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Onsager's Conjecture for the Incompressible Euler Equations in the Hölog Spaces $C^{0,\alpha}_{\lambda}(\bar\Omega)$

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Abstract. In this note we extend a 2018 result of Bardos and Titi (Arch Ration Mech Anal 228(1):197–207, 2018) to a new class of functional spaces $C^{0,\alpha}_{\lambda}(\bar\Omega)$. It is shown that weak solutions u satisfy the energy equality provided that $u \in L^3((0,T); C^{0,\alpha}_{\lambda}(\bar\Omega))$ with $\alpha \geq \frac{1}{3}$ and $\lambda > 0$. The result is new for $\alpha = \frac{1}{3}$. Actually, a quite stronger result holds. For convenience we start by a similar extension of a 1994 result of Constantin and Titi (Commun Math Phys 165:207–209, 1994), in the space periodic case. The proofs follow step by step those of the above authors. For the readers convenience, and completeness, proofs are presented in a quite complete form.

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1. Introduction

In this note, we are concerned with the Onsager's conjecture of incompressible Euler equations in a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0,T), \\ u(x,t) \cdot n(x) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

where T is a positive constant, and $n(x)$ is the outward unit normal vector field to the boundary $\partial\Omega$.

We say that $(u(x,t), p(x,t))$ is a weak solution of (1.1) in $\Omega \times (0,T)$, if $u \in L^\infty(0,T; L^2(\Omega))$, $p \in L^1_{loc}(\Omega \times (0,T))$, $\nabla \cdot u = 0$ in $\Omega \times (0,T)$, $u \cdot\ n = 0$ on $\partial\Omega \times (0,T)$ and, moreover,

$$\langle u, \partial_t \psi \rangle_x + \langle u \otimes u : \nabla \psi \rangle_x + \langle p, \nabla \cdot \psi \rangle_x = 0, \quad \text{in } L^1(0,T), \tag{1.2}$$

for all vector field $\psi(x,t) \in \mathcal{D}(\Omega \times (0,T))$. We have used the notation $\langle \cdot, \cdot \rangle_x$ in [1], which stands for the distributional duality with respect to the spatial variable x.

Onsager's conjecture for solutions to the Euler equations may be stated as follows: Conservation of energy holds if the weak solution $u \in L^3((0,T); C^{0,\alpha}(\bar\Omega))$, with $\alpha > \frac{1}{3}$; Dissipative solutions $u \in L^3((0,T); C^{0,\alpha}(\bar\Omega))$ should exist for $\alpha < \frac{1}{3}$. See [12]. This conjecture has been intensively studied by many mathematicians for the last two decades. In the absence of a physical boundary (namely the case of whole space $\mathbb{R}^n$ or the case of periodic boundary conditions in the torus $\mathbb{T}^n$), Eyink in [8] proved that Onsager's conjecture holds if $\alpha > \frac{1}{2}$. Later, a complete proof was established by Constantin and Titi in [7], for $\alpha > \frac{1}{3}$, under slightly weaker regularity assumptions on the solution. In [6] Cheskidov, Constantin, Friedlander, and Shvydkoy proved energy equality in the space periodic case for solutions

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$u \in L^3([0,T]; B^{\frac{1}{3}}_{3,c(N)})$, where $B^{\frac{1}{3}}_{3,c(N)}$ is a Besov type space for which $B^{\frac{1}{3}}_{3,p} \subset B^{\frac{1}{3}}_{3,c(N)} \subset B^{\frac{1}{3}}_{3,\infty}$, for $1 \le p < \infty$, see reference [6] for details. See also the end of this section.

Recently, Bardos and Titi [1] considered the Onsager's conjecture in bounded domains under the non-slip boundary condition. They proved energy conservation if $u \in L^3((0,T); C^{0,\alpha}(\bar{\Omega}))$, for $\alpha > \frac{1}{3}$. Later on, Bardos, Titi and Wiedemann [3] relax this assumption, requiring only interior Hölder regularity and continuity of the normal component of the energy flux near the boundary. See also [11]. The result obtained in [3] is particularly significant from the physical point of view. A very interesting extension of Onsager's conjecture to a class of conservation laws that possess generalized entropy is shown in by Bardos et al. in Ref. [2].

Concerning the second part of Onsager's conjecture, in a series of papers, Isett [9], Buckmaster et al. [5], see references therein, by using the convex integration machinery, proved the existence of dissipative energy weak solutions for any $\alpha < \frac{1}{3}$. Furthermore, Isett [10] constructed energy non-conserving solutions under the assumption

$$|u(x+y,t) - u(x,t)| \le C|y|^{\frac{1}{3} - B\sqrt{\frac{\log\log|y|^{-1}}{\log|y|^{-1}}}}$$

for some constants C and B and for all (x,t) and all $|y| \le 10^{-2}$.

In this note we will study Onsager's conjecture in a new class of functional spaces, Hölog spaces, which have been considered by the first author in [4]. To state our main result, we first introduce the definition of Hölog spaces.

Definition 1.1. For each $0 \le \alpha < 1$ and each $\lambda \in \mathbb{R}$, set

$$C^{0,\alpha}_{\lambda}(\bar{\Omega}) = \{f \in C(\bar{\Omega}) : [f]_{C^{0,\alpha}_{\lambda}(\bar{\Omega})} < \infty\},$$

where

$$[f]_{C^{0,\alpha}_{\lambda}(\bar{\Omega})} = \sup_{x,y\in\bar{\Omega}, 0<|x-y|<1} \frac{|f(x) - f(y)|}{\left(\log \frac{1}{|x-y|}\right)^{-\lambda} |x-y|^{\alpha}}. \tag{1.3}$$

A norm is introduced in $C^{0,\alpha}_{\lambda}(\bar{\Omega})$ by setting $\|f\|_{C^{0,\alpha}_{\lambda}(\bar{\Omega})} \equiv [f]_{C^{0,\alpha}_{\lambda}(\bar{\Omega})} + \|f\|_{C(\bar{\Omega})}$.

Now we can state our main theorem.

Theorem 1.2. *Assume that*

$$u \in L^3((0,T); C^{0,\alpha}_{\lambda}(\bar{\Omega})), \tag{1.4}$$

with $\alpha \ge \frac{1}{3}$ *and* $\lambda > 0$. *Then the weak solution of* (1.1) *satisfies the energy conservation:*

$$\|u(\cdot,t_2)\|_{L^2(\Omega)} = \|u(\cdot,t_1)\|_{L^2(\Omega)}, \quad \textit{for any } t_1, t_2 \in (0,T). \tag{1.5}$$

Clearly, for $\alpha > \frac{1}{3}$ the above results follow immediately from the relation $C^{0,\alpha}_{\lambda}(\bar{\Omega}) \subset C^{0,\alpha}(\bar{\Omega})$. The new results are obtained for $\alpha = \frac{1}{3}$.

As still remarked in the abstract, the proof of the above result is a step by step adaptation of that in Ref. [1]. So we are aware that the merit of the results goes in a greater part to the above authors. However the new results are significantly stronger then the previous ones, in particular in the form stated in the following theorem.

Theorem 1.3. *Theorem* 1.2 *still holds if one replace in* (1.3) *the function* $\left(\log \frac{1}{|x-y|}\right)^{-\lambda}$ *by* $\omega(|x-y|)$, *where* $\omega(s)$ *is a positive and non-decreasing function for* $s > 0$, *and* $\lim_{s\to 0} \omega(s) = \omega(0) = 0$.

The reason that led us to put in light the $C^{0,\frac{1}{3}}_{\lambda}(\bar{\Omega})$ case instead of the stronger case considered in Theorem 1.3 is due to the effort employed by us to try to prove the first case, before realizing that the way followed in Ref. [1] could be applied successfully.

Let's end this section by a comparison between the distinct results. Concerning Theorem 1.2, the gap between the set consisting of all Hölder spaces $C^{0,\alpha}(\bar\Omega)$, with $\alpha > \frac{1}{3}$, and a fixed Hölog space $C_\lambda^{0,\frac{1}{3}}(\bar\Omega)$ is wide. In fact, the union of all the above Hölder spaces is contained in the single space $C_{2\lambda}^{0,\frac{1}{3}}(\bar\Omega)$, which is away from $C_\lambda^{0,\frac{1}{3}}(\bar\Omega)$. Nevertheless, in comparison to the result stated in Theorem 1.3, also the spaces $C_\lambda^{0,\frac{1}{3}}(\bar\Omega)$ are far from $C^{0,\frac{1}{3}}(\bar\Omega)$. In fact, roughly speaking, we may say that there is few "free space" between the set of spaces considered in this last theorem, and $C^{0,\frac{1}{3}}(\bar\Omega)$. Recall also the sharp result, still referred above, obtained for the space periodic case in Ref. [6]. Concerning this point, let's consider the relation between $B^{\frac{1}{3}}_{3,c(\mathbb{N})}$ and Hölog spaces $C_\lambda^{0,\frac{1}{3}}(\bar\Omega)$. The Besov space $B^{\frac{1}{3}}_{3,\infty}$ can be characterized as follows, see Proposition 8′ in [14]:

$$B^{\frac{1}{3}}_{3,\infty} =: \left\{ f \in L^3 : \|f\|_3 + \sup_{|y|>0} \frac{\|f(x+y) + f(x-y) - 2f(x)\|_3}{|y|^{\frac{1}{3}}} < \infty \right\}.$$

Hence one has $C_\lambda^{0,\frac{1}{3}} \subset B^{\frac{1}{3}}_{3,\infty}$, for any $\lambda > 0$. From Shvydkoy [13], $c(\mathbb{N})$ stands to indicate

$$\frac{1}{|y|} \int_{\mathbb{T}^n} |f(x-y) - f(x)|^3 dx \to 0, \quad \text{as } |y| \to 0,$$

which implies that $C_\lambda^{0,\frac{1}{3}} \subset B^{\frac{1}{3}}_{3,c(\mathbb{N})}$. Hence, in the case of period domain, our $C_\lambda^{0,\frac{1}{3}}$ result is covered by that of Cheskidov, Constantin, Friedlander, and Shvydkoy's.

2. Theorem 1.2 for the Period Domain $\mathbb{T}^n$

Before proving Theorem 1.2 we consider a simpler situation, the period domain case. This helps us to understand the proof of the general bounded domain case. In this case, as in [7], taking in (1.2) $\psi = (u^\epsilon)^\epsilon$, one can get

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}^n} |u^\epsilon|^2 dx + \int_{\mathbb{T}^n} (u \otimes u)^\epsilon : \nabla u^\epsilon dx = 0,$$

which shows that

$$\|u^\epsilon(t_2)\|^2 - \|u^\epsilon(t_1)\|^2 = -2 \int_{t_1}^{t_2} \int_{\mathbb{T}^n} (u \otimes u)^\epsilon : \nabla u^\epsilon dx, \tag{2.1}$$

where, as [7], we introduce a nonnegative radially symmetric $C^\infty(\mathbb{R}^n)$ mollifier, $\phi(x)$, with support in $|x| \le 1$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$, and for any $0 < \epsilon < 1$ we define $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$ and set $u^\epsilon = u * \phi_\epsilon$.

Now, we estimate the term on the right side in (2.1). Firstly, it is well known that, see [7],

$$(u \otimes u)^\epsilon(x) - (u^\epsilon \otimes u^\epsilon)(x) = \int_{\mathbb{T}^n} (\delta_y u \otimes \delta_y u)(x) \phi_\epsilon(y) dy - (u - u^\epsilon)(x) \otimes (u - u^\epsilon)(x),$$

where

$$(\delta_y u)(x) = u(x-y) - u(x).$$

Secondly, one has, for almost all $t \in (0, T)$,

$$|u(x-y) - u(x)| \le \left(\log \frac{1}{|y|}\right)^{-\lambda} |y|^\alpha \|u\|_{C_\lambda^{0,\alpha}}, \quad \text{for any } 0 < |y| < 1, \tag{2.2}$$

which gives

$$|u(x) - u^\epsilon(x)| = \left| \int_{\mathbb{T}^n} (u(x) - u(x-y)) \phi_\epsilon(y) dy \right| \le \left(\log \frac{1}{\epsilon}\right)^{-\lambda} \epsilon^\alpha \|u\|_{C_\lambda^{0,\alpha}}. \tag{2.3}$$

Furthermore, one has

$$\begin{aligned} |\nabla u^\epsilon(x)| &= \left|\int_{\mathbb{T}^n} \nabla\phi_\epsilon(z)\cdot u(x-z)dz\right| \\ &= \left|\int_{\mathbb{T}^n} \nabla\phi_\epsilon(z)\cdot (u(x-z)-u(x))dz\right| \\ &\le C\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda}\int_{\mathbb{T}^n}|\nabla\phi_\epsilon(z)|dz \\ &\le C\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^{\alpha-1}\|u\|_{C^{0,\alpha}_\lambda}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \left|\int_{\mathbb{T}^n}(\delta_y u\otimes\delta_y u)(x)\phi_\epsilon(y)dy\right| &\le C\left[\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda}\right]^2\int_{\mathbb{T}^n}\phi_\epsilon(y)dy, \\ &= C\left[\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda}\right]^2. \end{aligned} \tag{2.5}$$

Note that the estimates (2.3)–(2.4) are point-wise. In this sense they are stronger than the related estimates (6)–(8) in [7].

Finally, noting that

$$\int_{\mathbb{T}^n} u^\epsilon\otimes u^\epsilon : \nabla u^\epsilon dx = \int_{\mathbb{T}^n} u^\epsilon\cdot\nabla\frac{1}{2}|u^\epsilon|^2dx = \int_{\mathbb{T}^n}\frac{1}{2}|u^\epsilon|^2\nabla\cdot u^\epsilon dx = 0,$$

one can deduce from (2.3)–(2.5) that

$$\begin{aligned} &\left|\int_{t_1}^{t_2}\int_{\mathbb{T}^n}(u\otimes u)^\epsilon : \nabla u^\epsilon dxdt\right| \\ &\quad\le \int_{t_1}^{t_2}\int_{\mathbb{T}^n}\left(\left|\int(\delta_y u\otimes\delta_y u)(x)\phi_\epsilon(y)dy\right| + |u-u^\epsilon|^2\right)|\nabla u^\epsilon(x)|dxdt \\ &\quad\le C\int_{t_1}^{t_2}\left[\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda}\right]^2\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^{\alpha-1}\,\|u\|_{C^{0,\alpha}_\lambda}dt \\ &\quad= C\left(\log\frac{1}{\epsilon}\right)^{-3\lambda}\epsilon^{3\alpha-1}\int_{t_1}^{t2}\|u\|^3_{C^{0,\alpha}_\lambda}dt. \end{aligned}$$

From this estimate, letting $\epsilon\to 0$ in (2.1), we obtain the Theorem 1.2, for the periodic domain case, since $\alpha\ge\frac{1}{3}$ and $\lambda>0$.

3. Preliminary Results

When we consider a bounded domain, due to the boundary effect, one can not take $(u^\epsilon)^\epsilon$ as test function. To overcome this difficulty, Bardos and Titi [1] introduced a distance function: For any $x\in\bar\Omega$ one defines $d(x)=\inf_{y\in\partial\Omega}|x-y|$, and set $\Omega_h=\{x\in\Omega : d(x)<h\}$. As in [1], since $\partial\Omega$ is a C^2 compact manifold, there exists $h_0(\Omega)>0$ with the following properties:

- For any $x\in\overline{\Omega_{h_0}}$, the function $x\mapsto d(x)$ belongs to $C^1(\overline{\Omega_{h_0}})$;
- for any $x\in\overline{\Omega_{h_0}}$, there exists a unique point $\sigma(x)\in\partial\Omega$ such that

$$d(x)=|x-\sigma(x)|,\quad \nabla d(x)=-n(\sigma(x)). \tag{3.1}$$

Now, let $0 \leq \eta(s) \leq 1$ be a $C^\infty(\mathbb{R})$ nondecreasing function such that $\eta(s) = 0$, for $s \in (-\infty, \frac{1}{2}]$, and $\eta(s) = 1$, for $s \in [1, \infty)$. Then $\theta_h(x) = \eta(\frac{d(x)}{h})$ is a $C^1(\Omega)$ function, compactly supported in Ω. Denote by the same symbol θ_h its extension by zero outside Ω. Similarly, for any $w \in L^\infty(\Omega)$, the extension $\theta_h w$ by zero outside Ω is well defined over all $\mathbb{R}^n$, and will be also denoted by $\theta_h w$.

It is natural to take $\theta_h\left((\theta_h u)^\epsilon\right)^\epsilon$ as a text function. Contrarily to the period domain case, now $\nabla\cdot\psi \neq 0$. Hence we will need to estimate the pressure in a suitable way. Actually, due to $C^{0,\alpha}_\lambda(\bar{\Omega}) \subset C^{0,\alpha}(\bar{\Omega})$, we can get the following result from Proposition 1.2 in [1].

Proposition 3.1. *Under the assumption of Theorem 1.2 the pair (u, p) satisfies the following regularity properties:*

$$u \otimes u \in L^3((0,T); L^2(\Omega)), \quad p \in L^{\frac{3}{2}}((0,T); C^{0,\alpha}(\bar{\Omega})),$$

and

$$\partial_t u = -\nabla \cdot (u \otimes u) - \nabla p \in L^{\frac{3}{2}}((0,T); H^{-1}(\Omega)).$$

Furthermore, one has

$$\int_0^T \|p\|^{\frac{3}{2}}_{C^{0,\alpha}(\bar{\Omega})} dt \leq C \int_0^T \|u\|^3_{C^{0,\alpha}(\bar{\Omega})} dt \leq C \int_0^T \|u\|^3_{C^{0,\alpha}_\lambda(\bar{\Omega})} dt. \tag{3.2}$$

Remark 3.1. In [1], although the authors assume $\alpha > \frac{1}{3}$, it follows from the proof of their proposition 1.2 that the result holds for any $\alpha > 0$, especially for $\alpha = \frac{1}{3}$.

Remark 3.2. According to Proposition 4.3 below, since

$$\int_0^T \|p\|_{L^\infty} \|u\|_{C^{0,\alpha}_\lambda} dt \leq \left(\int_0^T \|p\|_{L^\infty} dt\right)^{\frac{2}{3}} \left(\int_0^T \|u\|^3_{C^{0,\alpha}_\lambda} dt\right)^{\frac{1}{3}},$$

to obtain Theorem 1.2, we merely need to have the estimate $\int_0^T \|p\|^{\frac{3}{2}}_{L^\infty} dt \leq C \int_0^T \|u\|^3_{C^{0,\alpha}_\lambda} dt$. Hence, the estimate (3.2) is enough to obtain our theorem.

Compared with the periodic domain case, since the test function include the function θ_h, we need some estimates for θ_h.

Lemma 3.2. *Let $h \in (0, \min\{h_0, 1\})$. For any vector field $w \in C^{0,\alpha}_\lambda(\bar{\Omega})$, with $w \cdot n = 0$ on $\partial\Omega$, there exists a constant C independent of h such that*

$$|w(x) \cdot \nabla\theta_h(x)| \leq C\|w\|_{C^{0,\alpha}_\lambda(\bar{\Omega})} \left(\log\frac{1}{h}\right)^{-\lambda} h^{\alpha-1}, \tag{3.3}$$

and

$$\int_{\mathbb{R}^n} |w(x) \cdot \nabla\theta_h(x)| dx \leq C\|w\|_{C^{0,\alpha}_\lambda(\bar{\Omega})} \left(\log\frac{1}{h}\right)^{-\lambda} h^{\alpha}. \tag{3.4}$$

Proof. The proof is completely similar to that of Lemma 1.3 in [1]. For completeness, and for the readers convenience, we give here the proof. When $x \in (\Omega_h)^c$, since $\nabla\theta_h(x) = 0$, one has $w(x) \cdot \nabla\theta_h(x) = 0$. When $x \in \Omega_h$, it follows from (3.1) that

$$\nabla\theta_h(x) = -\frac{1}{h}\eta'\left(\frac{d(x)}{h}\right) n(\sigma(x)).$$

Noting that $w(\sigma(x)) \cdot n(\sigma(x)) = 0$, one can get

$$\begin{aligned}|w(x) \cdot \nabla\theta_h(x)| =& \frac{1}{h}\eta'\left(\frac{d(x)}{h}\right)|(w(x) - w(\sigma(x))) \cdot n(\sigma(x))| \\ \leq& \frac{C}{h}\|w\|_{C^{0,\alpha}_\lambda}\left(\log\frac{1}{|x-\sigma(x)|}\right)^{-\lambda}|x-\sigma(x)|^\alpha \\ \leq& C\|w\|_{C^{0,\alpha}_\lambda}\left(\log\frac{1}{h}\right)^{-\lambda}h^{\alpha-1}.\end{aligned}$$

This gives (3.3). Integrating (3.3) over $\mathbb{R}^n$, combining with the facts that the support of $\nabla\theta_h$ is a subset of $\overline{\Omega_h}$, and $|\Omega_h| \leq Ch$, one obtains (3.4). □

4. Proof of Theorem 1.2

In this section, we focus on the proof of Theorem 1.2. First, we set $h \in (0, \min\{h_0, 1\})$ and $\epsilon \in (0, \frac{h}{4})$. As in [1], we take in (1.2) $\psi = \theta_h\left((\theta_h u)^\epsilon\right)^\epsilon$ as test function. Note that, due to Proposition 3.1, $\psi \in W^{1,3}((0,T); H^1_0(\Omega)$. So it can be used as test vector field function. So one shows that

$$\begin{aligned}&\langle u, \partial_t\left(\theta_h\left((\theta_h u)^\epsilon\right)^\epsilon\right)\rangle_x + \langle u\otimes u : \nabla\left(\theta_h\left((\theta_h u)^\epsilon\right)^\epsilon\right)\rangle_x \\ &\quad + \langle p, \nabla\cdot\left(\theta_h\left((\theta_h u)^\epsilon\right)^\epsilon\right)\rangle_x = 0, \quad \text{in } L^1(0,T).\end{aligned} \tag{4.1}$$

Next, as in [1], we establish three propositions to estimate the three terms on the left side of (4.1), denoted here by J_1, J_2, and J_3 respectively.

For J_1, by arguing as in [1] Proposition 2.1, one proves the following statement.

Proposition 4.1. *For any $(t_1, t_2) \in (0,T)$, one has*

$$\lim_{h\to 0}\int_{t_1}^{t_2} J_1 dt = \frac{1}{2}\|u(t_2)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(t_1)\|^2_{L^2(\Omega)}.$$

Next, we control J_2.

Proposition 4.2. *The following estimate holds.*

$$\begin{aligned}|J_2| =& |\langle u\otimes u : \nabla\left(\theta_h\left((\theta_h u)^\epsilon\right)^\epsilon\right)\rangle_x| \leq C\left(\log\frac{1}{h}\right)^{-\lambda}h^\alpha\|u\|_{C^{0,\alpha}_\lambda(\Omega)}\|u\|^2_{L^\infty} \\ &+ C\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^{\alpha-1}\|u\|_{C^{0,\alpha}_\lambda}\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right)^2.\end{aligned}$$

Proof. We first write J_2 as

$$J_2 = \langle u\otimes u : \nabla\theta_h \otimes \left((\theta_h u)^\epsilon\right)^\epsilon\rangle_x + \langle u\otimes u : \theta_h\nabla\left((\theta_h u)^\epsilon\right)^\epsilon\rangle_x =: J_{21} + J_{22}.$$

For J_{21}, by Lemma 3.2, one can get

$$\begin{aligned}|J_{21}| =& |\langle u\otimes u : \nabla\theta_h \otimes \left((\theta_h u)^\epsilon\right)^\epsilon\rangle_x| \\ =& \left|\int_{\Omega_h}(u\cdot\nabla\theta_h)\left(u\cdot\left((\theta_h u)^\epsilon\right)^\epsilon\right)dx\right| \\ \leq& C\left(\log\frac{1}{h}\right)^{-\lambda}h^\alpha\|u\|_{C^{0,\alpha}_\lambda}\|u\|^2_{L^\infty}.\end{aligned}$$

For J_{22}, one has

$$\begin{aligned} J_{22} &= \langle u \otimes u : \theta_h \nabla \left((\theta_h u)^\epsilon\right)^\epsilon \rangle_x | \\ &= |\langle u \otimes \theta_h u : \nabla \left((\theta_h u)^\epsilon\right)^\epsilon \rangle_x | \\ &= \langle (u \otimes \theta_h u)^\epsilon : \nabla (\theta_h u)^\epsilon \rangle_x \\ &= \langle ((u \otimes \theta_h u)^\epsilon - (u^\epsilon \otimes (\theta_h u)^\epsilon)) : \nabla (\theta_h u)^\epsilon \rangle_x, \end{aligned}$$

where we have used that

$$\int_{\mathbb{R}^n_x} u^\epsilon \otimes (\theta_h u)^\epsilon : \nabla (\theta_h u)^\epsilon dx = \int u^\epsilon \cdot \nabla \frac{1}{2} |(\theta_h u)^\epsilon|^2 dx = \int \frac{1}{2} |(\theta_h u)^\epsilon|^2 \nabla \cdot u^\epsilon dx = 0.$$

By using the identity

$$(v \otimes w)^\epsilon(x) - (v^\epsilon \otimes w^\epsilon)(x) = \int_{\mathbb{R}^n_y} (\delta_y v \otimes \delta_y w)(x) \phi_\epsilon(y) dy - (v - v^\epsilon)(x) \otimes (w - w^\epsilon)(x),$$

where

$$(\delta_y) v(x) = v(x-y) - v(x), \quad (\delta_y) w(x) = w(x-y) - w(x),$$

one can write $J_{22} = J_{221} + J_{222}$ with

$$\begin{aligned} J_{221} &= \int_{\mathbb{R}^n_x} \left(\int_{\mathbb{R}^n_y} (\delta_y u \otimes \delta_y(\theta_h u))(x) \phi_\epsilon(y) dy \right) : \left(\int_{\mathbb{R}^n_z} \nabla \phi_\epsilon(z) \otimes (\theta_h u)(x-z) dz \right) dx \\ &= \int_{\Omega} \left(\int_{\mathbb{R}^n_y} (\delta_y u \otimes \delta_y(\theta_h u))(x) \phi_\epsilon(y) dy \right) : \left(\int_{\mathbb{R}^n_z} \nabla \phi_\epsilon(z) \otimes (\theta_h u)(x-z) dz \right) dx, \end{aligned}$$

and

$$\begin{aligned} J_{222} &= \int_{\mathbb{R}^n_x} ((u - u^\epsilon) \otimes ((\theta_h u) - (\theta_h u)^\epsilon)) : \nabla (\theta_h u)^\epsilon dx \\ &= \int_{\Omega} ((u - u^\epsilon) \otimes ((\theta_h u) - (\theta_h u)^\epsilon)) : \nabla (\theta_h u)^\epsilon dx. \end{aligned}$$

For J_{221}, noting that supp $\phi_\epsilon \subset \{y : |y| \le \epsilon\}$, that $|\delta_y \theta_h(x)| \le C\frac{\epsilon}{h}$ for all $|y| \le \epsilon$, and that $\int_{\mathbb{R}^n_z} |\nabla \phi_\epsilon| dz \le C\epsilon^{-1}$, one shows that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n_y} (\delta_y u \otimes \delta_y(\theta_h u))(x) \phi_\epsilon(y) dy \right| \\ &= \left| \int_{\mathbb{R}^n_y} (\delta_y u \otimes (\theta_h(x-y)(\delta_y u)(x) + (\delta_y \theta_h)(x) u(x-y))\, \phi_\epsilon(y) dy \right| \\ &\le C \left(\log \frac{1}{\epsilon} \right)^{-\lambda} \epsilon^\alpha \|u\|_{C^{0,\alpha}_\lambda} \int_{\mathbb{R}^n_y} \left(\left(\log \frac{1}{\epsilon} \right)^{-\lambda} \epsilon^\alpha \|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h} \|u\|_{L^\infty} \right) \phi_\epsilon(y) dy \\ &= C \left(\log \frac{1}{\epsilon} \right)^{-\lambda} \epsilon^\alpha \|u\|_{C^{0,\alpha}_\lambda} \left(\left(\log \frac{1}{\epsilon} \right)^{-\lambda} \epsilon^\alpha \|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h} \|u\|_{L^\infty} \right), \end{aligned}$$

and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n_z} \nabla\phi_\epsilon(z) \otimes (\theta_h u)(x-z)dz\right| \\
&= \left|\int_{\mathbb{R}^n_z} \nabla\phi_\epsilon(z) \otimes ((\theta_h u)(x-z) - (\theta_h u)(x))\, dz\right| \\
&= \left|\int_{\mathbb{R}^n_z} \nabla\phi_\epsilon(z) \otimes (\delta_z\theta_h(x)u(x-z) - \theta_h(x)\delta_z u(x))\, dz\right| \\
&\le C\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right)\int_{\mathbb{R}^n_z}|\nabla\phi_\epsilon(z)|dz \\
&\le C\epsilon^{-1}\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right).
\end{aligned} \tag{4.2}
$$

Hence, one has

$$
|J_{221}| \le C\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^{\alpha-1}\|u\|_{C^{0,\alpha}_\lambda}\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right)^2.
$$

For J_{222}, it follows from (4.2) that

$$
\begin{aligned}
|\nabla(\theta_h u)^\epsilon(x)| &= \left|\int_{\mathbb{R}^n_z} \nabla\phi_\epsilon(z) \otimes (\theta_h u)(x-z)dz\right| \\
&\le C\epsilon^{-1}\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right).
\end{aligned} \tag{4.3}
$$

On the other hand, for all $x \in \operatorname{supp}\theta_{h+\epsilon}$, one has

$$
|u(x) - u^\epsilon(x)| = \left|\int_{\mathbb{R}^n_y}(u(x) - u(x-y))\phi_\epsilon(y)dy\right| \le \left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda}, \tag{4.4}
$$

and

$$
\begin{aligned}
&|(\theta_h u)(x) - (\theta_h u)^\epsilon(x)| \\
&= \left|\int_{\mathbb{R}^n_y}((\theta_h u)(x) - (\theta_h u)(x-y))\phi_\epsilon(y)dy\right| \\
&= \left|\int_{\mathbb{R}^n_y}\phi_\epsilon(y)\,(\delta_y\theta_h(x)u(x-y) - \theta_h(x)\delta_y u(x))\, dy\right| \\
&\le C\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right).
\end{aligned} \tag{4.5}
$$

Combining with (4.3)–(4.5), one gets

$$
|J_{222}| \le C\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^{\alpha-1}\|u\|_{C^{0,\alpha}_\lambda}\left(\left(\log\frac{1}{\epsilon}\right)^{-\lambda}\epsilon^\alpha\|u\|_{C^{0,\alpha}_\lambda} + \frac{\epsilon}{h}\|u\|_{L^\infty}\right)^2.
$$

Now, collecting the above estimates obtained for J_{21}, J_{221}, and J_{222}, one obtains the desired estimate for J_2. □

Finally, we estimate J_3.

Proposition 4.3. *One has*

$$|\langle p, \nabla \cdot (\theta_h ((\theta_h u)^\epsilon)^\epsilon)\rangle_x| \le C\|p\|_{L^\infty}\|u\|_{C^{0,\alpha}_\lambda}\left(\left(\log\left(\frac{1}{h}\right)\right)^{-\lambda} h^\alpha + \left(\log\left(\frac{1}{\epsilon}\right)\right)^{-\lambda}\epsilon^\alpha\right).$$

Proof. First, one has

$$\begin{aligned}\langle p, \nabla \cdot (\theta_h ((\theta_h u)^\epsilon)^\epsilon)\rangle_x &= \int_\Omega p\nabla\cdot(\theta_h ((\theta_h u)^\epsilon)^\epsilon)\,dx\\ &= \int_\Omega (p\theta_h)\nabla\cdot((\theta_h u)^\epsilon)^\epsilon\,dx + \int_\Omega p\nabla\theta_h\cdot((\theta_h u)^\epsilon)^\epsilon\,dx\\ &=: J_{31} + J_{32}.\end{aligned}$$

Concerning J_{31}, from (2.22) and (2.25) in Proposition 2.3 of [1], one obtains

$$J_{31} = \int_\Omega \Big(p(x)\theta_h(x)\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_z}\phi_\epsilon(x-y)\phi_\epsilon(z-y)u(z)\cdot\nabla\theta_h(z)dzdy\Big)dx,$$

by Lemma 3.2, which implies that

$$|J_{31}| \le C\|p\|_{L^\infty}\|u\|_{C^{0,\alpha}_\lambda}\left(\log\left(\frac{1}{h}\right)\right)^{-\lambda}h^\alpha.$$

For J_{32}, as in [1], one has

$$\begin{aligned}J_{32} =& \int_{\Omega_h}\left(p(x)\nabla\theta_h(x)\cdot\int_{\mathbb{R}^n_z}\int_{\mathbb{R}^n_y}\theta_h(x-y+z)u(x-y+z)\phi_\epsilon(y)\phi_\epsilon(z)dydz\right)dx,\\ =& \int_{\Omega_h}p(x)\left(\int_{\mathbb{R}^n_z}\int_{\mathbb{R}^n_y}\phi_\epsilon(y)\phi_\epsilon(z)\theta_h(x-y+z)\left(u(x-y+z)-u(x)\right)\cdot\nabla\theta_h(x)dydz\right)dx\\ &+\int_{\Omega_h}p(x)\left(\int_{\mathbb{R}^n_z}\int_{\mathbb{R}^n_y}\phi_\epsilon(y)\phi_\epsilon(z)\theta_h(x-y+z)u(x)\cdot\nabla\theta_h(x)dydz\right)dx\\ =&: J_{321} + J_{322}.\end{aligned}$$

Noting that

$$|u(x-y+z)-u(x)| \le C\|u\|_{C^{0,\alpha}_\lambda}\left(\log\left(\frac{1}{\epsilon}\right)\right)^{-\lambda}\epsilon^\alpha,$$

for the relevant x, y, z for which the integrand in the definition of J_{321} is not zero, and that $\int_{\Omega_h}|\nabla\theta_h(x)|dx \le C$, one shows that

$$|J_{321}| \le C\|p\|_\infty\|u\|_{C^{0,\alpha}_\lambda}\left(\log\left(\frac{1}{\epsilon}\right)\right)^{-\lambda}\epsilon^\alpha.$$

Concerning J_{322}, it follows from Lemma 3.2 that

$$\begin{aligned}|J_{322}| &\le \int_{\Omega_h}|p(x)|\left(\int_{\mathbb{R}^n_z}\int_{\mathbb{R}^n_y}\phi_\epsilon(y)\phi_\epsilon(z)|u(x)\cdot\nabla\theta_h(x)|dydz\right)dx\\ &\le C\|p\|_\infty\|u\|_{C^{0,\alpha}_\lambda}\left(\log\left(\frac{1}{h}\right)\right)^{-\lambda}h^\alpha.\end{aligned}$$

Collecting the above estimates, one proves the proposition. □

Now, it follows from Propositions 4.2, 4.3 and the estimate (3.2) in Proposition 3.1 that

$$\begin{aligned}\int_{t_1}^{t_2}|J_2+J_3|dt \le C\left(\left(\log\left(\frac{1}{h}\right)\right)^{-\lambda}h^\alpha+\left(\log\left(\frac{1}{\epsilon}\right)\right)^{-\lambda}\epsilon^\alpha\right)\int_{t_1}^{t_2}\|u\|^3_{C^{0,\alpha}_\lambda}dt\\ +\left(\left(\log\left(\frac{1}{\epsilon}\right)\right)^{-3\lambda}\epsilon^{3\alpha-1}+\left(\log\left(\frac{1}{\epsilon}\right)\right)^{-\lambda}\frac{\epsilon^{\alpha+1}}{h^2}\right)\int_{t_1}^{t_2}\|u\|^3_{C^{0,\alpha}_\lambda}dt,\end{aligned} \tag{4.6}$$

by choosing $\epsilon = h^{\frac{2}{1+\alpha}}$ and by letting $h \to 0$, since $\alpha \ge \frac{1}{3}$ and $\lambda > 0$, one has $\int_{t_1}^{t_2}|J_2+J_3|dt \to 0$. Combining this fact with Proposition 4.1 and Eq. (4.1), one proves the Theorem 1.2.

Compliance with ethical standards

Conflict of interest The authors declared that they have no conflicts of interest.

References

[1] Bardos, C., Titi, E.S.: Onsager's conjecture for the incompressible Euler equations in bounded domains. Arch. Ration. Mech. Anal. **228**(1), 197–207 (2018)

[2] Bardos, C., Gwiazda, P., Świerczewska-Gwiazda, A., Titi, E.S., Wiedemann, E.: On the extension of Onsager's conjecture for general conservation laws. J. Nonlinear Sci. **29**(2), 501–510 (2019)

[3] Bardos, C., Titi, E.S., Wiedemann, E.: Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit. Commun. Math. Phys. **370**(1), 291–310 (2019)

[4] Beirão da Veiga, H.: Moduli of continuity, functional spaces, and elliptic boundary value problems, The full regularity spaces $C^{0,\lambda}_\alpha$. Adv. Nonlinear Anal. **7**(1), 15–34 (2018)

[5] Buckmaster, T., De Lellis, C., Székelyhidi, L.Jr., Vicol, V.: Onsager's conjecture for admissible weak solutions. arXiv:1701.08678

[6] Cheskidov, A., Constantin, P., Friedlander, S., Shvydkoy, R.: Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity **21**(6), 1233–1252 (2008)

[7] Constantin, P.W.E., Titi, E.S.: Onsager's conjecture on the energy conservation for solutions of Euler's equation. Commun. Math. Phys. **165**, 207–209 (1994)

[8] Eyink, G.L.: Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer. Phys. D **78**(3–4), 222–240 (1994)

[9] Isett, P.: A proof of Onsager's conjecture. Ann. Math. **188**(3), 871–963 (2018)

[10] Isett, P.: On the endpoint regularity in Onsager's conjecture. arXiv:1706.01549

[11] Nguyen, Q.H., Nguyen, P.T.: Onsager's conjecture on the energy conservation for solutions of Euler equations in bounded domains. J. Nonlinear Sci. **29**(1), 207–213 (2019)

[12] Onsager, L.: Statistical hydrodynamics. Nuovo Cimento **6**(Supplemento 2), 279–287 (1949)

[13] Shvydkoy, R.: On the energy of inviscid singular flows. J. Math. Anal. Appl. **349**, 583–595 (2009)

[14] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions, vol. 2. Princeton University Press, Princeton (1970)

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