslant \Phi(v_n) - \frac{1}{6}\epsilon\rho^2$; passing to the limit and using (2.14ii), one has $\frac{1}{2}(B(u_0),u_0) > \frac{1}{2}M\rho_1{}^2$ and consequently $\|u_0\| > \rho_1$.

Hence u_0 is a maximum point for Φ on $\mathbb{K}_\rho - \mathbb{K}_{\rho_1}$; moreover,

$$\Phi(u_0) \geqslant (\tfrac{1}{2}M - \tfrac{1}{6}\epsilon)\rho^2.$$

Continuing the proof as for Theorem 1.1 we see that $\Gamma(u_0) = \lambda u_0$ with $\|u_0\| = \rho$ and $\lambda \in [M-\epsilon, M + \frac{1}{6}\epsilon]$. Theorem 1.2 is proved.

Remark 2.3. Notice that in Theorem 1.1 we have

$$M_{\mathbb{K}}(B) = \limsup_{\substack{u \in \mathbb{K} \\ u \to 0}} \Phi(u)/\tfrac{1}{2}\|u\|^2$$

as follows from (2.1). Similarly, from (2.10), it follows that in Theorem 1.2,

$$M_{\mathbb{K}}(B) = \limsup_{\substack{u \in \mathbb{K} \\ \|u\| \to +\infty}} \Phi(u)/\tfrac{1}{2}\|u\|^2.$$

Remark 2.4. If B is a weakly continuous potential operator (in particular a linear, continuous, compact, and self-adjoint operator) satisfying assumption 1.1, then $M_{\mathbb{K}}(B)$ is the largest positive λ for which $B(u) = \lambda u$ admits a nonvanishing solution $u \in \mathbb{K}$.

3. On the Basic Conditions (1.1) and (1.2)

In this section we make some observations on hypotheses (1.1) and (1.2).

Definition 3.1. Let B and $\bar{B}$ be two positively homogeneous operators on $\mathbb{K}$. Then B and $\bar{B}$ are said to be *equivalent* (and we write $B \sim \bar{B}$) if $(B(u), u) = (\bar{B}(u), u)$, $\forall u \in \mathbb{K}$.

Obviously if $B \sim \bar{B}$, one has $m_{\mathbb{K}}(B) = m_{\mathbb{K}}(\bar{B})$ and $M_{\mathbb{K}}(B) = M_{\mathbb{K}}(\bar{B})$.

Remark 3.1. If B is a linear continuous operator, then there exists a *unique* self-adjoint, linear, continuous operator B_p such that $B_p \sim B$. More precisely, $B_p = \frac{1}{2}(B + B^*)$.

Definition 3.2. Suppose that Γ is defined on $\mathbb{K}_\rho$, $\rho > 0$ [respectively, on $\mathbb{K} - \mathbb{K}_{\rho_0}$] and $\Gamma = B + \omega$ with B positively homogeneous on $\mathbb{K}$ and ω verifying (1.1) [respectively, (1.2)]. Under these conditions $B + \omega$ is said to be an a.d. (*admissible decomposition*) for Γ (at the origin) [respectively, at infinity].

Proposition 3.1. *If $\Gamma = B + \omega$ is an a.d., then $\bar{B} \sim B$ if and only if there exists an a.d. $\Gamma = \bar{B} + \bar{\omega}$.*

The proof is left to the reader.

Now let $\Gamma = A + \omega$ with A *positively homogeneous* on $\mathbb{K}$ and ω verifying

$$\lim_{t \to 0^+} \omega(tu)/t = 0, \qquad \forall x \in \mathbb{K}. \tag{3.1}$$

We then say that A is the *Gateaux derivative of Γ at the origin* and we write

$D\Gamma(0) = A$. Similarly if one has (3.1) when $t \to +\infty$, we say that A is the *Gateaux derivative of Γ at infinity* and we write $D\Gamma(\infty) = A$.

PROPOSITION 3.2. *If $\Gamma = B + \omega$ is an a.d. at the origin [respectively, at infinity] and if Γ is Gateaux differentiable at the origin [respectively, at infinity], then $B \sim D\Gamma(0)$ [respectively, $B \sim D\Gamma(\infty)$].*

Thus $D\Gamma(0)$ [respectively $D\Gamma(\infty)$] is not always the simplest operator in its equivalence class. For instance, the operator ω defined by (0.10) [respectively (0.17)] admits the a.d. $\omega = 0 + \omega$, but $D\omega(0) = A$ [respectively, $D\omega(\infty) = A$] exists in the G-sense with $A \neq 0$ since one has $A(x) = \phi(0)\eta e_1$ if $\xi = 0$.

If we replace (3.1) by the stronger condition

$$\lim_{\substack{u \to 0 \\ u \in \mathbb{K}}} \|\omega(u)\|/\|u\| = 0, \tag{3.2}$$

then we say that A is the *Fréchet derivative of Γ at the origin* (a similar definition holds for infinity by putting, in (3.2), $\|u\| \to +\infty$ instead of $u \to 0$). Obviously $\Gamma = D\Gamma(0) + (\Gamma - D\Gamma(0))$ is then an a.d. To be more precise, if one decomposes the vector field $\omega(u)$ in its radial and tangential components

$$\omega_r(u) = [(\omega(u), u)/\|u\|^2]u, \qquad \omega_t(u) = \omega(u) - [(\omega(u), u)/\|u\|^2]u$$

and if one considers the two conditions

$$\lim_{\substack{u \to 0 \\ u \in \mathbb{K}}} \|\omega_r(u)\|/\|u\| = 0 \tag{3.3i}$$

$$\lim_{\substack{u \to 0 \\ u \in \mathbb{K}}} \|\omega_t(u)\|/\|u\| = 0, \tag{3.3ii}$$

then (3.2) is equivalent to (3.3i) plus (3.3ii) while (1.1) is equivalent to (3.3i) only. These facts hold if we replace (1.1) by (1.2) and if we put $\|u\| \to +\infty$ instead of $u \to 0$ [in (3.2) and (3.3i or ii)].

Finally, for a positively homogeneous remainder, the following result holds.

PROPOSITION 3.3. *Suppose that $\omega: \mathbb{K} \to H$ verifies, for some $\alpha > 0$,*

$$(\omega(tu), u) = t^\alpha(\omega(u), u), \qquad \forall u \in \mathbb{K}, \quad \forall t > 0,$$

with $(\omega(u), u)$ not identically zero in $\mathbb{K}$. Then ω verifies (1.1) [respectively, (1.2)] if and only if $\alpha > 1$ [respectively, $\alpha < 1$] and $(\omega(u), u)$ is bounded in S_ρ for one positive ρ.

The proof is left an easy exercise.

4. Systems of the Hammerstein Type

Assumptions and Notation

In this section we apply the preceding results to systems of the Hammerstein type. To study this problem Theorems 1.1 and 1.3 together with some classical methods (see [6] and [10]) are used. For the study of Hammerstein operators we refer also to the papers by Amann [1, and references] and to a paper by H. Brézis and F. E. Browder [4].

In the following B is a subset of finite Lebesgue measure of the Euclidean space $\mathbb{R}^n$. Let $y = (y_1, \ldots, y_m)$ denote a generic $\mathbb{R}^m$-element. The norm and scalar product in this space are defined as $|y|^2 = |y_1|^2 + \cdots + |y_m|^2$ and $(y, \bar{y}) = y_1\bar{y}_1 + \cdots + y_m\bar{y}_m$. Also $p \geqslant 2$ is a real number and q is defined by the equation $(1/p) + (1/q) = 1$. We consider the reflexive Banach space $L^p(B)$ with the usual norm

$$\|u_1\|_p = \left(\int_B |u_1(x)|^p \, dx\right)^{1/p},$$

and also the Banach space $L^q(B)$ with norm $\| \ \|_q$ and the Hilbert space $L^2(B)$ with norm $\| \ \|_2$. We denote by $\langle \ , \ \rangle$ both the scalar product on $L^2(B)$ and the pairing between $L^q(B)$ and the dual space $L^p(B)$:

$$\langle v_1, u_1 \rangle = \int_B v_1(x) u_1(x) \, dx$$

for $v_1(x), u_1(x) \in L^2(B)$ or $v_1(x) \in L^q(B)$ and $u_1(x) \in L^p(B)$. We consider also the reflexive Banach space $L_m{}^p(B) = L^p(B) \times \cdots \times L^p(B)$ (m times), the norm of which is, as for $L^p(B)$, denoted by $\| \ \|_p$ and defined by

$$\|u\|_p = \left(\int_B |u(x)|^p \, dx\right)^{1/p},$$

where $u = (u_1, \ldots, u_m) \in L_m{}^p(B)$. Similar definitions hold for $\| \ \|_q$ and $\| \ \|_2$. The scalar product on $L_m{}^2(B)$ and the pairing between $L_m{}^q(B)$ and the dual space $L_m{}^p(B)$ are denoted by the same symbol $\langle \ , \ \rangle$, which is defined by

$$\langle v, u \rangle = \sum_{i=1}^{m} \langle v_i, u_i \rangle$$

with $v = (v_1, \ldots, v_m)$ and $u = (u_1, \ldots, u_m)$ belonging to $L_m{}^2(B)$ or with $v \in L_m{}^q(B)$ and $u \in L_m{}^p(B)$.

Let $g_i(x, y)$, $i = 1, \ldots, m$, be real functions defined on $B \times \mathbb{R}^m$ and satisfying the Carathéodory conditions, i.e., continuous in y for almost all $x \in B$ and measurable in x for all $y \in \mathbb{R}^m$. Moreover, assume that there exists a real function $G(x, y)$ defined on $B \times \mathbb{R}^m$ such that for almost all $x \in B$,

$$g_i(x, y) = \partial G(x, y)/\partial y_i, \qquad i = 1, \ldots, m,$$

and suppose that $G(x, 0) = 0$ and $g_i(x, 0) = 0$, $i = 1, \ldots, m$, for almost all $x \in B$. Finally defined a function $g: B \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$g(x, y) = (g_1(x, y), \ldots, g_m(x, y)).$$

We denote also by the symbols g_i and g the *Nemytsky operators* defined by $[g_i(u)](x) = g_i(x, u(x))$ a.e. in B and $[g(u)](x) = g(x, u(x))$ a.e. in B, and we assume that each g_i acts from $L_m{}^p(B)$ into $L^q(B)$, i.e., that $g_i(x, u(x)) \in L^q(B)$, $\forall u(x) \in L_m{}^p(B)$. Under this condition the operator g_i is bounded and continuous (cf. Krasnoselskii [6, Chapter I, §2]) and $g: L_m{}^p(B) \to L_m{}^q(B)$ is bounded and continuous. Moreover g is a potential operator, i.e., $g = \nabla \Phi$ in the Fréchet sense where $\Phi: L_m{}^p(B) \to \mathbb{R}$ is defined by

$$\Phi(u) = \int_B G(x, u(x))\, dx$$

cf. [10, Theorem 21.1]).

Consider now m real symmetric kernels $K_i(x, \xi)$, measurable on $B \times B$, and the associated linear integral operators

$$[A_i(f)](x) = \int_B K_i(x, \xi) f(\xi)\, d\xi \quad \text{a.e. on } B. \tag{4.1}$$

We assume that the operators A_i act and are *completely continuous from $L^q(B)$ into $L^p(B)$*. Hence they are completely continuous as operators on $L^2(B)$. Moreover we assume that each A_i, $i = 1, \ldots, m$ is *self-adjoint and positive on $L^2(B)$*. Obviously the operator $A(v) = (A_1(v_1), \ldots, A_m(v_m))$ verifies the corresponding properties as an operator from $L_m{}^q(B)$ into $L_m{}^p(B)$ and as an operator on $L_m{}^2(B)$. Moreover the positive square root of A on $L_m{}^2(B)$ is given by $A^{1/2} = (A_1^{1/2}, \ldots, A_m^{1/2})$, where $A_i^{1/2}$ denotes the positive root of A_i on $L^2(B)$. Recall that each $A_i^{1/2}$ (consequently $A^{1/2}$) is self-adjoint and positive cf. [8, n. 104].

One can prove (see [10, Corollary 23.1] that the range of $A^{1/2}$ is contained in $L_m{}^p(B)$ and that *the operator $A^{1/2} = L_m{}^2(B) \to L_m{}^p(B)$ is completely continuous*. It follows that the adjoint operator is a completely continuous operator from $L_m{}^q(B)$ into $L_m{}^2(B)$. Since it coincides with $A^{1/2}$ on $L_m{}^2(B)$, it will be denoted by the same symbol $A^{1/2}$.

It is easy to see that the functional $\Phi A^{1/2}: L_m{}^2(B) \to \mathbb{R}$ is Fréchet differentiable with

$$\nabla(\Phi A^{1/2}) = A^{1/2}(\nabla \Phi) A^{1/2}. \tag{4.2}$$

We return now to the Nemytsky operator g. We assume that the functions g_i can be written in the form

$$g_i(x, y) = \sum_{j=1}^{m} g_{ij}(x) y_j + \omega_i(x, y), \tag{4.3}$$

where $g_{ij}(x)$, $i, j = 1, \ldots, m$, belongs to $L^s(B)$ with $s = p/(p-2)$, $g_{ij}(x) = g_{ji}(x)$[3] and the remainders $\omega_i(x, y)$ satisfy the Charathéodory conditions and act from $L_m{}^p(B)$ into $L^q(B)$. We define the Nemytsky operator $\omega(u)$ by

$$[\omega(u)](x) = (\omega_1(x, u(x)), \ldots, \omega_m(x, u(x))).$$

The Nemytsky operator g is then the sum of the linear operator g_0 defined by

$$[g_0(u)](x) = \left(\sum_{j=1}^{m} g_{1j}(x) u_j(x), \ldots, \sum_{j=1}^{m} g_{mj}(x) u_j(x) \right)$$

with the remainder ω.

The Main Results

The following results hold.

THEOREM 4.1. *Assume that the conditions described in this section hold and that*

$$\lim_{\|u\|_p \to 0} \langle \omega(u), u \rangle / \|u\|_p^2 = 0. \tag{4.4}$$

Then [*in the space* $L_m{}^p(B)$] *the largest positive eigenvalue for the linear problem*

$$A g_0(u) = \lambda u \tag{4.5}$$

is the largest positive bifurcation point for the problem

$$A g(u) = \lambda u. \tag{4.6}$$

The result holds again if we replace "largest positive" with "smallest negative."

THEOREM 4.2. *Assume that the conditions described in this section hold and that*

$$\lim_{\|u\|_p \to +\infty} \langle \omega(u), u \rangle / \|u\|_p^2 = 0. \tag{4.7}$$

[3] If the functions $g_i(x, y)$ are regular, then $g_{ij}(x) = [\partial g_i(x, y)/\partial y_j]_{y=0}$; an analogous remark holds for the asymptotic case.

Then [in the space $L_m{}^p(B)$] the largest positive eigenvalue for the linear problem (4.5) is the largest positive asymptotic bifurcation point for problem (4.6). The result holds again if one replace "largest positive" with "smallest negative."

Recall that (4.5) and (4.6) can be written more explicitly as

$$\int_B K_i(x,\xi)\left[\sum_{j=1}^{m} g_{ij}(\xi)u_j(\xi)\right]d\xi = \lambda u_i(x), \qquad i = 1, \ldots, m,$$

and

$$\int_B K_i(x,\xi)g_i(\xi, u_1(\xi), \ldots, u_m(\xi))\,d\xi = \lambda u_i(x), \qquad i = 1, \ldots, m,$$

respectively.

Remark 4.1. In Theorems 4.1 and 4.2 it is not necessary that the operators A_i, $1 \leqslant i \leqslant m$, are integral operators [see (4.1)]. In fact the results and proofs hold if these operators verify the following properties:

(a) A_i is completely continuous as an operator from $L^q(B)$ into $L^p(B)$;
(b) A_i is self-adjoint and positive as an operator on $L^2(B)$;
(c) $A_i^{1/2}(L^2(B)) \subset L^p(B)$, and $A_i^{1/2}$ is completely continuous as an operator from $L^2(B)$ into $L^p(B)$.

Proof of Theorem 4.1. The functional $\Phi A^{1/2}$ is weakly continuous on $L_m{}^2(B)$ since $A^{1/2}$ is completely continuous from $L_m{}^2(B)$ into $L_m{}^p(B)$. On the other hand, Φ is Fréchet differentiable and (4.2) holds. It follows that

$$\nabla(\Phi A^{1/2}) = A^{1/2}g_0A^{1/2} + A^{1/2}\omega A^{1/2},$$

where $A^{1/2}g_0A^{1/2}$ is linear and completely continuous on $L_m{}^2(B)$. Moreover from (4.4),

$$\lim_{\|u\|_2 \to 0} \langle (A^{1/2}\omega A^{1/2})(u), u\rangle / \|u\|_2^2 = 0$$

since $\|A^{1/2}(u)\|_p \geqslant c\|u\|_2$. Thus (1.1) holds. Then, by Theorem 1.1, the largest positive bifurcation point for

$$(A^{1/2}gA^{1/2})(u) = \lambda u \quad \text{in} \quad L_m{}^2(B) \tag{4.8}$$

is given by

$$M(A^{1/2}g_0A^{1/2}) \equiv \sup_{\|u\|_2 = 1} \langle (A^{1/2}g_0A^{1/2})(u), u\rangle,$$

if this quantity is positive. Since $A^{1/2}g_0A^{1/2}$ is a self-adjoint, completely continuous operator, M is the largest positive eigenvalue for

$$(A^{1/2}g_0A^{1/2})(u) = \lambda u \quad \text{in} \quad L_m{}^2(B). \tag{4.9}$$

A corresponding result holds for the smallest negative bifurcation point in (4.8).

To conclude the proof it suffices to verify that $\lambda \neq 0$ is a bifurcation point for (4.8) in $L_m{}^2(B)$ if and only if it is a bifurcation point for (4.6) in $L_m{}^p(B)$.[4] This can be done with a standard method: If λ is a bifurcation point for (4.8), one has $(A^{1/2}gA^{1/2})(u_\epsilon) = \lambda_\epsilon u_\epsilon$ with $0 \neq \|u_\epsilon\|_2 \to 0$ and $\lambda_\epsilon \to \lambda$. By putting $v_\epsilon = A^{1/2}(u_\epsilon)$, one has $(Ag)(v_\epsilon) = \lambda_\epsilon v_\epsilon$ with $\|v_\epsilon\|_p \to 0$ and $v_\epsilon \neq 0$ (since $(A^{1/2}g)(v_\epsilon) = \lambda_\epsilon u_\epsilon \neq 0$). Reciprocally, if λ is a bifurcation point for (4.6), one has $(Ag)(v_\epsilon) = \lambda_\epsilon v_\epsilon, \lambda_\epsilon \to \lambda$, $0 \neq \|v_\epsilon\|_p \to 0$. By putting $u_\epsilon = (1/\lambda_\epsilon)(A^{1/2}g)(v_\epsilon)$, it follows that $\|u_\epsilon\|_2 \leqslant c|\lambda_\epsilon|^{-1}\|g(v_\epsilon)\|_q \to 0$. Moreover $A^{1/2}(u_\epsilon) = v_\epsilon$, consequently $u_\epsilon \neq 0$ and

$$(A^{1/2}gA^{1/2})(u_\epsilon) = (A^{1/2}g)(v_\epsilon) = \lambda_\epsilon u_\epsilon.$$

Proof of Theorem 4.2. If $A^{1/2}\omega A^{1/2}$ does not verify (1.2), there exists a real $\epsilon_0 > 0$ and a sequence $u_k \in L_m{}^2(B)$ such that $\|u_k\|_2 \to +\infty$ and

$$\langle(\omega A^{1/2})(u_k), A^{1/2}(u_k)\rangle/\|u_k\|_2^2 \geqslant \epsilon_0, \qquad \forall k. \tag{4.10}$$

Since ω is a bounded operator from $L_m{}^p(B)$ into $L_m{}^q(B)$, it follows from (4.10) that the sequence $A^{1/2}(u_k)$ is unbounded in $L_m{}^p(B)$. Moreover (4.10) yields

$$|\langle(\omega A^{1/2})(u_k), A^{1/2}(u_k)\rangle|/\|A^{1/2}(u_k)\|_p^2 \geqslant c\epsilon_0, \qquad \forall k,$$

and with the help of (4.7) it follows that $A^{1/2}(u_k)$ is bounded in $L_m{}^p(B)$, which is in contradiction with the preceding conclusion. The remainder of the proof is similar to the preceding one.

Sufficient Conditions for (4.4) *and* (4.7)

Now we state some sufficient conditions for (4.4) and (4.7).

PROPOSITION 4.1. *Assume that the vector function ω verifies the condition*

$$|(\omega(x, y), y)| \leqslant b(x)|y|^2 + c_1|y|^p, \tag{4.11}$$

[4] In the same way one proves a similar result for the eigenvalues of (4.9) and (4.5).

where $b(x) \in L^s(B)$ *and* c_1 *is a constant, and also the condition*

$$\lim_{|y| \to 0} (\omega(x, y), y)/|y|^2 = 0 \tag{4.12}$$

for almost all $x \in B$. *Then* (4.4) *holds.*

Remark 4.2. A sufficient condition for (4.11) is

$$|\omega(x, y)| \leqslant b(x)|y| + c_1|y|^{p-1}, \tag{4.13}$$

where $b(x)$ and c_1 are defined as before (note that $p/q = p - 1$).

Moreover terms of the form $d(x)|y|^r$, with $1 \leqslant r \leqslant p - 1$ and $d(x) \in L^{p/(p-r-1)}(B)$, are implicitly included in the second term of (4.13) since they can be interpolated by $b(x)|y|$ and $c_1|y|^{p/q}$.

In connection with (4.3) recall that ω acts from $L_m{}^p(B)$ into $L_m{}^q(B)$ if and only if $|\omega(x, y)| \leqslant a(x) + b(x)|y| + c_1|y|^{p-1}$ with $a(x) \in L^q(B)$.

PROPOSITION 4.2. *Assume that the vector function* ω *verifies the condition*

$$|(\omega(x, y), y)| \leqslant a_0(x) + b(x)|y|^2, \tag{4.14}$$

where $a_0(x) \in L^1(B)$ *and* $b(x) \in L^s(B)$, *and also the condition*

$$\lim_{|y| \to +\infty} (\omega(x, y), y)/|y|^2 = 0 \tag{4.15}$$

for almost all $x \in B$. *Then* (4.7) *holds.*

PROPOSITION 4.3. *Assume that* (4.15) *holds unformly for almost all* $x \in B$. *Then* (4.7) *holds.*

Proof of Proposition 4.1. Assume that (4.4) fails. Then there exists and $\epsilon_0 > 0$ and a sequence $u_k(x) \in L_m{}^p(B)$ such that $\|u_k\|_p \to 0$ and

$$|\langle \omega(u_k), u_k \rangle| > (1 + c_2 + c_3)\epsilon_0 \|u_k\|_p^2, \qquad \forall k, \tag{4.16}$$

where the positive constants c_2 and c_3 are defined in the following. Consider now the real measurable functions defined on B by

$$f_k(x) = \begin{cases} |(\omega(x, u_k(x)), u_k(x))|/|u_k(x)|^2 & \text{if } |u_k(x)| \neq 0, \\ 0 & \text{if } |u_k(x)| = 0. \end{cases} \tag{4.17}$$

Since the integral is absolutely continuous with respect to the measure, there exists a $\delta_0 > 0$ such that

$$|E| \leqslant \delta_0 \Rightarrow \int_E |b(x)|^s \, dx \leqslant \epsilon_0{}^s, \tag{4.18}$$

where E is a measurable subset of B and $|E|$ denotes the measures of E.

On the other hand by assuming (without loss of generality) that $u_k(x) \to 0$ a.e. on B, it follows by (4.12) that $f_k(x) \to 0$ a.e. on B. Thus $f_k(x)$ converges in measure to 0. Hence there exists an integer k_0 such that

$$k > k_0 \Rightarrow |B_0^{(k)}| \leqslant \delta_0, \tag{4.19}$$

where

$$B_0^{(k)} = \{x \in B : f_k(x) \geqslant \epsilon_0\}.$$

Since

$$|\langle \omega(u_k), u_k \rangle| \leqslant \int_{B - B_0^{(k)}} |(\omega(x, u_k(x)), u_k(x))| \, dx + \int_{B_0^{(k)}} |(\omega(x, u_k(x)), u_k(x))| \, dx,$$

it follows from the definition of $B_0^{(k)}$ and (4.11) that

$$|\langle \omega(u_k), u_k \rangle| \leqslant \epsilon_0 \|u_k\|_2^2 + \int_{B_0^{(k)}} b(x) |u_k(x)|^2 \, dx + c_1 \|u_k\|_p^p,$$

and by Hölder's inequality,

$$|\langle \omega(u_k), u_k \rangle| \leqslant c_2 \epsilon_0 \|u_k\|_p^2 + c_1 \|u_k\|_p^p + \left(\int_{B_0^{(k)}} b(x)^s \, dx \right)^{1/s} \|u_k\|_p^2,$$

where c_2 is a positive constant such that $\| \ \|_2^2 \leqslant c_2 \| \ \|_p^2$. Assuming k sufficiently large in order that $k > k_0$ and $\|u_k\|_p^{p-2} \leqslant \epsilon_0$, it follows, with the help of (4.19) and (4.18), that

$$|\langle \omega(u_k), u_k \rangle| \leqslant (1 + c_1 + c_2) \epsilon_0 \|u_k\|_p^2,$$

which contradicts (4.16).

Proof of Proposition 4.2. Assume that (4.7) fails. Then there exists a positive ϵ_0 and a sequence $u_k \in L_m{}^p(B)$ such that $\|u_k\|_p \to +\infty$ and

$$|\langle \omega(u_k), u_k \rangle| > (2 + c_2) \epsilon_0 \|u_k\|_p^2, \qquad \forall k. \tag{4.20}$$

Let $\delta_0 > 0$ be such that (4.18) holds and define a function $f : B \times \mathbb{R}^m \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} |(\omega(x, y), y)| / |y|^2 & \text{if } \ y \neq 0, \\ 0 & \text{if } \ y = 0. \end{cases} \tag{4.21}$$

Then there exists a constant $M_0 > 0$ such that

$$|\{x \in B : f(x, u(x)) \geqslant \epsilon_0\}| \leqslant \delta_0 \tag{4.22}$$

holds for any measurable $u : B \to \mathbb{R}^m$ such that $|u(x)| \geqslant M_0$ a.e. in B. If not, to any positive integer M there corresponds a measurable function $u_M(x)$ such that $|u_M(x)| \geqslant M$ a.e. in B and

$$|\{x \in B : f(x, u_M(x)) \geqslant \epsilon_0\}| > \delta_0. \tag{4.23}$$

Since $|u_M(x)| \to +\infty$ as $M \to +\infty$, it follows from (4.15) that $f(x, u_M(x)) \to 0$ a.e. in B and in particular $f(x, u_M(x)) \to 0$ in measure. But this contradicts (4.23).

Now define $B_0^{(k)}$ and $C^{(k)}$ by

$$B_0^{(k)} = \{x \in B : f(x, u_k(x)) \geqslant \epsilon_0\} \qquad \text{and} \qquad C^{(k)} = \{x \in B : |u_k(x)| \geqslant M_0\},$$

respectively. We prove that

$$|C^{(k)} \cap B_0^{(k)}| \leqslant \delta_0, \qquad \forall k. \tag{4.24}$$

Put

$$\bar{u}_k(x) = \begin{cases} u_k(x) & \text{if } x \in C^{(k)}, \\ y_0 & \text{if } x \notin C^{(k)}, \end{cases}$$

where $y_0 \in \mathbb{R}^m$ verifies $|y_0| = M_0$. Since $|\bar{u}_k(x)| \geqslant M_0$ on B it follows, from (4.22), that

$$|\{x \in B : f(x, \bar{u}_k(x)) \geqslant \epsilon_0\}| \leqslant \delta_0.$$

Moreover one easily sees that $C^{(k)} \cap B_0^{(k)} \subset \{x \in B : f(x, \bar{u}_k(x)) \geqslant \varepsilon_0\}$ and consequently (4.24) holds.

Finally one has, for any k,

$$\begin{aligned} |\langle \omega(u_k), u_k \rangle| \leqslant & \int_{B - C^{(k)}} |(\omega(x, u_k(x)), u_k(x))|\, dx \\ & + \int_{C^{(k)} \cap (B - B_0^{(k)})} |(\omega(x, u_k(x)), u_k(x))|\, dx \\ & + \int_{C^{(k)} \cap B_0^{(k)}} |(\omega(x, u_k(x)), u_k(x))|\, dx. \end{aligned} \tag{4.25}$$

The first integral on the right is bounded by a constant (dependent on M_0) as follows from (4.14). The second is bounded by $\epsilon_0 \int_B |u_k(x)|^2\, dx$ as follows from the definitions of $B_0^{(k)}$ and f. The third integral on the right of (4.25) is less than or equal to

$$c + \left(\int_{C^{(k)} \cap B_0^{(k)}} b(x)^s\, dx \right)^{1/s} \|u_k\|_p^2$$

as follows from (4.14) and Hölder's inequality. Thus, by using (4.24) and (4.18), the referred integral is bounded by

$$c + \epsilon_0 \|u_k\|_p^2.$$

Hence (4.25) yields

$$|\langle \omega(u_k), u_k \rangle| \leqslant c_3 + (1 + c_2)\epsilon_0 \|u_k\|_p^2, \qquad \forall k,$$

which contradicts (4.20) for any index k such that $\|u_k\|_p \geqslant c_3 \epsilon_0^{-1}$.

Proof of Proposition 4.3. If (4.7) fails, there exists an $\epsilon_0 > 0$ and a sequence $u_k \in L_m{}^p(B)$ such that $\|u_k\|_p \to +\infty$ and (4.20) holds. On the other hand there exists $M_0 > 0$ such that

$$|y| > M_0 \Rightarrow |(\omega(x, y), y)| \leqslant \epsilon_0 |y|^2$$

for almost all $x \in B$. Hence

$$\begin{aligned} |\langle \omega(u_k), u_k \rangle| &\leqslant \int_{\{|u_k(x)| > M_0\}} |(\omega(x, u_k(x)), u_k(x))|\, dx \\ &\quad + \int_{\{|u_k(x)| \leqslant M_0\}} |(\omega(x, u_k(x)), u_k(x))|\, dx \\ &\leqslant c_2 \epsilon_0 \|u_k\|_p^2 + c(M_0). \end{aligned}$$

By choosing k sufficiently large in order that $\|u_k\|_p^2 \geqslant c(M_0)/2\epsilon_0$ one contradicts (4.20).

Examples with a Nondifferentiable Operator g

There exist operators g for which the sufficient conditions stated in Proposition 4.1 hold (hence Theorem 4.1 applies) but which are not F-differentiable at the origin. Similarly there exist operators g for which the sufficient conditions stated in Propositions 4.2 or 4.3 hold (hence Theorem 4.2 applies) but which are not F-differentiable at infinity. This holds if $m \geqslant 2$, even for functions g independent of the variable x. To show this we give two examples of remainders ω, one verifying the conditions of Proposition 4.1 but not

$$\lim_{\|u\|_p \to 0} \|\omega(u)\|_q / \|u\|_p = 0, \tag{4.26}$$

and the other verifying the conditions of Proposition 4.2 but not

$$\lim_{\|u\|_p \to +\infty} \|\omega(u)\|_q / \|u\|_p = 0. \tag{4.27}$$

Note that (4.26) means that $Dg(0) = g_0$ in the F-sense and consequently it implies that $D(A^{1/2} g A^{1/2})(0) = A^{1/2} g_0 A^{1/2}$ in the F-sense. A similar remark, concerning (4.27), holds at infinity.

EXAMPLE 1. Consider the vector function $\omega(y)$ whose components $\omega_1(y)$ and $\omega_2(y)$ are defined by (0.10) with $\xi = y_1$ and $\eta = y_2$; recall that ϕ, $\phi(0) \neq 0$, is an arbitrary, real, continuously differentiable function defined and with compact support on $\mathbb{R}$. This function $\omega(y)$ verifies the hypothesis of Proposition 4.1. In fact (4.12) and (4.11) hold (with $b(x)$ constant and $c_1 = 0$) as follows from (0.13) and (0.15), respectively. Defining $g_i(x, y)$, $i = 1, 2$, by (4.3) with arbitrary $g_{ij}(x) \in L^{p/(p-2)}(B)$, $g_{ij}(x) = g_{ji}(x)$ [note that $g_i(x, y) =$

$\partial G(x, y)/\partial y_i$ with G given by $G(x, y_1, y_2)=\frac{1}{2}\sum_{i,j=1}^{m} g_{ij}(x)y_i y_j+\phi(y_1/y_2^2)y_1 y_2$; see (0.9)], it follows from Theorem 4.1 that the largest positive and the smallest negative eigenvalues for the linear system

$$\int_B K_i(x, \xi)[g_{i1}(\xi)u_1(\xi) + g_{i2}(\xi)u_2(\xi)]\, d\xi = \lambda u_i(x), \qquad i = 1,2, \quad (4.28)$$

are, respectively, the largest positive and the smallest negative bifurcation points for the nonlinear system

$$\int_B K_i(x, \xi)g_i(\xi, u_1(\xi), u_2(\xi))\, d\xi = \lambda u_i(x), \qquad i = 1,2. \quad (4.29)$$

However, (4.26) does not hold. This is easily verified by considering in (4.26) the constant (vector) functions $u_t(x)$ whose two components are, respectively, the constant functions 0 and t ($t \neq 0$). Thus $\omega(u_t)$ has as components the constant functions $t\phi(0)$, and 0 and consequently

$$\|\omega(u_t)\|_q/\|u_t\|_p = |\phi(0)|(\text{mes } B)^{(p-2)/p} \quad (4.30)$$

does not verify (4.26) when $t \to 0$.

EXAMPLE 2. (Asymptotic case). In Example 1 replace (0.10) by (0.18) [and $\phi(y_1/y_2^2)$ by $\phi(y_1^2/y_2)$]. From (0.19) and (0.15) it follows that Proposition 4.2 (as well as Proposition 4.3) applies to our remainder ω. It follows from Theorem 4.2 that the largest positive and the smallest negative eigenvalues for the linear system (4.28) are, respectively, the largest positive and the smallest negative bifurcation points for the nonlinear system (4.29). However, condition (4.27) fails since (4.30) holds for the vector functions $u_t(x)$ used in Example 1.

Remark 4.3. Since Theorems 1.1 and 1.2 hold for nonlinear positively homogeneous operators B it follows that the results of this section apply also when g_0 is an operator of this type. For instance, let $m = 1$ and put $g(x, y) = g_0(x, y) + \omega(x, y)$ with $g_0(x, y) = g_0^+(x)y$ if $y \geqslant 0$, $g_0(x, y) = g_0^-(x)y$ if $y \leqslant 0$, where $g_0^+(x)$ and $g_0^-(x)$ are given functions and ω is a suitable remainder. Then the Nemytsky operator g_0 is positively homogeneous and consequently the results stated in Theorem 4.1 or in Theorem 4.2 hold with obvious changes. Roughly speaking, g_0 corresponds to the case in which $y \to g(x, y)$ is right and left differentiable at $y = 0$, for almost all $x \in B$, but is not necessarily differentiable. A similar example holds for the study of the largest negative bifurcation point λ for Problem (2.1) of [2][5]) if one assumes the graphs β and γ only right and left differentiable at the origin

[5] In this case λ is the principal bifurcation point for (2.1) and corresponds to the largest positive bifurcation point $1/\mu$ for Equation (2.4) of [2].

(differentiability is understood in the sense introduced in Beirão da Veiga [2]). This holds because the proof of Theorem I of Beirão da Veiga [2] holds (with obvious changes) if the graphs β and γ are right and left differentiable instead of differentiable (this was remarked to the author by J. Hernandez).

Remark 4.4. We use in the following the usual notations for Sobolev spaces. Consider the bifurcation problem

$$\begin{aligned} Lu(x) &= \mu g(x, u(x)) \quad \text{in} \quad B, \\ u(x) &= 0 \quad \text{on} \quad \partial B, \end{aligned} \tag{4.31}$$

where B is an open bounded set with boundary ∂B, and

$$Lu \equiv -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right), \qquad a_{ij}(x) = a_{ji}(x), \tag{4.32}$$

is an uniformly elliptic operator on B and g acts from $L^p(B)$ into $L^q(B)$ with $p < 2^* = 2n/(n-2)$ (assume for the sake of convenience that $n > 2$). It is well known that the Green's operator $A(f) = u$ related to the problem $Lu = f$ in B, $u = 0$ on ∂B, acts from $H^{-1,2}(B)$ onto $H_0^{1,2}(B)$ and, in particular, it is completely continuous from $L^q(B)$ into $L^p(B)$. Moreover (4.32) is equivalent to the problem (in $L^p(B)$)

$$(Ag)(u) = \lambda u, \qquad \lambda = 1/\mu^6, \tag{4.31$'$}$$

which can be reduced, by splitting the linear operator A, to

$$(A^{1/2}gA^{1/2})(u) = \lambda u. \tag{4.31$''$}$$

This last problem concerns a potential operator in the Hilbert space $L^2(B)$.

Assume now that instead of the linear operator L one has a nonlinear operator, as *for instance*,

$$Lu \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{r-2}\frac{\partial u}{\partial x_i}\right), \tag{4.33}$$

and assume that g acts from $L^p(B)$ into $L^q(B)$ with $p < r^* = rn/(r-n)$ (assume for convenience that $n > r$). The inverse operator $A(f) = u$ related to the problem $Lu = f$ in B, $u = 0$ on ∂B, acts from $H^{-1,r'}(B)$ onto $H_0^{1,r}(B)$ and is weakly continuous from $L^q(B)$ into $L^p(B)$. However, the problem does not reduce to (4.31$''$) since A is nonlinear. This leads to the study of problem (4.31) directly. Since L is defined on $H_0^{1,r}(B)$ but not on all of $L^p(B)$ (and g is defined in both of these spaces) one is lead to consider L and g as

[6] The largest positive λ for (4.31$'$) corresponds to the smallest positive μ for (4.31) which is the principal bifurcation point for a wide class of problems.

operators from $H_0^{1,r}(B)$ into $H^{-1,r'}(B)$. In this context the operator g is weakly continuous. Furthermore g and L are potential operators with $L = \nabla\Psi$ where

$$\Psi(u) = \frac{1}{r}\int_B |\nabla u|^r \, dx, \qquad u \in H_0^{1,r}(B).$$

is weakly lower semicontinuous.

Thus the bifurcation problem (4.31), (4.33) can be studied as a particular case ($\Lambda = L, \Gamma = g$) of the general problem

$$\Gamma(u) = \lambda\Lambda(u), \qquad \lambda = 1/\mu, \tag{4.34}$$

where $\Gamma = \nabla\Phi$ and $\Lambda = \nabla\Psi$ are potential operators acting from a Banach space V into the dual space V', Φ being weakly continuous (more generally weakly upper semicontinuous) and Ψ weakly lower semicontinuous. The results of the first part of this chapter can be extended to this more general problem and applied, in particular, to the referred example or to more general problems where $L = (L_1, \ldots, L_m)$ is a system of nonlinear operators and the boundary conditions are not of the Dirichlet type.

These results will be proved in a forthcoming paper and generalize in particular a result of Naumann which is applied by this author to the study of bifurcation buckling of thin elastic shells (cf. Naumann [7], Theorem 5.1'.[7]

APPENDIX

The proof of Theorem 1.1 is easily adapted to prove an analogous result concerning the case $M_{\mathbb{K}}(B) = 0$. Consider the following condition.

ASSUMPTION A.1. *For any* $u \in \mathbb{K}$, $u \neq 0$, *such that* $(\Gamma(u), u) \geq 0$ *and* $\Gamma(u) \neq \lambda u$, $\forall\lambda \in \mathbb{R}$, *there exists a* $v \in \mathbb{K}_\rho$ $(\rho = \|u\|)$, $v \neq u$, *such that* $(\Gamma(u), v) > (\Gamma(u), u)$.

One has, in particular, from Lemma 2.3:

Remark A.1. Proposition 1.3 and Corollary 1.1 hold if one replaces in these statements Assumption 1.1 by Assumption 5.1.

The following result holds (a similar result holds for the asymptotic bifurcation).

[7] Similar problems under unilateral conditions were studied by Do [5, and references]. It seems that our results regarding Equation (4.34) can be adapted in order to be applicable in this case.

THEOREM A.1. *Assume that the conditions of Theorem* 1.1 *hold with* $M_{\mathbb{K}}(B) = 0$ *and with Assumption* 1.1 *replaced by Assumption* 5.1. *Furthermore assume that:*

(i) *for any* $\rho > 0$ *there exists* $u \in \mathbb{K}_\rho$ *such that* $\Phi(u) > 0$;
(ii) *if* $\Phi(u) > 0$ *and* $u \in \mathbb{K}$, *then* $\Gamma(u) \neq 0$.

Then 0 *is the largest bifurcation point for* Γ *in* $\mathbb{K}$.

To finish this chapter we remark that the method used here is applicable if B is positively homogeneous of degree α (i.e., if $B(tu) = t^\alpha B(u), \forall u \in \mathbb{K}, \forall t > 0$) and if the remainder ω verifies (1.1) [respectively, (1.2)] with $\|u\|^\beta$ instead of $\|u\|^2$. In particular, one has the following results, the proofs of which are similar to those of Theorems 1.1 and 1.2.

THEOREM A.2. *Assume that the conditions of Theorem* 1.1 *hold with the following modifications: B is positively homogeneous of degree* $\alpha > 0$ (*instead of* $\alpha = 1$) *and*

$$\lim_{\substack{\|u\| \to 0 \\ u \in \mathbb{K}}} (\omega(u), u)/\|u\|^{1+\alpha} = 0$$

instead of (1.1).[8] *Then if* $\alpha > 1$, *they are not positive bifurcation points and* 0 *is a bifurcation point for* Γ (*unique if* $m_{\mathbb{K}}(B) > -\infty$). *If* $\alpha < 1$, *then* $+\infty$ *is a bifurcation point for* Γ (*unique if* $m_{\mathbb{K}}(B) > 0$).

THEOREM A.3. *Assume that the conditions of Theorem* 1.2 *hold with the following modifications: B is positively homogeneous of degree* $\alpha > -1$ (*instead of* $\alpha = 1$) *and*

$$\lim_{\substack{\|u\| \to +\infty \\ u \in \mathbb{K}}} (\omega(u), u)/\|u\|^{1+\alpha} = 0$$

instead of (1.2) *and* $\Phi(u) - [1/(\alpha + 1)](B(u), u)$ *replaces* $\Phi(u) - \frac{1}{2}(B(u), u)$.[8] *Then if* $\alpha < 1$, *they are not positive asymptotic bifurcation points and* 0 *is an asymptotic bifurcation point for* Γ (*unique if* $m_{\mathbb{K}}(B) > -\infty$). *If* $\alpha > 1$, *then* $+\infty$ *is an asymptotic bifurcation point* (*unique if* $m_{\mathbb{K}}(B) > 0$).

These results can be adapted to the case $M_{\mathbb{K}}(B) = 0$ as in Theorem 5.1.

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[8] We assume that $M_{\mathbb{K}}(B) > 0$.

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