PAPER

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On the partial regularity of suitable weak solutions in the non-Newtonian shear-thinning case

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Abstract
We study the partial regularity of suitable weak solutions to incompressible non-Newtonian fluids in the shear-thinning case $p < 2$. For the shear-thickening case $p > 2$ this problem was previously considered in 2002 by Guo and Zhu (*J. Differ. Equ.* **178** 281–97). By partially appealing to some of their ideas, we show that in the $p < 2$ case the singular points are concentrated on a closed set whose one dimensional Hausdorff measure is zero.

Keywords: partial regularity, non-Newtonian fluids, shear-thinning
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1. Introduction

In this paper, we are concerned with the following modified Navier–Stokes equations which describes the dynamics of incompressible mono-polar non-Newtonian fluids:

$$\begin{cases} u_t - \nabla\cdot\tau^{\mathrm{V}} + u\cdot\nabla u + \nabla\pi = 0, & \text{in } \Omega\times(0,T), \\ \operatorname{div} u = 0, & \text{in } \Omega\times(0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = a(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$\tau^{\mathrm{V}} = \left(\mu_0 + \mu_1|e(u)|^{p-2}\right) e(u), \tag{1.2}$$

and

$$e(u) = (e_{ij}(u)), \quad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.3}$$

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Here, Ω is a domain in $\mathbb{R}^3$, $u = u(x,t) = (u_1, u_2, u_3)^\top$ is the velocity and π is the pressure. τ^{V} denotes the viscous part of the stress tensor which depends only on the rate strain tensor $e(u)$. Furthermore, μ_0, μ_1 are positive constants.

When $\mu_1 = 0$ or $p = 2$ the system reduces to the famous Navier–Stokes equations. Navier–Stokes equations has been studied by many mathematicians. Leray [17] (unbounded domain) and Hopf [6] (bounded domain) showed the existence of weak solutions. But the global regularity and uniqueness are unknown until now. In a series of papers, Scheffer [24–27] studied the partial regularity of solutions of the Navier–Stokes equations which satisfy a local version of the energy inequality. Later on, Caffarelli *et al* [4] improved Scheffer's results. They proved that the set of possible interior singular points of a suitable weak solution is of one-dimensional parabolic Hausdorff measure zero if the force satisfies $f \in L^{\frac{5}{2}+\delta}$ for some $\delta > 0$. A simplified proof was proposed by Lin in [19]. Concerning the restriction of the force f, Ladyzhenskaya and Seregin [16] proved the CKN partial regularity result under the condition that the force satisfies a Morrey type condition

$$\sup_{Q_r(x,t)\subset\Omega\times(0,T)} \frac{1}{r^{1+\delta}} \iint_{Q_r(x,t)} f^2 < \infty$$

with $\delta > 0$. Kukavica [9] proved that the CKN partial regularity result holds under the assumption $f \in L^{\frac{5}{3}+\delta}$ where $\delta > 0$. Furthermore, see [10–12].

This paper focuses on the incompressible non-Newtonian fluids. It is worth noting that the literature on this subject is extremely wide. It would be out of place, even not possible, to try here such an engagement. The first mathematical investigations go back to Ladyzhenskaya's lecture at the International Mathematical Congress in 1966, where she proposed to study the system (1.1) with $p = 4$. Later on, these first results were extended, and presented in further contributions of Ladyzhenskaya, see [13–15]. Combining monotone operator theory and compactness arguments, she proved the existence of weak solutions to system (1.1) for the periodic boundary condition if $p \geqslant \frac{11}{5}$ and their uniqueness if $p \geqslant \frac{5}{2}$, see also [20]. For more results about this subject one can refer to the monograph Málek *et al* [21]. When one imposes the Dirichlet boundary condition Málek *et al* [22] established the existence of weak solutions for $p \geqslant 2$. Later on, Wolf [30] extended this result to $p > \frac{8}{5}$. Concerning the regularity of weak solutions, the global strong solutions were obtained by Málek *et al* [21] with the periodic boundary condition when $p \geqslant \frac{11}{5}$. Later on, Málek *et al* [22] proved the global existence of strong solutions under the Dirichlet boundary condition for $p \geqslant \frac{9}{4}$. Under this last boundary condition, in reference [2] regularity results up to the boundary were established, for $p \geqslant 2 + \frac{2}{5}$, by following ideas introduced in reference [1], for slip and non-slip boundary conditions.

It is natural to consider the partial regularity of the non-Newtonian system (1.1) by appealing to the Caffarelli–Kohn–Nirenberg results. An attempt in this direction is done in reference [5] where it is claimed that the set of singular points of the suitable weak solutions to the non-Newtonian system for $p > 2$ is of $5 - 2p$ dimensional Hausdorff measure zero ([5, theorem 1.1, item (i)]). See a related note in section 3.

2. Partial regularity for $p < 2$

From now on, without loss of generality, we assume that $\mu_0 = \mu_1 = 1$. Standard, or clear, notation will be not defined.

In this section, we consider the shear-thinning fluids, i.e., $p < 2$. We will prove that the singular points are concentrated on a closed set whose one dimensional Hausdorff measure is zero. Our main idea is to treat the term $\mathrm{div}(|e(u)|^{p-2}e(u))$ as a 'special force'. Note that we cannot regard this last term directly as a typical external force since it lacks the necessary integrability. So we adopt the argument developed by Kukavica in reference [9], where partial regularity is proved under a quite weak assumption on the forces, namely f is divergence-free and $f \in L^q(D)$, $q > \frac{5}{3}$.

We first give the definition of a suitable weak solution. Let $D = \Omega \times (0, T)$. The pair (u, π) is called a suitable weak solution to the system (1.1) on D if the following conditions are met:

(a) $u \in L_t^\infty L_x^2(D) \cap L_t^2 \left(W_x^{1,2} \cap W_x^{1,p}\right)(D)$ and $\pi \in L^{\frac{3}{2}}(D)$.

(b) The non-Newtonian system (1.1) is satisfied in D in the weak sense, i.e. for every $\psi \in C_0^1(D)$,

$$\int_D \partial_t u \cdot \psi - (u \otimes u) : \nabla\psi + \left(\mu_0 + \mu_1 |e(u)|^{p-2}\right) e(u) : e(\psi) \mathrm{d}x \, \mathrm{d}t = \int_\Omega \pi \ \mathrm{div} \ \psi \, \mathrm{d}x.$$

(c) The local energy inequality holds for any $t \in (0, T)$, i.e.

$$\begin{aligned} &\int_\Omega |u|^2 \phi|_t \, \mathrm{d}x + 2\int_0^t \int_\Omega \left(|\nabla u|^2 + |e(u)|^p\right) \phi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad \leqslant \int_0^t \int_\Omega |u|^2 (\phi_s + \Delta\phi) \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega (|u|^2 + 2\pi) u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad\quad - 2\int_0^t \int_\Omega |e(u)|^{p-2} e(u) : (u \otimes \nabla\phi) \, \mathrm{d}x \, \mathrm{d}s \end{aligned} \tag{2.1}$$

for all $\phi \in C_0^\infty(D)$ such that $\phi \geqslant 0$ in D.

Remark 2.1. Let's recall that the local energy inequality for the classical (Newtonian) Navier–Stokes equation with a force term f is as follows:

$$\begin{aligned} &\int_\Omega |u|^2 \phi|_t \, \mathrm{d}x + 2\int_0^t \int_\Omega |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad \leqslant \int_0^t \int_\Omega |u|^2 (\phi_s + \Delta\phi) \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega (|u|^2 + 2\pi) u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad\quad + 2\int_0^t \int_\Omega (f \cdot u) \, \phi \, \mathrm{d}x \, \mathrm{d}s. \end{aligned} \tag{2.2}$$

Note that in equation (2.1) the term $\mathrm{div}(|e(u)|^{p-2}e(u))$ gives rise to two integral terms. The first one (last term on the left-hand side of equation (2.1)) is a positive, helpful (even crucial) term. On the contrary, the second one (last term on the right-hand side of (2.1)) should play, roughly, a role similar to that played in (2.2) by the force term f. However, by regarding $|e(u)|^{p-2}e(u)$ as a force f, and by considering the last integrals on the right-hand sides of (2.1) and (2.2), we show that the other terms in the two integrals are still different. One is $u \otimes \nabla\phi$, another is $u\phi$. Moreover, in our case, $-\Delta\pi = \mathrm{div}(u \cdot \nabla u) - \mathrm{div}\,\mathrm{div}(|e(u)|^{p-2}e(u))$. So the pressure's expression has an additional part $-(-\Delta)^{-1}\,\mathrm{div}\,\mathrm{div}(|e(u)|^{p-2}e(u))$. Hence we need a more delicate procedure with respect to the classical one. This is the reason why we call the originating term $\mathrm{div}(|e(u)|^{p-2}e(u))$ a 'special force'.

In the sequel, we denote by $B_r(x_0)$ the standard euclidean ball with the centre x_0 and the radius r, and by $Q_r(x_0, t_0) = \bar{B}_r(x_0) \times [t_0 - r^2, t_0]$ the parabolic cylinder labelled by the top centre point $(x_0, t_0) \in D$. For simplicity, we write $Q_r = Q_r(0, 0)$ and $B_r = B_r(0)$.

We say that a point $(x_0, t_0) \in D$ is semi-regular if $u \in L^5(D_0)$ in an open neighbourhood $D_0 \subset D$ of (x_0, t_0). According to Serrin-type criterion, see [29], the solution is strong in D_1 for $\bar{D}_1 \subset D$ (note that Serrin-type criterion is valid for the system (1.1), since $\mu_0 = 1 > 0$). Here, by strong solution, we mean $u \in L_t^2 W_x^{2,2}(D_1) \cap L_t^\infty \left(W_x^{1,2} \cap W_x^{1,p}\right)(D_1) \cap L_t^p W_x^{1,3p}(D_1)$ and $u_t \in L_t^2 L_x^2(D_1)$. We call a point $(x_0, t_0) \in D$ singular if it is not semi-regular.

Remark 2.2. Unlike the Navier–Stokes equations, we don't know if the strong solution of system (1.1) has $C^{1,\alpha}$ regularity. This is the reason why, by following [21] p 214, we appeal to the terminology semi-regular instead of regular.

An interesting result to guarantee the local Hölder continuity of the velocity gradient for strong solutions to system (1.1) has been presented by Seregin in [28].

Next we give some notation: for $(x_0, t_0) \in D$, and all $r > 0$ such that $Q_r(x_0, t_0) \subset D$, set

$$A_{(x_0,t_0)}(r) = \sup_{(t_0 - r^2, t_0)} r^{-1} \int_{B_r(x_0)} |u|^2 \, \mathrm{d}x,$$

$$B_{(x_0,t_0)}(r) = r^{-1} \iint_{Q_r(x_0,t_0)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t,$$

$$G_{(x_0,t_0)}(r) = r^{-2} \iint_{Q_r(x_0,t_0)} |u|^3 \, \mathrm{d}x \, \mathrm{d}t,$$

$$D_{(x_0,t_0)}(r) = r^{-2} \iint_{Q_r(x_0,t_0)} |\pi|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t.$$

Theorem 2.1. *Let $\Omega = \mathbb{R}^3$, $1 < p < 2$, and assume that $u_0 \in W^{1,2}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Then there exists a suitable weak solution (u, π) of the modified Navier–Stokes system (1.1) on D.*

The result also holds for bounded, smooth, domains Ω under the additional assumption

$$\pi \in L^{\frac{3}{2}}(D). \tag{2.3}$$

The proof of theorem 2.1 is similar to [4, 19] (see also [5]), we omit its details, and give the following remarks.

Remark 2.3. When Ω has the boundaries, even for $p > 2$, the authors do not know how to prove (2.3) since in this case we merely know that π is a distribution. This is the reason why in theorem 2.1, we have restricted ourselves to the whole space.

Remark 2.4. The proof of theorem 2.1 refers to [4, 19]. Actually, following the arguments in [4, appendix] and [19, theorem 2.2], the suitable weak solutions of the modified Navier–Stokes system (1.1) can be constructed *a priori* estimates to the weak solutions obtained in references [23]. Furthermore, we remark that if $\mu_0 = 0$, one has to restrict $p > \frac{9}{5}$ since the corresponding estimates hold only when $p > \frac{9}{5}$ in this case, see the references [23, theorem 4.84]. However, for our case $\mu_0 > 0$, one has an independent estimate in $L_t^2 W_x^{1,2}(D)$ for any $p > 1$, and one can get that $\int_0^T \|\nabla u\|_{W^{1,p}}^{\frac{2}{7-2p}} \mathrm{d}t < \infty$ for any $T > 0$ as [23, theorem 4.86]. It follows from the arguments in [4, 19] that these priori estimates are sufficient to construct the suitable weak solutions, and therefore our theorem 2.1 is true for $p > 1$.

Our main result is as follows.

Theorem 2.2. *Let $1 < p < 2$. There exists a sufficiently small universal constant $\epsilon_0 > 0$ with the following property. If (u, π) is a suitable weak solution of the system (1.1) near*

$(x_0, t_0) \in D$, and if

$$\limsup_{r\to 0+} B^{\frac{1}{2}}_{(x_0,t_0)}(r) < \epsilon_0, \tag{2.4}$$

then (x_0, t_0) is a semi-regular point. In particular, the one dimensional parabolic Hausdorff measure of the set of singular points equals 0.

Now, we focus on the proof of theorem 2.2. The second part of the theorem follows from the first, see [4], pp 776–777. Hence it is sufficient to prove the first part. As in [9] let $0 < r < \frac{\rho}{2}$, and set $\kappa = \frac{r}{\rho}$, and

$$\theta_{(x,t)}(r) = A^{\frac{1}{2}}_{(x,t)}(r) + B^{\frac{1}{2}}_{(x,t)}(r) + \kappa^{-4} D^{\frac{2}{3}}_{(x,t)}(r).$$

Then we have the following lemma.

Lemma 2.3. *Assume $(0, 0) \in D$. Set $\theta(r) = \theta_{(x,t)}(r)$. Then we have*

$$\theta(r) \leqslant C\kappa^{\frac{2}{3}}\theta(\rho) + C\kappa^{-5}B^{\frac{1}{2}}(\rho)\theta(\rho) + C\kappa^{-1}\rho^{2-p}\theta^{\frac{p}{2}}(\rho) + C\rho^{4-2p}\kappa^{-\frac{8}{3}}\theta^{\,p-1}(\rho), \tag{2.5}$$

and

$$\theta(r) \leqslant C\kappa^{\frac{2}{3}}\theta(\rho) + C\kappa^{-5}\theta^{2}(\rho) + C\kappa^{-1}\rho^{2-p}\theta^{\frac{p}{2}}(\rho) + C\rho^{4-2p}\kappa^{-\frac{8}{3}}\theta^{p-1}(\rho), \tag{2.6}$$

for $0 < r \leqslant \frac{3\rho}{5}$ such that $Q_\rho \subset D$, where $C > 0$ is a universal constant.

Proof. Noting that $B^{\frac{1}{2}}(\rho) \leqslant \theta(\rho)$, it follows that (2.5) implies (2.6). So it is sufficient to prove (2.5). As in [9], we set

$$\psi(x, t) = r^2 G(x, r^2 - t)\,, \quad \text{for} \quad (x, t) \in \mathbb{R}^3 \times (-\infty, 0),$$

where $G(x, t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t})$ is the Gaussian kernel. For convenience, we list some estimates on the function ψ (see [9] for details on the proofs)

$$\begin{aligned}
\psi(x,t) &\geqslant \frac{1}{Cr}\,, \quad (x,t) \in Q_r,\\
\psi(x,t) &\leqslant \frac{C}{r}\,, \quad (x,t) \in Q_\rho,\\
|\nabla\psi(x,t)| &\leqslant \frac{C}{r^2}\,, \quad (x,t) \in Q_\rho,\\
\psi(x,t) &\leqslant \frac{Cr^2}{\rho^3}\,, \quad (x,t) \in Q_\rho \setminus Q_{\rho/2},\\
|\nabla\psi(x,t)| &\leqslant \frac{Cr^2}{\rho^4}\,, \quad (x,t) \in Q_\rho \setminus Q_{\rho/2}.
\end{aligned} \tag{2.7}$$

In addition, let $\eta : \mathbb{R}^3 \times \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that $\eta = 1$ on $Q_{\rho/2}$ and $\eta = 0$ on Q^c_ρ with

$$|\partial_t^b \partial_x^{\alpha_0} \eta| \leqslant \frac{C(|\alpha_0|, b)}{\rho^{|\alpha_0|+2b}}\,, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}\,,\ b \in \mathbb{N}_0\,,\ \alpha_0 \in \mathbb{N}_0^3.$$

Substituting $\phi(x,t) = \psi(x,t)\eta(x,t)$ in the energy inequality (2.1), we get for any $s \in [-r^2, 0]$,

$$\begin{aligned}
&\int_{B_r} |u|^2\psi|_s + 2\iint_{Q_r} \left(|\nabla u|^2 + |e(u)|^p\right)\psi \\
&\quad \leqslant \iint_{Q_\rho} |u|^2(\phi_t + \Delta\phi) + \iint_{Q_\rho} |u|^2 u\cdot\nabla\phi \\
&\qquad + 2\iint_{Q_\rho} \pi u\cdot\nabla\phi - 2\iint_{Q_\rho} |e(u)|^{p-2}e(u) : (u\otimes\nabla\phi) \\
&\quad := I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.8}$$

Estimates of I_1, I_2 and I_3 are as follows, see (2.14)–(2.16) in [9],

$$\begin{aligned}
&I_1 \leqslant C\kappa^2 A(\rho), \\
&I_2 \leqslant C\kappa^{-2}A^{\frac{1}{2}}(\rho)B^{\frac{1}{2}}(\rho)G^{\frac{1}{3}}(\rho), \\
&I_3 \leqslant C\kappa^{-2}D^{\frac{2}{3}}(\rho)G^{\frac{1}{3}}(\rho).
\end{aligned} \tag{2.9}$$

For I_4, by Hölder's inequality,

$$\begin{aligned}
I_4 &\leqslant \frac{C}{r^2}\rho^{\frac{5(7-3p)}{6}}\|\nabla u\|^{p-1}_{L^2(Q_\rho)}\|u\|_{L^3(Q_\rho)} \leqslant \frac{C}{r^2}\rho^{6-2p}B^{\frac{p-1}{2}}(\rho)G^{\frac{1}{3}}(\rho) \\
&= C\kappa^{-2}\rho^{4-2p}B^{\frac{p-1}{2}}(\rho)G^{\frac{1}{3}}(\rho),
\end{aligned} \tag{2.10}$$

where we have used $|\nabla\phi| \leqslant |\eta||\nabla\psi| + |\nabla\eta|\psi \leqslant \frac{1}{r^2}$ on Q_ρ.

From (2.7), we have

$$\sup_{s\in(-r^2,0)}\int_{B_r} |u|^2\psi|_s \geqslant C^{-1}A(r) \tag{2.11}$$

and

$$2\iint_{Q_r} |\nabla u|^2\psi \geqslant C^{-1}B(r). \tag{2.12}$$

By equations (2.8), (2.11), and (2.12) one shows that $A(r) + B(r) \leqslant C(I_1 + I_2 + I_3 + I_4)$.

By appealing to the estimates (2.9) and (2.10), one has

$$\begin{aligned}
A(r) + B(r) &\leqslant C\kappa^2 A(\rho) + C\kappa^{-2}A^{\frac{1}{2}}(\rho)B^{\frac{1}{2}}(\rho)G^{\frac{1}{3}}(\rho) \\
&\quad + C\kappa^{-2}D^{\frac{2}{3}}(\rho)G^{\frac{1}{3}}(\rho) + C\kappa^{-2}\rho^{4-2p}B^{\frac{p-1}{2}}(\rho)G^{\frac{1}{3}}(\rho),
\end{aligned}$$

which implies that

$$\begin{aligned}
A^{\frac{1}{2}}(r) + B^{\frac{1}{2}}(r) &\leqslant C\kappa A^{\frac{1}{2}}(\rho) + C\kappa^{-1}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho)G^{\frac{1}{6}}(\rho) \\
&\quad + C\kappa^{-1}D^{\frac{1}{3}}(\rho)G^{\frac{1}{6}}(\rho) + C\kappa^{-1}\rho^{2-p}B^{\frac{p-1}{4}}(\rho)G^{\frac{1}{6}}(\rho).
\end{aligned}$$

It is easy to show that

$$C\kappa^{-1}D^{\frac{1}{3}}(\rho)G^{\frac{1}{6}}(\rho) \leqslant C\kappa^{-3}D^{\frac{2}{3}}(\rho) + C\kappa G^{\frac{1}{3}}(\rho),$$

and from Gagliardo–Nirenberg inequality,

$$G^{\frac{1}{3}}(\rho) \leqslant CA^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho) + CA^{\frac{1}{2}}(\rho).$$

Hence, we have

$$\begin{aligned} A^{\frac{1}{2}}(r) + B^{\frac{1}{2}}(r) \leqslant\ & C\kappa A^{\frac{1}{2}}(\rho) + C\kappa^{-1}A^{\frac{3}{8}}(\rho)B^{\frac{3}{8}}(\rho) + C\kappa^{-1}A^{\frac{1}{2}}(\rho)B^{\frac{1}{4}}(\rho) \\ & + C\kappa^{-3}D^{\frac{2}{3}}(\rho) + C\kappa A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho) + C\kappa^{-1}\rho^{2-p}B^{\frac{p-1}{4}}(\rho) \\ & \times \left(A^{\frac{1}{8}}(\rho)B^{\frac{1}{8}}(\rho) + A^{\frac{1}{4}}(\rho)\right). \end{aligned} \tag{2.13}$$

Next, we focus on the pressure estimates. Using the equation $\Delta\pi = \partial_{ij}U_{ij} + \partial_{ij}\left(|e(u)|^{p-2}e_{ij}(u)\right)$, where $U_{ij} = -u_i\left(u_j - |B_\rho|^{-1}\int_{B_\rho} u_j\right)$, we get

$$\begin{aligned} \Delta(\tilde\eta\pi) =\ & \partial_{ij}(\tilde\eta U_{ij}) + (\partial_{ij}\tilde\eta)U_{ij} - \partial_j(U_{ij}\partial_i\tilde\eta) - \partial_i(U_{ij}\partial_j\tilde\eta) \\ & - \pi\Delta\tilde\eta + 2\partial_j((\partial_j\tilde\eta)\pi) + \partial_{ij}\left(\tilde\eta|e(u)|^{p-2}e_{ij}(u)\right) \\ & + (\partial_{ij}\tilde\eta)|e(u)|^{p-2}e_{ij}(u) - \partial_i((\partial_j\tilde\eta)|e(u)|^{p-2}e_{ij}(u)) \\ & - \partial_j((\partial_i\tilde\eta)|e(u)|^{p-2}e_{ij}(u)), \end{aligned}$$

where the function $\tilde\eta \in C_0^\infty(\mathbb{R}^3)$ verifies the assumptions $\tilde\eta = 1$ in a neighbourhood of $\bar B_{3\rho/5}$, $\tilde\eta = 0$ in a neighbourhood of $B^c_{4\rho/5}$, and

$$|\partial^{\alpha_0}\tilde\eta(x)| \leqslant \frac{C(|\alpha_0|)}{\rho^{|\alpha_0|}}, \quad x \in \mathbb{R}^3,\ \alpha_0 \in \mathbb{N}_0^3.$$

Further, we denote by N the kernel of Δ^{-1}. One has

$$\begin{aligned} \tilde\eta\pi =\ & -R_iR_j(\tilde\eta U_{ij}) + N*((\partial_{ij}\tilde\eta)U_{ij}) - \partial_jN*(U_{ij}\partial_i\tilde\eta) - \partial_iN*(U_{ij}\partial_j\tilde\eta) \\ & - N*(\pi\Delta\tilde\eta) + 2\partial_jN*((\partial_j\tilde\eta)\pi) + R_iR_j(\tilde\eta|e(u)|^{p-2}e_{ij}(u)) \\ & + N*((\partial_{ij}\tilde\eta)|e(u)|^{p-2}e_{ij}(u)) - \partial_jN*(|e(u)|^{p-2}e_{ij}(u)\partial_i\tilde\eta) \\ & - \partial_iN*(|e(u)|^{p-2}e_{ij}(u)\partial_j\tilde\eta) \\ =\ & \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 + \pi_9 + \pi_{10}, \end{aligned} \tag{2.14}$$

where R_i is the ith Riesz transform. Estimates of π_1–π_6 are as follows, see [9],

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_1\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{-\frac{1}{2}}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho),$$

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_2\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{-\frac{1}{2}}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho),$$

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_3\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{-\frac{1}{2}}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho),$$

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_4\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{-\frac{1}{2}}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho),$$

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_5\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{\frac{1}{3}}D^{\frac{1}{3}}(\rho),$$

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_6\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\kappa^{\frac{1}{3}}D^{\frac{1}{3}}(\rho). \tag{2.15}$$

For π_7, by the Calderón–Zygmund theorem, one has

$$\|\pi_7\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_r)} \leqslant \int_{B_\rho} |\nabla u|^{\frac{3(p-1)}{2}}\,\mathrm{d}x \leqslant \rho^{\frac{3(7-3p)}{4}}\left(\int_{B_\rho}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{3(p-1)}{4}},$$

which yields

$$\|\pi_7\|_{L^{\frac{3}{2}}(Q_r)} \leqslant \rho^{\frac{5(7-3p)}{6}}\left(\iint_{Q_\rho}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{p-1}{2}} \leqslant \rho^{\frac{16-6p}{3}}B^{\frac{p-1}{2}}(\rho),$$

so we have

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_7\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant \rho^{2-p}\left(\frac{\rho}{r}\right)^{\frac{2}{3}}B^{\frac{p-1}{4}}(\rho). \tag{2.16}$$

For π_8, since $|N(x)| \leqslant \frac{C}{|x|}$, one has

$$|\pi_8(x)| \leqslant C\left|\int_{B_\rho}\frac{1}{|x-y|}\left|\left(|e(u)|^{p-2}e_{ij}(u)\partial_{ij}\tilde{\eta}\right)(y)\right|\mathrm{d}y\right|,$$

for each $x \in B_r$. By noting that $\partial_{ij}\tilde{\eta} = 0$ on $\bar{B}_{3\rho/5}$ and on $B^c_{4\rho/5}$, and that $|x-y| \geqslant \frac{4\rho}{5} - r \geqslant \frac{3\rho}{10}$ if $x \in B_r$ and $y \in B^c_{4\rho/5}$, it follows that

$$\begin{aligned}\|\pi_8\|_{L^\infty(B_r)} &\leqslant \frac{C}{\rho}\||e(u)|^{p-2}e_{ij}(u)\partial_{ij}\tilde{\eta}\|_{L^1(B_\rho)} \leqslant \frac{C}{\rho^3}\||e(u)|^{p-1}\|_{L^1(B_\rho)}\\ &\leqslant \rho^{\frac{3-3p}{2}}\|\nabla u\|^{p-1}_{L^2(B_\rho)}\end{aligned}$$

for all $t \in (-r^2, 0)$. Hence

$$\|\pi_8\|_{L^{\frac{3}{2}}(B_r)} \leqslant r^2\|\pi_8\|_{L^\infty(B_r)} \leqslant C\rho^{\frac{7-3p}{2}}\|\nabla u\|^{p-1}_{L^2(B_\rho)}.$$

Thus it follows that

$$\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_8\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} \leqslant C\rho^{2-p}\left(\frac{\rho}{r}\right)^{\frac{2}{3}}B^{\frac{p-1}{4}}(\rho). \tag{2.17}$$

Similarly, we have

$$\begin{aligned}\left(\frac{1}{r^{\frac{4}{3}}}\|\pi_9\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} &\leqslant C\rho^{2-p}\left(\frac{\rho}{r}\right)^{\frac{2}{3}}B^{\frac{p-1}{4}}(\rho),\\ \left(\frac{1}{r^{\frac{4}{3}}}\|\pi_{10}\|_{L^{\frac{3}{2}}(Q_r)}\right)^{\frac{1}{2}} &\leqslant C\rho^{2-p}\left(\frac{\rho}{r}\right)^{\frac{2}{3}}B^{\frac{p-1}{4}}(\rho).\end{aligned} \tag{2.18}$$

From (2.14), and by appealing to (2.15)–(2.18), one gets

$$D^{\frac{1}{3}}(r) \leqslant C\kappa^{-\frac{1}{2}}A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho) + C\kappa^{\frac{1}{3}}D^{\frac{1}{3}}(\rho) + C\rho^{2-p}\kappa^{\frac{2}{3}}B^{\frac{p-1}{4}}(\rho), \tag{2.19}$$

for $0 < r \leqslant \frac{3\rho}{5}$.

Now (2.19) and (2.13) imply that

$$\begin{aligned}\theta(r) &\leqslant C\kappa A^{\frac{1}{2}}(\rho) + C\kappa^{-1}A^{\frac{3}{8}}(\rho)B^{\frac{3}{8}}(\rho) + C\kappa^{-1}A^{\frac{1}{2}}(\rho)B^{\frac{1}{4}}(\rho) + C\kappa A^{\frac{1}{4}}(\rho)B^{\frac{1}{4}}(\rho)\\
&\quad + C\kappa^{-1}\rho^{2-p}A^{\frac{1}{8}}(\rho)B^{\frac{p-1}{4}+\frac{1}{8}}(\rho) + C\kappa^{-1}\rho^{2-p}A^{\frac{1}{4}}(\rho)B^{\frac{p-1}{4}}(\rho)\\
&\quad + C\kappa^{-5}A^{\frac{1}{2}}(\rho)B^{\frac{1}{2}}(\rho) + C\kappa^{-\frac{10}{3}}D^{\frac{2}{3}}(\rho) + C\rho^{4-2p}\kappa^{-\frac{8}{3}}B^{\frac{p-1}{2}}(\rho)\\
&\leqslant C\kappa\theta(\rho) + C\kappa^{-1}B^{\frac{1}{4}}(\rho)\theta(\rho) + C\kappa^{-1}B^{\frac{1}{4}}(\rho)\theta(\rho) + C\kappa\theta(\rho)\\
&\quad + C\kappa^{-1}\rho^{2-p}\theta^{\frac{p}{2}}(\rho) + C\kappa^{-5}B^{\frac{1}{2}}(\rho)\theta(\rho) + C\kappa^{\frac{2}{3}}\theta(\rho) + C\rho^{4-2p}\kappa^{-\frac{8}{3}}\theta^{p-1}(\rho),\end{aligned} \tag{2.20}$$

which gives (2.5). □

By the same argument of lemma 2 in [9], we can prove the following lemma. Since the proof is essentially same, we omit its proof.

Lemma 2.4. *Let $0 < r < R$ and $t_1 < t_2$ be such that $\bar{B}_R \times [t_1, t_2] \subset D$. Then we have*

$$\lim_{\delta\to 0+}\sup_{t\in[t_1,\,t_2+\delta]}\int_{B_r}|u(x,t)|^2\,\mathrm{d}x \leqslant \sup_{t\in[t_1,\,t_2]}\int_{B_R}|u(x,t)|^2\,\mathrm{d}x. \tag{2.21}$$

Next, we prove the following lemma.

Lemma 2.5. *There exists a sufficiently small universal constant $\epsilon_0 > 0$ with the following property. If*

$$\lim\sup_{r\to 0+} B^{\frac{1}{2}}_{(x_0,t_0)}(r) < \epsilon_0,$$

then for every $\delta \in (0, \frac{2}{3})$ there exist $r_2, r_3 > 0$ and $\bar{M} > 0$ such that

$$\max\left\{A^{\frac{1}{2}}_{(x,t)}(r), B^{\frac{1}{2}}_{(x,t)}(r), G^{\frac{2}{3}}_{(x,t)}(r)\right\} \leqslant \bar{M}r^{\delta}$$

for $(x,t) \in \mathcal{B}_{(x_0,r_0)}(r_2) = \{(x,t) : |x-x_0|^2 + |t-t_0|^2 < r_2^2\}$ and $r \in (0, r_3)$.

Proof. Recall that $\kappa = \frac{r}{\rho}$. Without loss of generality, we assume $(x_0, t_0) = (0, 0)$. Let $\tilde{\theta}(r) = \frac{\theta_{(0,0)}(r)}{r^{\delta}}$, then by lemma 2.3, we have

$$\begin{aligned}\tilde{\theta}(r) &\leqslant C\kappa^{\frac{2}{3}-\delta}\tilde{\theta}(\rho) + C\kappa^{-5-\delta}B^{\frac{1}{2}}(\rho)\tilde{\theta}(\rho) + C\kappa^{-1-\delta}\rho^{(2-\delta)(1-\frac{p}{2})}\tilde{\theta}^{\frac{p}{2}}(\rho)\\
&\quad + C\rho^{(2-\delta)(2-p)}\kappa^{-\frac{8}{3}-\delta}\tilde{\theta}^{p-1}(\rho),\end{aligned} \tag{2.22}$$

which implies that

$$\begin{aligned}\tilde{\theta}(r) &\leqslant C\kappa^{\frac{2}{3}-\delta}\tilde{\theta}(\rho) + C\kappa^{-5-\delta}B^{\frac{1}{2}}(\rho)\tilde{\theta}(\rho) + \frac{1}{6}\tilde{\theta}(\rho)\\
&\quad + C\rho^{2-\delta}\left(\kappa^{-\frac{1}{2-p}(\frac{8}{3}+\delta)} + \kappa^{-\frac{2(1+\delta)}{2-p}}\right).\end{aligned} \tag{2.23}$$

Similarly, from (2.6), we have

$$\tilde{\theta}(r) \leqslant C_0 \kappa^{\frac{2}{3}-\delta}\tilde{\theta}(\rho) + C\kappa^{-5-\delta}\rho^{\delta}\tilde{\theta}^2(\rho) + \frac{1}{6}\tilde{\theta}(\rho) + C\rho^{2-\delta}\left(\kappa^{-\frac{1}{2-p}(\frac{8}{3}+\delta)} + \kappa^{-\frac{2(1+\delta)}{2-p}}\right). \tag{2.24}$$

Having obtained the estimates (2.23) and (2.24), by lemma 2.4, by the smallness assumption (2.4), and by following the inductive argument in the proof of lemma 3 in [9], we can prove the lemma. For the sake of completeness, we give some details.

Let $\kappa = \min\{\frac{1}{2}, (6C_0)^{\delta-\frac{2}{3}}\}$. Then we have $C_0\kappa^{\frac{2}{3}-\delta} \leqslant \frac{1}{6}$ and $r \leqslant \frac{\rho}{2}$. Next, by choosing $\epsilon_0 \leqslant \frac{\kappa^{5+\delta}}{6C}$, one has $\frac{C\epsilon_0}{\kappa^{5+\delta}} \leqslant \frac{1}{6}$. From assumption (2.4), there exists a sufficient small $r_4 > 0$, satisfying $Q_{r_4} \subset D$, and such that

$$B^{\frac{1}{2}}(r) \leqslant \epsilon_0, \quad 0 < r < r_4,$$

and $\max\{2Cr_4^{2-\delta}\kappa^{-\frac{1}{2-p}(\frac{8}{3}+\delta)}, 2Cr_4^{2-\delta}\kappa^{-\frac{2(1+\delta)}{2-p}}, Cr_4^{\delta}\kappa^{-(5+\delta)}\} \leqslant \frac{1}{8}$. Now set

$$\rho = R_n =: \kappa^n r_4, \quad r = R_{n+1} =: \kappa^{n+1} r_4.$$

Note that $\kappa = \frac{r}{\rho}$. Define $\tilde{\theta}_n =: \tilde{\theta}(R_n)$, $n = 0, 1, 2, \ldots$. Note that $\tilde{\theta}_0 = \tilde{\theta}(r_4)$. Then by (2.23) it follows:

$$\begin{aligned}\tilde{\theta}_{n+1} = \tilde{\theta}(R^{n+1}) &\leqslant C_0\,\kappa^{\frac{2}{3}-\delta}\tilde{\theta}_n + C\kappa^{-5-\delta}B^{\frac{1}{2}}(R_n)\tilde{\theta}_n + \frac{1}{6}\tilde{\theta}_n \\ &\quad + C(\kappa^n r_4)^{2-\delta}\kappa^{-\frac{2}{2-p}(\frac{8}{3}+\delta)} + C(\kappa^n r_4)^{2-\delta}\kappa^{-\frac{2(1+\delta)}{2-p}} \\ &\equiv A_1 + A_2 + A_3 + A_4 + A_5.\end{aligned} \tag{2.25}$$

Furthermore,

$$\begin{aligned}&A_1 \leqslant \frac{1}{6}\tilde{\theta}_n,\\ &A_2 \leqslant \epsilon_0\kappa^{-5-\delta}\tilde{\theta}_n \leqslant \frac{1}{6}\tilde{\theta}_n,\\ &A_3 \leqslant \frac{1}{6}\tilde{\theta}_n,\\ &A_4 \leqslant C\kappa^{(2-\delta)n}\cdot\kappa^{-\frac{2}{2-p}(\frac{8}{3}+\delta)}r_4^{(2-\delta)} \leqslant \frac{1}{16}\kappa^{(2-\delta)n} \leqslant \frac{1}{16},\\ &A_5 \leqslant \frac{1}{16}\kappa^{(2-\delta)n} \leqslant \frac{1}{16}.\end{aligned}$$

In estimating A_4 we took into account that $\kappa^{(2-\delta)n} \leqslant 1$, since $\delta < \frac{2}{3}$.

The above estimates show that

$$\tilde{\theta}_{n+1} \leqslant \left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)\tilde{\theta}_n + \frac{2}{16} \leqslant \frac{1}{2}\tilde{\theta}_n + \frac{1}{8}, \quad n = 0, 1, 2, \ldots,$$

which gives

$$\tilde{\theta}_n \leqslant \frac{1}{2^n}\tilde{\theta}_0 + \frac{1}{8}\frac{1-(1/2)^n}{1/2} \leqslant \frac{1}{2^n}\tilde{\theta}_0 + \frac{1}{4}, \quad n = 1, 2, \ldots .$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $\tilde{\theta}_{n_0} \leqslant \frac{1}{3}$, i.e. $\tilde{\theta}_{(0,0)}(\kappa^{n_0} r_4) \leqslant \frac{1}{3}$.

We have obtained $\tilde{\theta}_{(0,0)}(\kappa^{n_0} r_4) \leqslant \frac{1}{3}$, i.e.,

$$\begin{aligned}(\kappa^{n_0} r_4)^{-\delta}\theta_{(0,0)}(\kappa^{n_0} r_4) &= (\kappa^{n_0} r_4)^{-\delta}\,[A^{\frac{1}{2}}_{(0,0)}(\kappa^{n_0} r_4) + B^{\frac{1}{2}}_{(0,0)}(\kappa^{n_0} r_4) \\ &\quad + \kappa^{-4} D^{\frac{2}{3}}_{(0,0)}(\kappa^{n_0} r_4)] \leqslant \frac{1}{3}.\end{aligned}$$

Next, we will prove that, there exists $r_2 > 0$ and $r_3 \in (0, r_4)$ such that

$$\tilde{\theta}_{(x,t)}(\kappa^{n_0} r_3) \leqslant \frac{1}{2}, \quad \text{for any } (x,t) \in \mathcal{B}_{(0,0)}(r_2) = \{(x,t) : |x-0|^2 + |t-0|^2 < r_2^2\},$$

i.e., for any $(x,t) \in \mathcal{B}_{(0,0)}(r_2)$, we have

$$\begin{aligned}(\kappa^{n_0} r_3)^{-\delta}\theta_{(x,t)} &= (\kappa^{n_0} r_3)^{-\delta}\,[A^{\frac{1}{2}}_{(x,t)}(\kappa^{n_0} r_3) + B^{\frac{1}{2}}_{(x,t)}(\kappa^{n_0} r_3) \\ &\quad + \kappa^{-4} D^{\frac{2}{3}}_{(x,t)}(\kappa^{n_0} r_3)] \leqslant \frac{1}{2}.\end{aligned}$$

Next, for convenience, we set $\tilde{r}_4 = \kappa^{n_0} r_4$ and $\tilde{r}_3 = \kappa^{n_0} r_3$. First, for any $\tilde{r}_3 \in (0\,, \tilde{r}_4)$, choose r_2 such that $0 < r_2 < \tilde{r}_4^2 - \tilde{r}_3^2$. Then for any $(x,t) \in \mathcal{B}_{(0,0)}(r_2)$, we have $t - \tilde{r}_3^2 \geqslant 0 - \tilde{r}_4^2$, hence

$$A_{(x,t)}(\tilde{r}_3) = \sup_{s \in [t - r_3^2\,,\, t]} \frac{1}{\tilde{r}_3} \int_{B_{\tilde{r}_3}(x)} |u(y,s)|^2\,\mathrm{d}y \leqslant \frac{1}{\tilde{r}_3} \sup_{s \in [-\tilde{r}_4^2\,,\, t]} \int_{B_{\tilde{r}_3}(x)} |u(y,s)|^2\,\mathrm{d}y.$$

From lemma 2.4, if $B_{r_3}(x) \subset B_{r_4}(0) = B_{r_4}$, then

$$\lim_{t \to 0+} \sup_{s \in [-\tilde{r}_4^2\,,\, t]} \int_{B_{\tilde{r}_3}(x)} |u(y,s)|^2\,\mathrm{d}y \leqslant \sup_{s \in [-\tilde{r}_4^2\,,\, 0]} \int_{B_{\tilde{r}_4}} |u(y,s)|^2\,\mathrm{d}y,$$

which implies that for any $\epsilon > 0$, there exits $\delta_\epsilon > 0$, such that for any $|t - 0| \leqslant \delta_\epsilon$, we have

$$\sup_{s \in [-\tilde{r}_4^2\,,\, t]} \int_{B_{\tilde{r}_3}(x)} |u(y,s)|^2\,\mathrm{d}y \leqslant \sup_{s \in [-\tilde{r}_4^2\,,\, 0]} \int_{B_{\tilde{r}_4}} |u(y,s)|^2\,\mathrm{d}y + \epsilon.$$

Now choose r_2 such that $|t - 0| \leqslant r_2 \leqslant \delta_\epsilon$ and $r_2 < \tilde{r}_4 - \tilde{r}_3$ such that $B_{\tilde{r}_3}(x) \subset B_{\tilde{r}_4}(0) = B_{\tilde{r}_4}$. Then

$$\begin{aligned}A_{(x,t)}(\tilde{r}_3) &\leqslant \frac{1}{\tilde{r}_3} \sup_{s \in [-\tilde{r}_4^2\,,\, 0]} \int_{B_{\tilde{r}_4}} |u(y,s)|^2\,\mathrm{d}y + \frac{1}{\tilde{r}_3}\epsilon \\ &= \frac{\tilde{r}_4}{\tilde{r}_3} \sup_{s \in [-\tilde{r}_4^2\,,\, 0]} \frac{1}{\tilde{r}_4} \int_{B_{\tilde{r}_4}} |u(y,s)|^2\,\mathrm{d}y + \frac{1}{\tilde{r}_3}\epsilon = \frac{\tilde{r}_4}{\tilde{r}_3} A_{(0,0)}(\tilde{r}_4) + \frac{1}{\tilde{r}_3}\epsilon.\end{aligned}$$

Hence, we have

$$\tilde{r}_3^{-\delta} A^{\frac{1}{2}}_{(x,t)}(\tilde{r}_3) \leqslant \left(\frac{\tilde{r}_4}{\tilde{r}_3}\right)^{\frac{1}{2}-\delta} \tilde{r}_4^{-\delta} A^{\frac{1}{2}}_{(0,0)}(\tilde{r}_4) + \frac{1}{\tilde{r}_3^{\frac{1}{2}+\delta}} \epsilon^{\frac{1}{2}}. \tag{2.26}$$

On the other hand, for $B_{(x,t)}(\tilde{r}_3)$ and $D_{(x,t)}(\tilde{r}_3)$, we argue as follows. By the continuity of the integral, for any $\epsilon > 0$ there exist δ_ϵ such that, if $Q_{\tilde{r}_3}(x,t) \subset Q_{\tilde{r}_4}(0,0)$ and $|Q_{\tilde{r}_4}(0,0) \setminus Q_{\tilde{r}_3}(x,t)| \leqslant \delta_\epsilon$, then

$$B_{(x,t)}(\tilde{r}_3) = \tilde{r}_3^{-1} \iint_{Q_{\tilde{r}_3}(x,t)} |\nabla u|^2 \,\mathrm{d}y\,\mathrm{d}t \leqslant \tilde{r}_3^{-1} \left(\iint_{Q_{\tilde{r}_4}(0,0)} |\nabla u|^2 \,\mathrm{d}y\,\mathrm{d}t + \epsilon \right),$$

and

$$D_{(x,t)}(\tilde{r}_3) = \tilde{r}_3^{-2} \iint_{Q_{\tilde{r}_3}(x,t)} |\pi|^{\frac{3}{2}} \,\mathrm{d}y\,\mathrm{d}t \leqslant \tilde{r}_3^{-2} \left(\iint_{Q_{\tilde{r}_4}(0,0)} |\pi|^{\frac{3}{2}} \,\mathrm{d}y\,\mathrm{d}t + \epsilon \right).$$

If we choose $0 < \tilde{r}_2 < |\tilde{r}_4 - \tilde{r}_3|$, and $3(\tilde{r}_4^5 - \tilde{r}_3^5) \leqslant \delta_\epsilon$, then $Q_{\tilde{r}_3}(x,t) \subset Q_{\tilde{r}_4}(0,0)$ and $|Q_{\tilde{r}_4}(0,0) \setminus Q_{\tilde{r}_3}(x,t)| \leqslant \delta_\epsilon$. Hence

$$\tilde{r}_3^{-\delta} B^{\frac{1}{2}}_{(x,t)}(\tilde{r}_3) \leqslant \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{1}{2}-\delta} \tilde{r}_4^{-\delta} B^{\frac{1}{2}}_{(0,0)}(\tilde{r}_4) + \frac{1}{\tilde{r}_3^{\frac{1}{2}+\delta}} \epsilon^{\frac{1}{2}}, \tag{2.27}$$

and

$$\tilde{r}_3^{-\delta} \kappa^{-4} D^{\frac{2}{3}}_{(x,t)}(\tilde{r}_3) \leqslant \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{2}{3}-\delta} \tilde{r}_4^{-\delta} \kappa^{-4} D^{\frac{2}{3}}_{(0,0)}(r_4) + \frac{1}{\tilde{r}_3^{1+\delta}} \kappa^{-4} \epsilon^{\frac{1}{2}}. \tag{2.28}$$

From the above analysis, choose r_2 and $\tilde{r}_3 \in (0, \tilde{r}_4)$ such that

$$r_2 \leqslant \min\{\tilde{r}_4^2 - \tilde{r}_3^2, \tilde{r}_4 - \tilde{r}_3, \delta_\epsilon\}, \qquad 3(\tilde{r}_4^5 - \tilde{r}_3^5) \leqslant \delta_\epsilon.$$

Then, collecting (2.26)–(2.28), we have

$$\tilde{\theta}_{(x,t)}(\kappa^{n_0} r_3) \leqslant \max \left\{ \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{1}{2}-\delta}, \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{2}{3}-\delta} \right\} \tilde{\theta}_{(0,0)}(\kappa^{n_0} r_4) + \frac{2}{\tilde{r}_3^{\frac{1}{2}+\delta}} \epsilon^{\frac{1}{2}} + \frac{1}{\tilde{r}_3^{1+\delta}} \kappa^{-4} \epsilon^{\frac{1}{2}}.$$

Now, let ϵ and δ_ϵ be sufficiently small, such that

$$\max \left\{ \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{1}{2}-\delta}, \left(\frac{\tilde{r}_4}{\tilde{r}_3} \right)^{\frac{2}{3}-\delta} \right\} \leqslant \frac{10}{9}, \qquad \frac{2}{\tilde{r}_3^{\frac{1}{2}+\delta}} \epsilon^{\frac{1}{2}} + \frac{1}{\tilde{r}_3^{1+\delta}} \kappa^{-4} \epsilon^{\frac{1}{2}} \leqslant \frac{1}{10}.$$

Then for any $(x,t) \in \mathcal{B}_{(0,0)}(r_2) = \{(x,t) : |x-0|^2 + |t-0|^2 < r_2^2\}$, we have

$$\tilde{\theta}_{(x,t)}(\kappa^{n_0} r_3) \leqslant \frac{10}{9} \tilde{\theta}_{(0,0)}(\kappa^{n_0} r_4) + \frac{1}{10} \leqslant \frac{10}{27} + \frac{1}{10} < \frac{1}{2}.$$

From (2.24), by setting $r = \kappa^{n+1} r_3$ and $\rho = \kappa^n r_3$,

$$\begin{aligned} \tilde{\theta}_{(x,t)}(\kappa^{n+1} r_3) &\leqslant C_0 \kappa^{\frac{2}{3}-\delta} \tilde{\theta}_{(x,t)}(\kappa^n r_3) + C\kappa^{-5-\delta} (\kappa^n r_3)^\delta \tilde{\theta}^2_{(x,t)}(\kappa^n r_3) \\ &\quad + \frac{1}{6} \tilde{\theta}_{(x,t)}(\kappa^n r_3) + C(\kappa^n r_3)^{2-\delta} \kappa^{-\frac{1}{2-p}(\frac{8}{3}+\delta)} \\ &\quad + C(\kappa^n r_3)^{2-\delta} \kappa^{-\frac{2(1+\delta)}{2-p}} \\ &\equiv \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + \tilde{A}_4 + \tilde{A}_5. \end{aligned}$$

Similar to (2.25), we have

$$\begin{aligned}
\tilde{A}_1 &\leqslant \frac{1}{6}\tilde{\theta}_{(x,t)}(\kappa^n r_3),\\
\tilde{A}_2 &\leqslant \frac{1}{8}\kappa^{n\delta}\tilde{\theta}_{(x,t)}(\kappa^n r_3) \leqslant \frac{1}{8}\frac{1}{2^{n\delta}}\tilde{\theta}_{(x,t)}(\kappa^n r_3) \leqslant \frac{1}{8}\tilde{\theta}_{(x,t)}(\kappa^n r_3),\\
\tilde{A}_3 &\leqslant \frac{1}{6}\tilde{\theta}_{(x,t)}(\kappa^n r_3),\\
\tilde{A}_4 &\leqslant C\kappa^{(2-\delta)n}\cdot \kappa^{-\frac{2}{2-p}(\frac{8}{3}+\delta)} r_4^{(2-\delta)} \leqslant \frac{1}{16}\kappa^{(2-\delta)n} \leqslant \frac{1}{16},\\
\tilde{A}_5 &\leqslant \frac{1}{16}\kappa^{(2-\delta)n} \leqslant \frac{1}{16}.
\end{aligned}$$

Hence,

$$\tilde{\theta}_{(x,t)}(\kappa^{n+1} r_3) \leqslant \frac{1}{2}\tilde{\theta}_{(x,t)}(\kappa^n r_3) + \frac{1}{8}\tilde{\theta}^2_{(x,t)}(\kappa^n r_3) + \frac{1}{8}, \quad n = n_0\,, n_0+1\,, \dots.$$

Since

$$\tilde{\theta}_{(x,t)}(\kappa^{n_0} r_3) \leqslant \frac{1}{2}\,, \quad (x,t) \in \mathcal{B}_{(0,0)}(r_2),$$

we have

$$\tilde{\theta}_{(x,t)}(\kappa^{n_0+1} r_3) \leqslant \frac{7}{16} < \frac{1}{2}. \tag{2.29}$$

By induction, we have

$$\tilde{\theta}_{(x,t)}(\kappa^{n+1} r_3) \leqslant \frac{1}{2}\,, \quad n = n_0\,, n_0+1\,, \dots,$$

for $(x,t) \in \mathcal{B}_{(0,0)}(r_2)$. Note that (see [9], (2.23))

$$\tilde{\theta}_{(x,t)}(\rho_1) \leqslant C\left(\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}+\delta} + \left(\frac{\rho_2}{\rho_1}\right)^{\frac{4}{3}+\delta}\right)\tilde{\theta}_{(x,t)}(\rho_2), \quad 0 < \rho_1 < \rho_2.$$

Hence

$$\tilde{\theta}_{(x,t)}(r) \leqslant C\,, \quad r \in (0\,, r_3)$$

for $(x,t) \in \mathcal{B}_{(0,0)}(r_2)$. Thus, we have proved the lemma. □

Lemma 2.6 (proposition 6 of [18]). *Let $\mathcal{V} \subset \mathbb{R}^3 \times \mathbb{R}$ be a bounded domain. Assume that*

(*a*) $\sup_{(x,t)\in\mathcal{V}} \sup_{\rho>0} \rho^{-\lambda} \iint_{\mathcal{V}\cap\mathcal{B}_\rho(x,t)} |g(y,s)|^q \,\mathrm{d}y\,\mathrm{d}s < \infty$ *and*
(*b*) $g \in L^m(\mathcal{V})$

for some $m \geqslant q > 1$, and $0 \leqslant \lambda < 5$. For $\alpha > 0$, define

$$h(x,t) = \iint_{\mathcal{V}} \frac{g(y,s)}{(|x-y| + \sqrt{t-s})^{5-\alpha}}\,\mathrm{d}y\,\mathrm{d}s.$$

Then for all $\tilde{m} \in (m\,, \infty)$ such that

$$\frac{1}{\tilde{m}} > \frac{1}{m}\left(1 - \frac{q\alpha}{5-\lambda}\right),$$

we have $h \in L^{\tilde{m}}(\mathcal{V})$.

Proof of theorem 2.2. From lemma 2.5 it follows that there exist $r_2, r_3 > 0$, and $\bar{M} > 0$, such that

$$\max\left\{A^{\frac{1}{2}}_{(x,t)}(r)\,, B^{\frac{1}{2}}_{(x,t)}(r)\,, D^{\frac{2}{3}}_{(x,t)}(r)\right\} \leqslant \bar{M}r^{\delta}, \tag{2.30}$$

for $(x,t) \in \mathcal{B}_{(x_0,t_0)}(r_2)$ and $r \in (0, r_3)$. Without loss of generality, we assume that $r_2 = r_3$. Note that

$$G^{\frac{1}{3}}_{(x,t)}(r) \leqslant CA^{\frac{1}{4}}_{(x,t)}(r)B^{\frac{1}{4}}_{(x,t)}(r) \leqslant C\bar{M}r^{\delta} \tag{2.31}$$

for $(x,t) \in \mathcal{B}_{(x_0,t_0)}(r_2)$. Now let be

$$\begin{aligned} v_k(x,t) = &\int_{-\infty}^{t}\int \partial_j G(x-y,t-s)\eta(y,s)u_j(y,s)u_k(y,s)\mathrm{d}y\,\mathrm{d}s \\ &+ \int_{-\infty}^{t}\int \partial_k G(x-y,t-s)\eta(y,s)\pi(y,s)\mathrm{d}y\,\mathrm{d}s \\ &+ \int_{-\infty}^{t}\int \partial_j G(x-y,t-s)\eta(y,s)|e(u)|^{p-2}e_{jk}(u)\mathrm{d}y\,\mathrm{d}s, \end{aligned}$$

where $\eta \in C_0^{\infty}(\mathbb{R}^3)$ is a function identically to 1 on a neighbourhood of $\bar{\mathcal{B}}_{(x_0,t_0)}(3r_2/4)$ and identically to 0 on a neighbourhood of $\mathcal{B}^c_{(x_0,t_0)}(9r_2/10)$. Clearly, $u - v \in C^{\infty}\left(\mathcal{B}_{x_0,t_0}(3r_2/4)\right)$. Note that $|\nabla G(x,t)| \leqslant C(|x| + \sqrt{t})^{-4}$ for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. By (2.31), we have

$$\sup_{(x,t)\in\mathcal{V}}\sup_{r>0}\frac{1}{r^{2+3\delta}}\iint_{\mathcal{B}_r(x,t)\cap\mathcal{V}}|u|^3 < \infty,$$

where $\mathcal{V} = \mathcal{B}_{(x_0,t_0)}(r_2)$. By lemma 2.6, note that $u \in L^{\frac{10}{3}}(D)$, by letting $q = \frac{3}{2}$, $\alpha = 1$, $m = \frac{5}{3}$, $\lambda = 2 + 3\delta$ with $\delta > \frac{1}{4}$, we get $v^{(1)} \in L^5(\mathcal{V})$. Similarly, we have $v^{(2)} \in L^5(\mathcal{V})$. Concerning $v^{(3)}$, by appealing to (2.30), we show that

$$\sup_{(x,t)\in\mathcal{V}}\sup_{r>0}\frac{1}{r^{1+2\delta}}\iint_{\mathcal{B}_r(x,t)\cap\mathcal{V}}|\nabla u|^2 < \infty.$$

By lemma 2.6, note that $\nabla u \in L^{p-1}(D)$, by letting $q = \frac{2}{p-1}$, $\alpha = 1$, $m = \frac{2}{p-1}$, and $\lambda = 1 + 2\delta$, we get $v^{(3)} \in L^{\tilde{m}}(\mathcal{V})$, where

$$\frac{1}{\tilde{m}} > \frac{1}{m}\left(1 - \frac{\frac{2}{p-1}}{4 - 2\delta}\right).$$

By choosing $\delta > 2 - \frac{5}{5p-7}$, we get $v^{(3)} \in L^5(\mathcal{V})$. Hence $u \in L^5(\mathcal{V})$. □

3. Final remarks

We start by noting that, as long as the proofs depend heavily on CKN's argument and the term $\mathrm{div}(|e(u)|^{p-2}e(u))$ is regarded as an external force, the strict positiveness of the parameter μ_0 looks essential (concerning the singular case $\mu_0 = 0$ we refer the reader to [3], where the local in time existence of strong solutions for $\frac{7}{5} < p \leqslant 2$ was established).

In [19] the author appeals to the following property: for suitably small ϵ_0 there exists a constant C_0 such that $A(r) + D(r) \leqslant \epsilon_0$ implies $|u(x,t)| \leqslant \frac{C_0}{r}$.

Unfortunately, this property seems not applicable to non-Newtonian fluids since one has not the scaling invariance property. The lack of this property could be a high obstacle to prove sharp results in non-Newtonian cases.

In our proof of lemma 2.5 the positivity of the power of ρ in (2.22) is crucial. This leads to assumption $p < 2$. We are not able to overcome this condition. This point seems in some contrast with calculations in reference [5], p 293 up to equation (4.14).

In any case, when $p \geqslant \frac{11}{5}$, the Hausdorff dimension of the set of singular points of the suitable weak solutions should be zero since in this case global strong solutions exists. We expect that one can verify this fact from the point of view of partial regularity.

Another interesting problem is the regularity of strong solutions. As still noted in remark 2.2, we do not know if the strong solutions of system (1.1) are necessarily smooth. The results on this subject are not too many. It is worth noting that, in dimension two, it was showed in reference [8] that strong solution are $C^{1,\alpha}$ regular. But extension to three dimensions has not been made so far. Recently, an interesting result was obtained by Kang *et al* [7]. They considered existence of regular solutions for non-Newtonian fluids in dimension three, and proved local existence of unique regular solutions, and global existence for small initial data.

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