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Differentiability for Green's Operators of Variational Inequalities and Applications to the Determination of Bifurcation Points

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We study Fréchet differentiability at the origin, in the Hilbert space $L^2(\Omega)$, for the Green's operator P and we apply these results to the calculus of bifurcation points.

INTRODUCTION

A classical method for the study of questions concerning the eigenvalues of linear differential operators is to reduce this problem to the analogous one for the inverse operator. The problem becomes easier because this operator is in general compact. We can apply this method when the (nonlinear) operator T is maximal monotone, by considering the operator $P = (I + T)^{-1}$; this was used in [10] for studies concerning nonlinear spectral analysis and also in [6]. Using this method, the problem of finding the eigenvalues for (2.1) reduces easily to the same problem for the Green's operator P defined in (1.3).

Then it becomes natural to try to use for P the known abstract theorems for compact nonlinear operators. Obviously this will be interesting when the hypotheses that underlie these theorems are not trivially verified by P. This will be the case when one tries to apply the celebrated theorem of Krasnosel'skii (see [8]) because we must then investigate under what conditions the operator P is Frechet differentiable at the origin. This is the main purpose of our paper (see Theorem I). A theorem on the bifurcation points is also derived as a consequence of Theorem I (see Theorem II).

We remark that the method introduced in this paper to prove Theorem I is applicable to problems more general than (1.3); in particular it is independent from monotonicity properties and from the Hilbert structure of H.

The results stated in this paper, and corresponding proofs, were presented in [1].

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I

Let Ω be an open bounded set in the n-dimensional Euclidean space $\mathbf{R}^n$ and let Γ be the boundary of Ω. We assume Ω to be sufficiently smooth.

We shall assume that the spaces $L^p(\Omega)$, $1 < p < +\infty$, and the Sobolev spaces $W^{k,p}(\Omega)$, k positive integer, are familiar to the reader; we denote by $\|\ \|_p$ and $\|\ \|_{k,p}$ the usual norms in these spaces and we put $H = L^2(\Omega)$ and $\|\ \| = \|\ \|_2$. As usual $W_0^{1,2}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $W^{1,2}(\Omega)$, and $W^{-1,2}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$.

We shall consider also the spaces $L^p(\Gamma)$ and the Sobolev spaces $W^{k-(1/2),2}(\Gamma)$, k positive integer; we denote the usual norms in these spaces by $|\ |_p$ and $|\ |_{k-(1/2),2}$, respectively. Finally $W^{-1/p,p}(\Gamma)$ is the dual space of $W^{1/p,p'}(\Gamma)$, $p' = p/(p-1)$.

The following definitions are well known: If $\Theta: H \to H$ is a multivalued operator (a graph) we put $D(\Theta) = \{u \in H: \Theta(u) \neq \varnothing\}$. We say that Θ is monotone if $(v' - v, u' - u) \geqslant 0$ for all $u, u' \in H$ and all $v \in \Theta(u)$, $v' \in \Theta(u')$. We say that Θ is maximal monotone (and we write m.m.) if Θ is maximal in the class of monotone multivalued operators.

Let now $\alpha: \mathbf{R} \to 2^{\mathbf{R}}$ and suppose that $0 \in \alpha(0)$; we say that α is differentiable at the origin with finite derivative α' if the following condition holds:

(1.1) For any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$| z - \alpha' y | \leqslant \epsilon | y |, \qquad \forall z \in \alpha(y),$$

for all $y \in D(\alpha) \cap]-\delta_\epsilon, \delta_\epsilon[$.

Obviously if α is monotone then $\alpha' \geqslant 0$.

We say that α is differentiable at the origin with $\alpha' = +\infty$ if

(1.2) For any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$| y | \leqslant \epsilon | z |, \qquad \forall z \in \alpha(y),$$

for all $y \in D(\alpha) \cap]-\delta_\epsilon, \delta_\epsilon[$.

In the sequel β and γ are two m.m graphs on $\mathbf{R}$ verifying $0 \in \beta(0)$, $0 \in \gamma(0)$. The following result is due to Brézis (cf. [4, Corollary 13]):

For every $u \in H$ there exists a unique function $Pu \in W^{2,2}(\Omega)$ satisfying

$$\begin{aligned} -\Delta Pu + \gamma(Pu) + Pu \ni u, &\qquad \text{a.e. in } \Omega, \\ -\partial Pu/\partial n \in \beta(Pu), &\qquad \text{a.e. on } \Gamma, \end{aligned} \tag{1.3}$$

where $\partial/\partial n$ is the outward normal derivative; moreover $\| Pu \| \leqslant \| u \|$. In addition we can see that

$$\| Pu \|_{2,2} \leqslant c \| u \|. \tag{1.4}$$

We denote by c constants depending only on Ω, n, β, and γ.

In this section we prove the following theorem:

THEOREM I. (i) *If γ is differentiable at the origin with $\gamma' = +\infty$ then the operator $P: H \to H$ defined in* (1.3) *is Fréchet differentiable and* $\nabla P(0) = 0$.

(ii) *If β and γ are differentiable at the origin with $\beta' = +\infty$ and $\gamma' < +\infty$ then the operator P is Fréchet differentiable at the origin and $\nabla P(0) = A$ is the Green's operator for the linear Dirichlet problem*

$$\begin{aligned} -\Delta Au + Au + \gamma' Au &= u \quad \text{in } \Omega, \\ Au &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.5}$$

(iii) *If $\beta' < +\infty$ and $\gamma' < +\infty$ then $\nabla P(0) = A$ is the Green's operator for the linear problem*

$$\begin{aligned} -\Delta Au + Au + \gamma' Au &= u \quad \text{in } \Omega, \\ -\partial Au/\partial n &= \beta' Au \quad \text{on } \Gamma. \end{aligned} \tag{1.6}$$

We suppose without loss of generality that $n \geqslant 3$. If $n \leqslant 3$ the proofs are trivial since $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$. In order to prove Theorem I we state some lemmas:

(1.7) LEMMA. *Assume that α satisfies* (1.1) *with $\alpha' = 0$, $v, w \in L^p(\Gamma)$ with $p > q \geqslant 1$, $v(x) \in D(\alpha)$ a.e. on Γ, and $w(x) \in \alpha(v(x))$ a.e. on Γ; then*

$$|w|_q^q \leqslant \epsilon^q |v|_q^q + [p/(p-q)]\, \delta_\epsilon^{q-p}(|v|_p^p + |w|_p^p), \qquad \forall \epsilon > 0. \tag{1.8}$$

In particular if $v, w \in W^{1/2,2}(\Gamma)$ then

$$|w|_2 \leqslant \epsilon |v|_2 + c\delta_\epsilon^{(2-p)/2}(|v|_{1/2,2}^{p/2} + |w|_{1/2,2}^{p/2}), \qquad \forall \epsilon > 0. \tag{1.9}$$

with $p = 2(n-1)/(n-2)$.

Proof. Put $\delta_\epsilon = \delta$ and

$$\Gamma_{\delta,v} = \{x \in \Gamma : |v(x)| > \delta\};$$

using (1.1) with $\alpha' = 0$ we get

$$\begin{aligned} |w|_q^q &= \int_{\Gamma - \Gamma_{\delta,v}} |w|^q \, d\Gamma + \int_{\Gamma_{\delta,v}} |w|^q \, d\Gamma \\ &\leqslant \epsilon^q |v|_q^q + \int_{\Gamma_{\delta,v}} |w|^q \, d\Gamma. \end{aligned} \tag{1.10}$$

Furthermore,

$$\int_{\Gamma_{\delta,v}} |w|^q \, d\Gamma = \int_{\Gamma_{\delta,v}\cap\{x:|w(x)|\leqslant|v(x)|\}} |w|^q \, d\Gamma + \int_{\Gamma_{\delta,v}\cap\{x:|w(x)|>|v(x)|\}} |w|^q \, d\Gamma$$
$$\leqslant \int_{\Gamma_{\delta,v}} |v|^q \, d\Gamma + \int_{\Gamma_{\delta,w}} |w|^q_\epsilon \, d\Gamma. \tag{1.11}$$

On the other hand, denoting by $\mu(\sigma)$ the measure of the set $\Gamma_{\sigma,v}$, it is well known that

$$\mu(\sigma) \leqslant \left(\frac{|v|_p}{\sigma}\right)^p, \qquad \forall \sigma > 0.$$

Furthermore,

$$\int_{\Gamma_{\delta,v}} |v|^q \, d\Gamma = q \int_{\delta}^{+\infty} \mu(\sigma)\, \sigma^{q-1} \, d\sigma + \delta^q \mu(\delta),$$

and as a consequence a straightforward computation yields

$$\int_{\Gamma_{\delta,v}} |v|^q \, d\Gamma \leqslant q/(p-q)\, \delta^{q-p} |v|_p^p + \delta^{q-p} |v|_p^p = [p/(p-q)]\, \delta^{q-p} |v|_p^p . \tag{1.12}$$

From (1.10), (1.11), and (1.12) we find that (1.8) holds.

Finally (1.9) follows from (1.8) with $q = 2$ and from Sobolev's embedding theorem $W^{1/2,2}(\Gamma) \hookrightarrow L^p(\Gamma)$.

Similarly we obtain the following result:

(1.13) LEMMA. *Assume that* $\alpha' = +\infty$, $v \in L^p(\Gamma)$, $p > q \geqslant 1$, $w \in L^q(\Gamma)$, *and* $v(x) \in D(\alpha)$, $w(x) \in \alpha(v(x))$ *a.e. on* Γ. *Then, for any* $\epsilon > 0$, *we have*

$$|v|_q^q \leqslant \epsilon^q |w|_q^q + [p/(p-q)]\, \delta_\epsilon^{q-p} |v|_p^p . \tag{1.14}$$

If in particular $v \in W^{1/2,2}(\Gamma)$ *then*

$$|v|_2 \leqslant \epsilon |w|_2 + c\delta_\epsilon^{(2-p)/2} |v|_{1/2,2}^{p/2} , \tag{1.15}$$

with p defined as in Lemma (1.7).

Proof. The proof of (1.14) is analogous to the proof of (1.8); now we use the estimate

$$\int_\Gamma |v|^q \, d\Gamma = \int_{\Gamma-\Gamma_{\delta,v}} |v|^q \, d\Gamma + \int_{\Gamma_{\delta,v}} |v|^q \, d\Gamma$$
$$\leqslant \epsilon^q \int_\Gamma |w|^q \, d\Gamma + [p/(p-q)]\, \delta^{q-p} |v|_p^p$$

that follows from (1.2) and (1.12). The proof of (1.15) is immediate.

We see immediately from the proofs of (1.8) and (1.14) that Γ can be replaced by Ω. Hence, in particular, we can state the following lemmas:

(1.16) LEMMA. *Let $q < 2$ and assume that $\alpha' = 0$. If $v, w \in L^2(\Omega)$, $v(x) \in D(\alpha)$ and $w(x) \in \alpha(v(x))$ a.e. on Ω then we have, for any $\epsilon > 0$,*

$$\| w \|_q^q \leqslant \epsilon^q \| v \|_q^q + [2/(2-q)]\, \delta_\epsilon^{q-2}(\| v \|^2 + \| w \|^2). \tag{1.17}$$

(1.18) LEMMA. *Assume that $\alpha' = +\infty$. If $v \in L^p(\Omega)$, $p > 2$, $w \in L^2(\Omega)$, $v(x) \in D(\alpha)$ and $w(x) \in \alpha(v(x))$ a.e. on Ω then we have, for any $\epsilon > 0$,*

$$\| v \|^2 \leqslant \epsilon^2 \| w \|^2 + [p/(p-2)]\, \delta_\epsilon^{2-p} \| v \|_p^p . \tag{1.19}$$

For the remainder of the proof of Theorem I the reader is referred to [2] where a simplified version of the proof of [1] is indicated.

II

We are interested in the study of solutions $u \neq 0$ for the problem

$$\begin{aligned} -\Delta u + \gamma(u) + \lambda u \ni 0, &\qquad \text{a.e. in } \Omega, \\ -\partial u/\partial n \in \beta(u), &\qquad \text{a.e. on } \Gamma. \end{aligned} \tag{2.1}$$

For $\lambda > 0$ the only solution to problem (2.1) is the null solution; hence we may assume, without loss of generality, that $\lambda \leqslant 0$.

If λ, u is a solution of (2.1) with $u \neq 0$ we say that λ is an eigenvalue. We say that λ_0 is a bifurcation point for (2.1) if for any $\epsilon > 0$ there exists a solution λ, u with $0 < \| u \| < \epsilon$ and $| \lambda - \lambda_0 | < \epsilon$.

In this section, as a consequence of Theorem I, we prove the existence and we characterize completely the bifurcation points for (2.1), when β and γ verify the hypothesis of Theorem I. More precisely we prove the following result:

THEOREM II. (i) *If γ is differentiable at the origin and $\gamma' = +\infty$ then problem (2.1) has no bifurcation points.*

(ii) *If β and γ are differentiable at the origin with $\beta' = +\infty$, $\gamma' < +\infty$, the bifurcation points of (2.1) are exactly the eigenvalues for the Dirichlet problem*

$$\begin{aligned} -\Delta u + (\gamma' + \lambda) u = 0 &\qquad \text{in } \Omega, \\ u = 0 &\qquad \text{on } \Gamma, \end{aligned} \tag{2.2}$$

i.e., they are given by $\lambda_0 = \lambda_1 - \gamma'$, where λ_1 are the eigenvalues for the Dirichlet problem $-\Delta u + \lambda u = 0$ in Ω, $u = 0$ on Γ.

(iii) *If $\beta' < +\infty$ and $\gamma' < +\infty$ then the bifurcation points for* (2.1) *are the* *igenvalues for the linear problem*

$$-\Delta u + (\gamma' + \lambda)u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n + \beta' u = 0 \quad \text{on } \Gamma, \tag{2.3}$$

.e., they are given by $\lambda_0 = \lambda_1 - \gamma'$, where λ_1 are the eigenvalues for the linear *roblem $-\Delta u + \lambda u = 0$ in Ω, $\partial u/\partial u + \beta' u = 0$ on Γ.*

By using the method referred to in the Introduction we see easily that the ifurcation points λ of (2.1) are transformed in the bifurcation points μ of

$$v = \mu P v \tag{2.4}$$[1]

y means of the change of variables (used also in [6])

$$\mu = 1 - \lambda, \qquad v = \mu u. \tag{2.5}$$

Moreover $\mu = 1 - \lambda$ gives a correspondence between the eigenvalues λ of 2.2) [resp. (2.3)] and the characteristic values μ of

$$v = \mu A u, \tag{2.6}$$

here A is the linear operator defined in (1.5) [resp. (1.6)].

Therefore to prove Theorem II in case (ii) [resp. (iii)] it is sufficient to prove hat the bifurcation points μ for Eq. (2.4) are the characteristic values μ for the near equation (2.6). But this statement holds since by Krasnosel'skii's theorem see [8, Sect. VI, Theorem 2.2, Sect. IV, Lemma 2.1]) the bifurcation points or Eq. (2.4) are the characteristic values for the linearized equation $v = \mu \nabla P(0) v$, nd by Theorem I(ii) [resp. (iii)] we have $\nabla P(0) = A$.

In case (i) we have, by Theorem I, $\nabla P(0) = 0$ and the null operator has no haracteristic values.

We remark that the supplementary hypotheses required to apply rasnosel'skii's theorem are verified as consequences of some results due ssentially to Brézis and Moreau.

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[1] Note that $\mu \geqslant 1$ since P is a contraction.

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