On the truth, and limits, of a full equivalence $\mathbf{p \cong v^2}$ in the regularity theory of the Navier-Stokes equations. A point of view.

H. Beirão da Veiga *
Department of Mathematics,
Pisa University (Italy)
email: bveiga@dma.unipi.it

May 26, 2018

Abstract

The motivation at the origin of this note is the well known sufficient condition for regularity of solutions to the evolution Navier-Stokes equations, sometimes referred to in the literature as Ladyzhenskaya-Prodi-Serrin's condition. Such a condition requires that the velocity field v, alone, satisfies sufficiently strong integrability requirements in space-time. On the other hand, a relation like $p \cong |v|^2$, with p pressure field, is loosely suggested by the Navier-Stokes equations themselves. In three papers published nearly twenty years ago we have considered this problem. The results obtained there immediately suggest new interesting questions. In this paper, we propose, and solve, some of them, while many other related problems remain still open.

1 Introduction and main results.

This note concerns sufficient conditions of the so called Ladyzhenskaya–Prodi–Serrin's type (in the sequel simply denoted by LPS), for regularity of weak solutions of the evolution Navier-Stokes equations

$$\begin{cases} \partial_t\, v + (v\cdot\nabla)\, v - \,\mu\,\triangle v + \nabla\pi = \,f\,, \\ \nabla\cdot v = 0\,, \quad \text{in } \Omega\times(0,T]\,; \\ v(x,0) = \,v_0(x) \qquad \text{in } \Omega\,, \\ v = \,0, \quad \text{on } \Gamma\times(0,T]\,, \end{cases} \tag{1}$$

*Partially supported by FCT (Portugal) under grant UID/MAT/04561/3013.

where $v_0 \in H^1_0(\Omega)$ is divergence free. Here Ω is a smooth open, bounded, subset of $\mathbb{R}^n$, $n \geq 3$, and Γ is its boundary. We assume that the reader is familiar with the classical literature on these equations. In particular, we will not recall the meaning of usual notation such as, for instance, Lebesgue spaces, Sobolev spaces, and so on. Moreover, we shall skip the standard proof of some peripheral results, and leave it to the reader the task of showing the details.

Weak solutions are characterized by the property

$$v \in L^\infty(0, T; L^2(\Omega) \cap L^2(0, T; H^1(\Omega))\,,$$

and by being weakly continuous with values in $L^2(\Omega)$.

The choice of boundary conditions plays an important role, since not all extensions of LPS's type conditions to more general functional spaces seem to be possible, or at least quite difficult to prove, under different boundary conditions. In particular, under the classical no-slip ones assumed here, reflection techniques are not suitable.

Our aim is to propose, justify, discuss and solve some open problems related to our old contributions [2], [3], and [4].

We start by briefly recalling the LPS condition. It states that weak solutions v of (1) satisfying

$$v \in L^r(0, T; L^q(\Omega))\,, \quad \frac{2}{r} + \frac{n}{q} = 1\,, \quad q > n \tag{2}$$

are *strong*. This means here that

$$v \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))\,. \tag{3}$$

It is well known that strong solutions are *smooth*, if data and domain are also smooth.

Obviously, in condition (2), as in many similar assumptions made later on, we may replace " $=$ " with " $\leq$ ". However, in our opinion, this substitution may generate some confusion between sharp and non sharp regularity results. See also the appendix.

Let's come to our main problem. The well know "interior" equation

$$-\Delta p = \sum_{i,j=1}^{n} \partial_i\, \partial_j (v_i\, v_j) \tag{4}$$

roughly suggests the formal relation

$$p \cong v^2\,. \tag{5}$$

Actually, (4) suggests, more appropriately, $p \lessapprox v^2$, rather than $v^2 \lessapprox p$. However, the latter inequality is just the one related to the results proved later on. Note that from $p \lessapprox v^2$ it merely would follow

$$\frac{|p|}{1+|v|} \leq \frac{|p|}{|v|} \leq |v|\,,$$

but not the reverse. So, at least formally, assumption (18) stated below looks weaker than (2).

Formally, (5) may suggest the following generalization

$$\frac{|p|}{(1+|v|)^\theta} \cong |\,v\,|^{2-\theta}\,, \tag{6}$$

even though the non-local character of the above relations should be considered.

Problem 1.1. *Assume that a weak solution* $(v,\,p)$ *of the Navier-Stokes equations* (1) *satisfies*

$$\frac{|p|}{(1+|v|)^\theta} \in L^r(0,\,T;\,L^q(\Omega)\,)\,, \tag{7}$$

for some $\theta \in [0,\,2]$, *where*

$$\frac{2}{r} + \frac{n}{q} = 2 - \theta\,. \tag{8}$$

Question: Does (3) *hold? If not, may one replace* (8) *by weaker, but significant, assumptions?*

Definition 1.1. We say that a regularity result is *sharp* if weak solutions (v,p) are strong under the couple of assumptions (7) and (8).

Note the difference between the meanings of "strong solution" and "sharp result".

In three papers published nearly twenty years ago, [2], [3], and [4], we have considered the above kind of problems. The results proved there suggest a positive answer to Problem 1.1, at least for $\theta < 1$. The answer seems to be quite different in the case $\theta > 1$, see section 4. Hence, our main interest here is to consider the case $\theta \leq 1$. For $\theta = 1$, sharp regularity was already shown in reference [4], where $q > n$ was assumed (the result, however, continue to hold also for $q = n$).

Our main result, showed next, is the following.

Theorem 1.1. *Let* $v_0 \in L^n(\Omega) \cap H^1_0(\Omega)$, *be divergence free and* $f \in L^1(0, T; L^n(\Omega))$. *Assume that a weak solution of the Navier-Stokes equations* (1) *satisfies*

$$\frac{|p|}{(1+|v|)^\theta} \in L^r(0, T; L^q(\Omega)), \tag{9}$$

where $0 \le \theta \le 1$, *and the exponents* $r, q \in (2, +\infty)$ *verify the condition*

$$\frac{2}{r} + \frac{n}{q} = 2 - \theta. \tag{10}$$

If $2 \le q < n$ *we also assume that*

$$r \le \frac{(n-2)\,q}{n-q} \equiv \frac{n-2}{(n/q)-1}. \tag{11}$$

Under the above hypothesis one has

$$v \in L^\infty(0, T; L^n(\Omega)) \cap L^n(0, T; L^{\frac{n^2}{n-2}}(\Omega)). \tag{12}$$

Furthermore

$$\nabla |v|^{n/2} \in L^2(0, T; L^2(\Omega)). \tag{13}$$

In particular, the solution is strong. Additional smoothness of solution follows from related smoothness of the data.

The particular cases in which r or q take one of the values 2 or $+\infty$, left to the interested reader, should be treated separately.

Note that if $q \ge n$ then r has the full range $(2, \infty)$. But for values $q < n$ the range of r shrinks as q decreases. For some considerations and examples that may help a better understanding of the pair of assumptions (10), (11), we reffer the reader to the appendix.

For further use we state here a corollary of theorem 1.1 in the case $r = q = \gamma$. Assumption (11) holds in this case.

Corollary 1.1. *Let* v_0 *and* f *be as in Theorem 1.1. Assume that a weak solution of the Navier-Stokes equations* (1) *satisfies*

$$\frac{|p|}{(1+|v|)^\theta} \in L^\gamma(Q_T) = L^\gamma(0, T; L^\gamma(\Omega)), \tag{14}$$

where $0 \le \theta \le 1$, *and the exponent* $\gamma \in (2, +\infty)$ *verifies the condition*

$$\frac{n+2}{\gamma} = 2 - \theta. \tag{15}$$

Then, the solution v *is strong.*

The basic idea of proof of Theorem 1.1 generalizes, by following the same line of thought, the procedure introduced in reference [4], which has also been used later on by other authors. The fundamental estimate (2.3) in [4] (see (33) below, where $\alpha = n$), was previously shown in reference [1], in particular Lemmas 1.1 and 1.2. Actually, related, similar, estimates were obtained earlier in [11], in a different context (set in equations (4) and (5) there, $q = \alpha$, $g = 1$, and $v = 0$).

Notations in [1] and in [4] are different. The quantities denoted in [1] by the symbols $N_\alpha(v)$ and $M_\alpha(v)$ are the $\alpha-$powers of the quantities denoted by the same symbols $N_\alpha(v)$ and $M_\alpha(v)$ in reference [4]. Here, see definitions in (31) below, we follow the notation used in [4]. It is worth noting that below we appeal to the results stated in [4] only for the particular value $\alpha = n$. So we denote here $N_n(v)$ simply by $N(v)$, and so on.

The particular case $\theta = 0$ (a condition on the pressure alone), is not considered here. We just quote here the pioneering reference [7] and two main references, [6] and [8], where the proofs generalized that in [4]; see [6] also for a rather complete bibliographic reference. We also refer the reader to [9] where, for the Cauchy problem in $\mathbb{R}^3$, it is shown that weak solutions are smooth if the pressure is non-negative (actually, the assumption is a little more general).

In reference [8] the case $\theta > 1$ is also considered, see section 4 below.

2 On some our old results.

In this section we briefly recall some results stated in references [2], [3], and [4]. Even though not necessary to understand the proof of theorem 1.1, we strongly encourage readers not skip this section.

We start by recalling Theorem 1.1 in [2]. As far as we know, this theorem is the first result where assumptions of type (7) were considered. It is one of the pioneering papers applying the truncation method to the Navier-Stokes equations. The presence of pressure and divergence free fields make this application non trivial.

Theorem 2.1. *Let v be a weak solution to the Navier-Stokes equations* (1)*, where the initial data v_0 is assumed to be bounded. Further, let*

$$\frac{|p|}{1+|v|} \in L^r(0,\,T;\,L^q(\Omega)\,)\,, \tag{16}$$

where

$$\frac{2}{r}+\frac{n}{q} < 1\,, \quad q \in (n,\,\infty\,]\,. \tag{17}$$

Then v is bounded in $Q_T \equiv \Omega \times (0, T)$. In particular v is smooth.

The classical sufficient condition for regularity (2) led us to investigate whether in the above theorem one can replace assumption (17) with (19). A positive answer was given in [4, Theorem 1], where the following result was proved.

Theorem 2.2. *Let v be a solution to the Navier-Stokes equations* (1)*, where, for some $\alpha > n$, $v_0 \in L^\alpha(\Omega)$, is divergence-free, with normal component vanishing at Γ. Further, let*

$$\frac{|p|}{1+|v|} \in L^r(0, T; L^q(\Omega)), \tag{18}$$

where

$$\frac{2}{r} + \frac{n}{q} = 1, \quad q \in (n, \infty]. \tag{19}$$

Then

$$v \in C(0, T; L^\alpha(\Omega)) \quad \textit{and} \quad |v|^{\alpha/2} \in L^2(0, T; H_0^1(\Omega)). \tag{20}$$

In particular v is smooth in Q_T.

Notice that, as done in [4], in $(20)_1$ we can equivalently replace $L^\alpha(\Omega)$ with $H_\alpha(\Omega)$, completion in the $L^\alpha(\Omega)$-norm of the subset of divergence free vector fields belonging to $C_0^\infty(\Omega)$, since, according to $(20)_2$, the divergence free vector field v vanishes on the boundary. We keep the notation L^α as it is more consistent with the notation used in the papers quoted later on.

In [2] and [4] we have assumed that $\theta = 1$. In reference [3] $\theta \leq 1$ was allowed, and $r = s$ was assumed. Furthermore, instead of Lebesgue spaces L^s we have used weak$-L^s$ spaces (also called Marcinkiewicz spaces, or Lorentz spaces), denoted below by the symbol L^s_*. Recall that, in a bounded domain, and with obvious notation, one has

$$L_*^{q+\epsilon} \subset L^q \subset L_*^{q-\epsilon}, \quad \text{and equivalently,} \quad L^{q+\epsilon} \subset L_*^q \subset L^{q-\epsilon}. \tag{21}$$

Following [3] we set

$$N = n + 2. \tag{22}$$

N is precisely the integrability exponent for which (8) holds for $r = s = N$.

As in [2], the proofs given in [3] made use of the truncation method, but with a different approach. In [3, Theorem 1.1] the following result was proved.

Theorem 2.3. *Let* $v_0 \in L^\infty(\Omega) \cap H^1_0(\Omega)$ *be a divergence field vector field, and assume that* $(v,\, p)$ *is a weak solution to the Navier-Stokes equations* (1)*. Assume that for some* $\theta \in [0,\, 1)$ *, and some*

$$2 < \gamma < N\,, \tag{23}$$

one has

$$\frac{|p|}{(1+|v|)^\theta} \in L^\gamma_*(Q_T)\,. \tag{24}$$

Then

$$v \in L^\mu_*(Q_T\,)\,, \tag{25}$$

where

$$\mu = (1-\theta)\,\frac{N\,\gamma}{N-\gamma}\,. \tag{26}$$

In particular the solution is smooth in Q_T *if*

$$\frac{2}{\gamma} + \frac{n}{\gamma} < 2-\theta\,, \quad \theta \in [0,\, 1]\,. \tag{27}$$

In reference [3] the condition (27) was written in the equivalent form $\gamma > N/(2-\theta)$. Furthermore, it was formally assumed that $\gamma > 2\,N/(2\,\theta + (1-\theta)\,N)$. This assumption is superfluous, as already explained in [3, Remark 1.5]. However it (implicitly) implied $\gamma > 2$, a condition required in [3, equation (2.4)], and claimed here in equation (23).

Smoothness under assumption (27) follows immediately from the first part of the above theorem, as pointed out in [3, Remark 1.2]. In fact, in this case one has $\mu > N$ in equation (25). Hence, thanks to (21), $v \in L^s(Q_T\,)$, for any s satisfying $N < s < \mu$. It then follows that the classical LPS condition for regularity applies, leading to regularity of the weak solution.

The reader should compare the last statement in Theorem 2.3 with that stated in the Corollary 1.1. Roughly speaking, we want to compare the results obtained by replacing the Marcinkiewicz spaces $L^\gamma_*(Q_T)$ and $L^\mu_*(Q_T)$ with the Lebesgue spaces $L^\gamma(Q_T)$ and $L^\mu(Q_T)$ respectively, an idea suggested by the inclusions (21). Note that, in the first case, the thesis is weaker, but the assumption is weaker as well. So, to easily compare the two results, it looks less rigorous but more useful to replace (27) by the limit situation

$$\frac{2}{\gamma} + \frac{n}{\gamma} = 2-\theta\,, \quad \theta \in [0,\, 1]\,. \tag{28}$$

This condition is just the assumption (15) in the Corollary 1.1. The fact that two completely methods of proof requires the same assumption lead us to believe that the assumption is more than accidental.

The practical advantage of a Lebesgue-type approach is that smoothness can be obtained under the equality sign " = " in assumption (27), while in Marcinkiewicz spaces the equality sign is not known to yield regularity. Hence, in the Marcinkiewicz framework, we were forced to appeal to the LPS assumption in Lebesgue spaces, at the price of an arbitrary small increment of the integrability exponent γ. Note that, if the LPS condition for regularity also holds for Marcinkiewicz spaces (an open problem), then the corresponding version of the regularity result would be stronger than the sharp regularity version in Lebesgue spaces. Note that some results aimed at extending the LPS condition to Marcinkiewicz spaces, are indeed known. See for instance [5] and [12].

In conclusion to this section, we wish to recall [3, Corollary 1.7], even though this result is marginal in our context, since it assumes $\theta = 0$. The use of Marcinkiewicz spaces in the framework of the corollary was, at that time, quite new, which provides a good reason to recall this old result.

Corollary 2.1. *Let $(v,\, p)$ be a weak solution to problem* (1)*. Assume that*

$$p \in L^{\gamma}_{*}(Q_T)\,, \tag{29}$$

for some $\gamma \in (2,\, N)$. Then

$$v \in L^{\mu}_{*}(Q_T)\,, \quad \mu = \frac{N\,\gamma}{N-\gamma}\,. \tag{30}$$

In particular, if $p \in L^{N/2}_{*}(Q_T)$ then $v \in L^{\mathbb{N}}_{*}(Q_T)$, and if $p \in L^{\gamma/2}_{*}(Q_T)$, $\gamma > N$, then v is smooth in Q_T.

3 Proof of Theorem 1.1.

We start by recalling some fundamental relations already proved in [4], to which the reader is referred. In [4] we have considered dependence on a parameter $\alpha \geq n$. Actually this assumption was used only to obtain smoothness of solutions, while the derivation of the equations only require $\alpha \geq 2$. Below we assume everywhere that $\alpha = n$. Hence, we drop the symbol $\alpha = n$ when unnecessary. For instance, we denote $N_n(v)$ simply by $N(v)$, and so on.

Following [4, equation (2.1)], we set

$$N(v) = \Big(\int_\Omega |\nabla v|^2 \, |v|^{n-2} \, dx \Big)^{1/n}$$

$$M(v) = \Big(\int_\Omega \big| \nabla |v|^{n/2} \big|^2 \, dx \Big)^{1/n} \tag{31}$$

Note that

$$M(v) \leq N(v) \,. \tag{32}$$

As in [4] and [1], the above two quantities play a leading role here.

The following lemma was shown in [1, Lemmas 1.1 and 1.2], and recovered in [4, Lemma 2.1]. As already remarked, related estimates were obtained in the previous reference [11], to which the reader is referred.

Lemma 3.1. *Let* $(v,\, p)$ *be a regular solution to problem* (1) *in* $\Omega \times [0,\, T]$. *Then*

$$\frac{1}{n}\,\frac{d}{dt}\,\|v\|_n^n + \frac{\mu}{2}\,N^n(v) + 4\mu\,\frac{n-2}{n^2}\,M^n(v) \leq$$

$$\frac{(n-2)^2}{2\mu} \int_\Omega p^2\,|v|^{n-2}\,dx + \|\, f\,\|_n \,\|\, v\,\|_n^{n-1} \,. \tag{33}$$

The next result is an immediate consequence of a Sobolev's inequality, see the Lemma 2.2 in reference [4].

Lemma 3.2. *Let* $|v|^{n-2}$ *belong to* H^1_0 *(so,* v *vanishes on the boundary). Then*

$$\|v\|^n_{\frac{n^2}{n-2}} \leq c_0\, M^n(v) \,. \tag{34}$$

For convenience we set

$$B \equiv \int_\Omega p^2\,|v|^{n-2}\,dx \,. \tag{35}$$

Clearly,

$$B \leq \int_\Omega p^2\,(1+|v|)^{n-2}\,dx = \int_\Omega P^2\,V^{n+2(\theta-1)}\,dx \,, \tag{36}$$

where

$$P \equiv \frac{|p|}{(1+|v|)^\theta}\,, \quad V \equiv 1+|v| \,. \tag{37}$$

By appealing to Hőlder's inequality with exponents $q/2$ and $q/(q-2)$, where $q \in (2, +\infty)$, one gets

$$B \leq \|P\|_q^2 \tag{38}$$

where

$$\widehat{n} \equiv n + 2(\theta - 1), \tag{39}$$

and $0 \leq \theta \leq 2$.

Lemma 3.3. *Set*

$$\beta = \frac{n}{2} - n\,\frac{n}{\widehat{n}}\Big(\frac{1}{2} - \frac{1}{q}\Big). \tag{40}$$

One has $\beta \geq 0$ for all couple $q,\, r \in [2, \infty)$.
On the other hand, $\beta \leq 1$ if $q \geq n$. If $2 \leq q < n$, then $\beta \leq 1$ if and only if r satisfies condition (11).

Proof. Assumption $\beta \geq 0$ is equivalent to

$$\frac{\widehat{n}}{n} \geq 1 - \frac{2}{q}.$$

By appealing to (39) we prove equivalence to the condition

$$\theta \geq 1 - \frac{n}{q}. \tag{41}$$

Further, from (10) it follows that

$$\theta = (1 - \frac{2}{r}) + (1 - \frac{n}{q}) \geq (1 - \frac{n}{q}), \tag{42}$$

since $r \geq 2$. This shows (41).

On the other hand one easily shows that assumption $\beta \leq 1$ is equivalent to

$$\frac{1}{2} - \frac{1}{n} \leq \frac{n}{\widehat{n}}\,(\frac{1}{2} - \frac{1}{q}).$$

By appealing to (39), straightforward calculations show equivalence to the condition

$$\theta \leq 1 + n\,\frac{(q-n)/\,q}{n-2}. \tag{43}$$

If $q \geq n$ this condition is obviously verified.

Assume now that $q < n$. By appealing to the equality relation stated in (42), straightforward calculations show equivalence between (43) and

$$\frac{1}{r} \geq \frac{1}{n-2}\,\frac{n-q}{q}.$$

Since the right hand side is negative, this is equivalent to assumption (11). ☐

The next interpolation result is an extension of a similar result proved in [4].

Lemma 3.4. *Let* β *be defined by* (40). *Then*

$$B \leq \|P\|_q^2 \, \|V\|_n^{\widehat{n}\,(1-\beta)} \, \|V\|_{\frac{n\,n}{n-2}}^{\widehat{n}\,\beta} \,. \tag{44}$$

Furthermore, for $r \in (2, +\infty)$ *and for arbitrary positive values of* ϵ*, one has*

$$B \leq \epsilon^{-\frac{r}{2}} \, \|P\|_q^r \, \|V\|_n^{\widehat{n}\,(1-\beta)\frac{r}{2}} + \epsilon^{\frac{r}{r-2}} \, \|V\|_{\frac{n^2}{n-2}}^{\widehat{n}\,\beta\,\frac{r}{r-2}} \,. \tag{45}$$

Proof. The term $\|P\|_q^2$ has no rule here. We merely interpolate in equation (38) the norm $\|V\|_{\frac{q\,\widehat{n}}{q-2}}$ between $\|V\|_n$ and $\|V\|_{\frac{n^2}{n-2}}$, and note that the above value of β satisfies the equation

$$\frac{q-2}{q\,\widehat{n}} = \frac{(1-\beta)}{n} + \frac{\beta}{n^2/(n-2)} \,.$$

Hence,

$$\|V\|_{\frac{q\,\widehat{n}}{q-2}} \leq \|V\|_n^{1-\beta} \, \|V\|_{\frac{n^2}{n-2}}^{\beta} \,. \tag{46}$$

The estimate (45) follows from (44) by Hőlder's inequality with exponents $r/2$ and $r/(r-2)$. ☐

The next lemma also holds for $0 \leq \theta \leq 2$.

Lemma 3.5. *Let be* $0 \leq \theta \leq 1$, *and* $q, r > 2$. *Then*

$$\widehat{n}\,\beta\,\frac{r}{r-2} = n \,. \tag{47}$$

Proof. By setting

$$\lambda \equiv \frac{\widehat{n}}{n} = 1 - \frac{2\,(1-\theta)}{n} \,, \tag{48}$$

equation (47) reads

$$\lambda\,\beta = 1 - \frac{2}{r} \,. \tag{49}$$

By (40) it readily follows that (49) is equivalent to

$$\lambda = (1 - \frac{2}{q}) + (1 - \frac{2}{r})\,\frac{2}{n} \,.$$

By appealing to the expression of λ shown in the right hand side of (48) it follows, after some straightforward calculations, that equation (47) is equivalent to (10). $\square$

Next we want to replace the second exponent in the right hand side of equation (45) simply by n.

Lemma 3.6. *One has*

$$\widehat{n}\,(1-\beta)\frac{r}{2} \leq n \tag{50}$$

if and only if

$$\theta \leq 1\,. \tag{51}$$

Proof. By (47) and (48) we get

$$\widehat{n}\,(1-\beta)\,\frac{r}{2} = n\left(\frac{r}{2}(\lambda-1)+1\right). \tag{52}$$

It readily follows that (50) is equivalent to $\lambda \leq 1$. Next, by the expression of λ in (48), we show that (50) holds if and only if $\theta \leq 1$. $\square$

The next result follows from equations (53) and (45).

Proposition 3.1. *Let be* $0 \leq \beta \leq 1$, $q, r > 2$, $0 \leq \theta \leq 1$. *Then*

$$\begin{gathered}\frac{1}{n}\frac{d}{dt}\|v\|_n^n + \frac{\mu}{2}\,N^n(v) + 4\mu\,\frac{n-2}{n^2}\,M^n(v) \leq \\ c_0\,\epsilon^{-\frac{r}{2}}\,\|P\|_q^r\,\|V\|_n^n + c_0\,\epsilon^{\frac{r}{r-2}}\,\|V\|^n_{\frac{n^2}{n-2}} + \|\,f\,\|_n\,\|\,v\,\|_n^{n-1}\,,\end{gathered} \tag{53}$$

where $c_0 = \frac{n-2)^2}{2\mu}$.

In what follows by the the symbol C we denote positive constants, that may depend on n, μ, r and q. The values of these constants can be easily estimated.

Since $\|V\|_s \leq |\Omega|^{1/s} + \|v\|_s$, for arbitrary exponents s, we may replace the $\|V\|$ norms by $\|v\|$ norms, up to not significant terms. Furthermore, due to (34), and by choosing a sufficiently small ϵ, we may drop the term $\|v\|^{\alpha}_{\frac{\alpha\,n}{n-2}}$ on the right hand side of (53). It follows that

$$\begin{gathered}\frac{d}{dt}\|v\|_n^n + \|v\|^n_{\frac{n^2}{n-2}} + N^n(v) + M^n(v) \leq C\,\|P\|_q^r\,\|v\|_n^n + C\,\|\,f\,\|_n\,\|\,v\,\|_n^{n-1} \\ + C\,\|P\|_q^r\,|\Omega| + C\,|\Omega|^{\frac{n-2}{n}}\,.\end{gathered} \tag{54}$$

This estimate generalizes [[4], estimate (2.16)] to values $\theta < 1$. Theorem 1.1 follows easily.

Proof. The proof of the Theorem, based on the estimate (54), follows a standard way. We start by appealing to Gronwall's lemma, supported by assumption (7), which shows that $\|P(t)\|_q^r \in L^1(0, T)$. This leads to the first claim in (12). The second claim follows from the integration in $(0, T)$ of the second term on the right hand side of (54). Equation (13) follows from the integration in time of the $M(v)$ term. Smoothness of solutions follows by simply appealing to

$$v \in L^n(0, T; L^{\frac{n^2}{n-2}}(\Omega)),$$

where

$$\frac{2}{n} + \frac{n}{\frac{n^2}{n-2}} = 1.$$

Since

$$\frac{n^2}{n-2} > n,$$

the "classical" LPS condition (2) shows smoothness. □

4 Remarks on the case $\theta > 1$.

Let us make some consideration concerning the case $\theta > 1$. Note that in references [2], [3], and [4], one always has assumed $\theta \leq 1$. If $\theta > 1$ there is no evidence of a positive answer to Problem 1.1. Actually, the constraint (51) imposed in Lemma 3.6 goes in the direction of a negative answer to the equivalence $p = v^2$. Furthermore, the interesting results stated in [8] still go in the direction of the same negative reply. Since the proofs follow by closely using ideas introduced in reference [4], we guess that the results are the best possible attainable by the present method. In reference [8], Theorem 1, point (H3), the author shows ($n = 3$) that if the solution v satisfies (7) for some $\theta \in [1, 5/3]$, where

$$\frac{2}{r} + \frac{3}{q} = \frac{5}{2} - \frac{3}{2}\theta, \tag{55}$$

with

$$\frac{6}{5-3\theta} < q \leq \infty \tag{56}$$

then the solution is regular. For $\theta = 1$ this result is in agreement with [4]. However, for $\theta > 1$ it is weaker than the result proposed in Problem 1.1 since the right hand side of (56) is strictly smaller then that of (8). It would be of interest to have an explanation to this "loss of regularity".

5 Appendix.

Assume that the triad (q, r, θ) satisfies assumptions (10) and (11), and let (q_1, r_1, θ_1) be a triad such that $q_1 \geq q$, $r_1 \geq r$, and $\theta_1 \leq \theta$. It follows that if a solution (v, p) satisfies

$$\frac{|p|}{(1 + |v|)^{\theta_1}} \in L^{r_1}(0, T; L^{q_1}(\Omega)),$$

then it is regular, since

$$\frac{|p|}{(1 + |v|)^{\theta}} \leq \frac{|p|}{(1 + |v|)^{\theta_1}} \in L^{r_1}(0, T; L^{q_1}(\Omega)) \subset L^{r}(0, T; L^{q}(\Omega)).$$

This situatioin corresponds to having equation (10) with the sign "=" replaced by the sign "≤". However this regularity result is not sharp. Actually, it can be obtained as a consequence of sharp results and related proofs.

Let's show a simple example concerning an application of condition (11). Assume the main case $n = 3$. The values $q = 5/2$ (note that $q < n$), $r = 10$, $\theta = 3/5$, verify (10) but not (11). This last condition requires $r \leq 5$. So, if we have in hands a $\theta = 3/5$ case, we may assume, for instance, the above value $r = 5$, by setting $q = 3$. If we want $q = 5/2$ $(q < n)$ we may choose $r = 5/2$, $\theta = 3/2$.

References

[1] H. Beirão da Veiga, *Existence and asymptotic behaviour for strong solutions of the Navier–Stokes equations in the whole space,* Indiana Univ. Math. J., **36** (1987), 149–166.

[2] H. Beirão da Veiga, *Concerning the regularity of the solutions to the Navier–Stokes equations via the truncation method; Part I*, Diff. Int. Eq., **10** (1997), 1149–1156.

[3] H. Beirão da Veiga, *Concerning the regularity of the solutions to the Navier–Stokes equations via the truncation method. Part II*, Équations

aux Dérivées Partielles et Applications; Articles dédiés à J.L. Lions à l'occasion de son 70. anniversaire; Gauthier-Villars, Paris (1998), 127-138.

[4] H. Beirão da Veiga, *A sufficient condition on the pressure for the regularity of weak solutions to the Navier–Stokes equations*, J.Math. Fluid Mech. **2** (2000), 99–106.

[5] L.C. Berselli and R. Manfrin, *On a theorem of Sohr for the Navier–Stokes equations*, J. Evol. Eq., **4** (2004), 193–211.

[6] L.C. Berselli and G.P. Galdi, *Regularity criteria involving the pressure for weak solutions to the Navier–Stokes equations*, Proc. Am. Math. Soc., **130** (2002), 3585–3595.

[7] S. Kaniel, *A sufficient condition for smoothness of solutions of Navier–Stokes equations*, Israel J.Math., **6** (1969), 354–358.

[8] Y. Zhou, *Regularity criteria in terms of pressure for the $3-D$ Navier–Stokes equations*, Math. Ann., **328** (2004), 173–192.

[9] G. Seregin and V. Sverak, *Navier-Stokes equations with lower bounds on the pressure*, Arch. Rat. Mech. Anal.**163** (2002), 65–86.

[10] C.L.M.H. Navier, *Memoire sur les lois du mouvement des fluides*,Mem. Acad. Sci. Inst. de France (2),**6** (1823), 389–440.

[11] S. Rionero and G.P. Galdi, *The weight function approach to uniquiness of viscous flows in unbounded domains*, Arch. Rat. Mech. Anal., **69** (1979), 37–52.

[12] H. Sohr, *Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes*, Math. Z. **184** (1983), 359–375.