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On the extension to slip boundary conditions of a Bae and Choe regularity criterion for the Navier–Stokes equations. The half-space case

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A R T I C L E I N F O

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A B S T R A C T

This note concerns the sufficient condition for regularity of solutions to the evolution Navier–Stokes equations known in the literature as Prodi–Serrin's condition. H.-O. Bae and H.J. Choe proved in a 1997 paper that, in the whole space $\mathbb{R}^3$, it is merely sufficient that two components of the velocity satisfy the above condition. Below, we extend the result to the half-space case $\mathbb{R}^n_+$ under slip boundary conditions. We show that it is sufficient that the velocity component *parallel* to the boundary enjoys the above condition. Flat boundary geometry is not essential, as shown in a forthcoming paper in cylindrical domains, prepared in collaboration with J. Bemelmans and J. Brand.

1. Introduction

This note concerns sufficient condition for regularity of weak solutions to the evolution Navier–Stokes equations related to the so called Prodi–Serrin's condition, see (3). Weak solutions are characterized by

$$u \in L^\infty(0, T; L^2(\Omega) \cap L^2(0, T; H^1(\Omega))\,.$$

Weak solutions are assumed to be weakly continuous with values in $L^2(\Omega)$.

In reference [1] the authors proved, in the whole space case, that it is sufficient that two components of the velocity satisfy the above condition (there are also similar results concerning two components of the vorticity, see [9]). Below we extend the result proved in reference [1] to the half-space case $\mathbb{R}^n_+$ under slip boundary conditions. However, the choice of the components to be controlled is not arbitrary. We have to consider the components parallel to the boundary.

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The structure of the proof follows Bae and Choe paper, by adding a suitable control of some boundary integrals which, clearly, were not present in the whole space case.

We assume that readers are acquainted with the main literature on the subject. In particular, we will not repeat well know notation as, for instance, Sobolev spaces notation, and so on.

In the sequel we are interested in the evolution Navier–Stokes equations in the half-space $\mathbb{R}^n_+ = \{x : x_n > 0\}$, $n \geq 3$,

$$\begin{cases} \partial_t\, u + (u\cdot\nabla)\, u - \mu \triangle u + \nabla\pi = 0\,, \\ \nabla\cdot u = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,T]\,; \\ u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^n_+\,, \end{cases} \tag{1}$$

under the classical Navier slip boundary conditions without friction. See [16] and [18]. On flat portions of the boundary this condition reads

$$\begin{cases} u_n = 0, \\ \partial_n\, u_j = 0, \quad 1 \leq j \leq n-1\,. \end{cases} \tag{2}$$

In the half-space case we will use this formulation. Let us recall that, for $n = 3$, the above slip boundary condition may also be written in the form $u_n = 0$, plus $\omega_j = 0$, for $j = 1, 2$, where $\omega = \nabla \times u$ is the vorticity field.

It is well know that weak solutions u satisfying the so called Prodi–Serrin's condition

$$u \in L^q(0,\, T;\, L^p(\mathbb{R}^n_+)\,)\,, \qquad \frac{2}{q} + \frac{n}{p} \leq 1\,, \qquad p > n \tag{3}$$

are strong, namely

$$u \in L^\infty(0,\, T;\, H^1(\mathbb{R}^n_+) \cap L^2(0,\, T;\, H^2(\mathbb{R}^n_+)\,)\,. \tag{4}$$

The proof is classical. Furthermore, strong solutions are smooth, if data and domain are also smooth.

It is well known that the above results hold in a very large class of domains Ω, under suitable boundary conditions. We assume this kind of results well known to the reader. In particular, the result is well known in the whole space $\mathbb{R}^n$, which is our departure point. In fact, consider the Navier–Stokes equations (1) with $\mathbb{R}^n_+$ replaced by $\mathbb{R}^n$. Differentiating both sides of the first equation (1) with respect to x_k, taking the scalar product with $\partial_k\, u$, adding over k, and integrating by parts over $\mathbb{R}^n$, one shows that

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla\, u|^2\, dx + \mu \int_{\mathbb{R}^n} |\nabla^2\, u|^2\, dx = -\int_{\mathbb{R}^n} \nabla\,[(u\cdot\nabla)\, u\,]\cdot\nabla\, u\, dx\,, \tag{5}$$

where obvious integrations by parts have been done. Clearly, no boundary integrals appear. A last integration by parts shows that

$$\Big|\int_{\mathbb{R}^n} \nabla\,[(u\cdot\nabla)\, u\,]\cdot\nabla\, u\, dx\Big| \leq\, c(n)\int_{\mathbb{R}^n} |\,u\,|\,|\nabla\, u|\,|\nabla^2\, u|\, dx\,. \tag{6}$$

So

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla\, u|^2\, dx + \mu \int_{\mathbb{R}^n} |\nabla^2\, u|^2\, dx \leq\, c(n)\int_{\mathbb{R}^n} |\,u\,|\,|\nabla\, u|\,|\nabla^2\, u|\, dx \tag{7}$$

follows.

By appealing to the Prodi–Serrin's assumption (3) applied to the term $|\,u\,|$ present in the right hand side of (7), well known devices lead to the desired regularity result (4) in $\mathbb{R}^n$ (these same devices are shown in section 2, in connection with the similar estimate (19)). In the proof, the crucial property is that the term $|\,u\,|$ in the right hand side of estimate (6) (hence also in (7)) enjoys the Prodi–Serrin's condition. In reference [1], see also [2], H.-O. Bae and H.J. Choe succeed in replacing, in the right hand side of (6), the term $|u|$ simply by $|\overline{u}|$, where $\overline{u}$ is an arbitrary $n-1$ dimensional component

$$\overline{u} = (u_1, \,\ldots,\, u_{n-1},\, 0) \tag{8}$$

of the velocity u. In other words, they succeed in improving the quite obvious estimate (6), by showing the much stronger estimate

$$\Big| \int_{\mathbb{R}^n} \nabla\,[(u\cdot\nabla)\,u\,]\cdot\nabla\,u\,dx\Big| \leq\, c(n) \int_{\mathbb{R}^n} |\overline{u}|\,|\nabla\,u|\,|\nabla^2\,u|\,dx\,. \tag{9}$$

Hence the estimate (7) holds with $|\,u|$ replaced by $|\,\overline{u}|$. The classical $|\,u|$-proof applies as well after this substitution. In this way the authors proved that (4) holds if merely $\overline{u}$ (instead of u) satisfies the Prodi–Serrin's condition. A quite unexpected result, at that time, may be not yet sufficiently exploited. Clearly, in the whole space case, $\overline{u}$ may be any $n-1$ dimensional component of the velocity.

The proof of the estimate (9) is based on a clever analysis of the structure of the integral on the left hand side of this equation.

The first aim of these notes is to prove equation (9) in the half space $\mathbb{R}^n_+$

$$\Big| \int_{\mathbb{R}^n_+} \nabla\,[(u\cdot\nabla)\,u\,]\cdot\nabla\,u\,dx\Big| \leq\, c(n) \int_{\mathbb{R}^n_+} |\overline{u}|\,|\nabla\,u|\,|\nabla^2\,u|\,dx\,, \tag{10}$$

under slip boundary conditions. As a consequence, the estimate (7) holds with $|u|$ replaced by $|\overline{u}|$ and $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$. It readily follows, as in the classical case, that solutions to the above boundary value problem are regular provided that $\overline{u}$ satisfies the Prodi–Serrin's condition (3).

Theorem 1.1. *Let u be a solution to the Navier–Stokes equations (1) in $\mathbb{R}^n_+$ under the slip boundary conditions (2). Furthermore, let $\overline{u}$ be the parallel to the boundary component of the velocity u, given by (8). If*

$$\overline{u} \in\, L^q(0,\,T;\,L^p(\mathbb{R}^n_+)\,)\,, \quad \frac{2}{q} + \frac{n}{p} \leq 1\,, \quad p > \,n\,, \tag{11}$$

then (4) holds.

Alternatively, the proof of the above result could be done by appealing to a reflection principle, see [7]. However this does not help extension to non-flat boundaries.

Concerning Prodi–Serrin's condition under slip boundary conditions we recall here references [2] and [3]. We end this section by quoting the very recent paper [4] where the authors proved the local, interior, regularizing effect of the Prodi–Serrin's condition only on two velocity components. It would be of interest to extend this result to arbitrary, smooth, coordinates (orthogonal for instance).

2. Extension to boundary value problems

In this section we prove equation (10). Our approach adds to that followed in reference [1] an accurate control of the boundary integrals, clearly not present in the whole space case. To obtain the explicit form of these integrals, we have to turn back to the volume integrals.

For notational convenience we set

$$\Gamma = \{ x : x_n = 0 \} .$$

The first steep is to prove (5), now with $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$, namely

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx + \mu \int_{\mathbb{R}^n_+} |\nabla^2 u|^2 \, dx = - \int_{\mathbb{R}^n_+} \nabla [(u \cdot \nabla) u] \cdot \nabla u \, dx . \tag{12}$$

By following the $\mathbb{R}^n$ case, we differentiate both sides of the first equation (1) with respect to x_k, take the scalar product with $\partial_k u$, add over k, and integrate by parts over $\mathbb{R}^n_+$. Now additional boundary integrals appear. We start from the $\triangle u$ term. One has

$$- \int_{\mathbb{R}^n_+} \nabla(\triangle u) \cdot \nabla u \, dx \equiv - \int_{\mathbb{R}^n_+} \partial_k(\partial_i^2 u_j) \, \partial_k u_j \, dx = \int_{\mathbb{R}^n_+} |\nabla^2 u|^2 \, dx - I$$

where

$$I \equiv \int_\Gamma (\partial_i \partial_k u_j)(\partial_k u_j) \nu_i \, d\Gamma = - \int_\Gamma (\partial_k \partial_n u_j)(\partial_k u_j) \, d\Gamma ,$$

since ν, the unit external normal to Γ, has components $(0, ..., 0, -1)$. If $j < n$ and $k = n$ the terms $\partial_k u_j$ vanish, due to the boundary conditions (2). If $j < n$, but $k < n$, the terms $\partial_k \partial_n u_j$ vanish, since $\partial_n u_j = 0$ on the boundary, and ∂_k is a tangential derivative. Hence we merely have to consider the $j = n$ terms, namely $(\partial_n \partial_k u_n) \partial_k u_n$. If $k < n$, it follows $\partial_k u_n = 0$. On the other hand, if $k = n$, by appealing to the divergence free condition, one has

$$\partial_n \partial_n u_n = - \sum_{j<n} \partial_n (\partial_j u_j) = 0 , \tag{13}$$

since $\partial_j(\partial_n u_j) = 0$ on Γ, for $j < n$. We have shown that

$$- \mu \int_{\mathbb{R}^n_+} \nabla(\triangle u) \cdot \nabla u \, dx = \mu \int_{\mathbb{R}^n_+} |\nabla^2 u|^2 \, dx ,$$

the boundary integral related to the viscous term vanishes.

Next we consider the pressure term. One has, by an integration by parts,

$$\int_{\mathbb{R}^n_+} (\nabla(\nabla \pi)) \cdot \nabla u \, dx \equiv \int_{\mathbb{R}^n_+} \partial_k(\partial_j \pi) \, \partial_k u_j \, dx = - \int_{\mathbb{R}^n_+} (\nabla \pi) \cdot \nabla (\nabla \cdot u) \, dx + A,$$

where

$$A \equiv \int_\Gamma (\partial_k \pi)(\partial_k u_j) \nu_j \, d\Gamma = - \int_\Gamma (\partial_k \pi) \partial_k u_n \, d\Gamma = - \int_\Gamma (\partial_n \pi)(\partial_n u_n) \, d\Gamma ,$$

since $\partial_k u_n = 0$ on the boundary for $k < n$. Furthermore, the volume integral on the right hand side vanishes, due to the divergence free condition.

Let's see that $A = 0$ by showing that $\partial_n \pi = 0$ on Γ. By appealing to the nth equation (1) we show that $\partial_n \pi = -\partial_t u_n - (u \cdot \nabla) u_n + \mu \triangle u_n$. So, by appealing to boundary condition $u_n = 0$, one easily shows that

$$\partial_n \pi = \mu \triangle u_n \quad \text{on}\, \Gamma\,.$$

Note that $(u \cdot \nabla) u_n = 0$ on Γ. By taking into account that the second order tangential derivatives of u_n vanish on the boundary, we show that $\triangle u_n = 0$, by appealing to (13). So $\partial_n \pi = 0$ on Γ, as desired. We have shown that

$$\int_{\mathbb{R}^n_+} (\nabla(\nabla \pi)) \cdot \nabla u \, dx = 0\,. \tag{14}$$

Equation (12) is proved.

Note that equation (14) holds under the non-slip boundary condition, with a simpler proof. In fact, in this case, $A = 0$ follows immediately from $\partial_n u_n = 0$ on Γ, which is an immediate consequence of the divergence free property and the non-slip boundary assumption.

The next, and main, step is to consider the non-linear term. We start by showing that

$$\int_{\mathbb{R}^n_+} \nabla\,[(u \cdot \nabla)\, u] \cdot \nabla\, u \, dx = \int_{\mathbb{R}^n_+} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\, dx\,. \tag{15}$$

This follows from the identity

$$\nabla\,[(u \cdot \nabla)\, u] \cdot \nabla\, u = (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) + \frac{1}{2}\, u_i\, \partial_i \Big(\sum_{j,\,k} (\partial_k u_j)^2 \Big) \tag{16}$$

since, by an integration by parts, we show that the integral of the second term on the right hand side of the (16) vanishes, as follows from the divergence free and the tangential to the boundary properties (unfortunately, in the cylindrical coordinates case, the counterpart of this main point is much more involved).

Next we prove the main estimate (10). Following [1], we consider separately the three cases $i \neq n$; $i = n$ and $j \neq n$; $i = j = n$.

If $i \neq n$, one has

$$\begin{aligned} &\int_{\mathbb{R}^n_+} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\, dx = \\ &- \int_{\mathbb{R}^n_+} u_i\, \partial_k \big((\partial_k u_j)(\partial_i u_j) \big)\, dx + \int_{\Gamma} u_i\, (\partial_k u_j)(\partial_i u_j)\, \nu_k\, dx \end{aligned} \tag{17}$$

The boundary integral is equal to

$$- \int_{\Gamma} u_i\, (\partial_n u_j)(\partial_i u_j)\, dx\,.$$

If $j \neq n$, one has $\partial_n u_j = 0$. If $j = n$, one has $\partial_i u_j = 0$, since ∂_i is a tangential derivative and $u_n = 0$. Hence the boundary integral in equation (17) vanishes. On the other hand, since $i \neq n$, the volume integral on the right hand side of equation (17) is bounded by the right hand side of inequality (10). After all, if $i \neq n$, the left hand side of equation (17) is bounded by the right hand side of inequality (10).

Next we assume that $i = n$ and $j \neq n$. In this case, by an integration by parts, one gets

$$\begin{aligned} &\int_{\mathbb{R}^n_+} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j)\, dx = \\ &- \int_{\mathbb{R}^n_+} (\triangle u)(\partial_i u_j)\, u_j\, dx \\ &- \int_{\mathbb{R}^n_+} (\partial_k u_n)(\partial_i \partial_k u_j)\, u_j\, dx\,, \end{aligned} \tag{18}$$

since the boundary integral, which appears after the above integration by parts, vanishes. In fact, the terms $(\partial_i u_j)$ vanish on the boundary, for $i = n$ and $j \neq n$. From (18) it follows that the left hand side of this equation is bounded by the right hand side of inequality (10), as desired.

If $i = j = n$, we have to estimate the integral

$$B \equiv \int_{\mathbb{R}^n_+} (\partial_k u_n)^2 (\partial_n u_n)\, dx = - \int_{\mathbb{R}^n_+} (\partial_k u_n)^2 (\sum_{j \neq n} \partial_j u_j)\, dx\,.$$

By integration by parts one gets

$$B = 2 \int_{\mathbb{R}^n_+} (\partial_k u_n)(\sum_{j \neq n} \partial_j \partial_k u_n)\, u_j\, dx - \int_{\Gamma} (\partial_k u_n)^2 \sum_{j \neq n} u_j\, \nu_j\, d\Gamma\,.$$

Since the above boundary integral vanishes, the absolute value of B is bounded by the right hand side of inequality (10). The proof of (10) is accomplished. From now on the proof of Theorem 1.1 follows a very classical way. For the readers' convenience we recall how to prove (5). From (12) and (10) it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n_+} |\nabla u|^2\, dx + \mu \int_{\mathbb{R}^n_+} |\nabla^2 u|^2\, dx \leq c(n)\, \|\, |\overline{u}|\, \nabla u\, \|_2\, \|\, \nabla^2 u\, \|_2\,. \tag{19}$$

On the other hand, by Hőlder's inequality,

$$\|\, |\overline{u}|\, \nabla u\, \|_2 \leq \|\, \overline{u}\, \|_p\, \|\, \nabla u\, \|_{\frac{2p}{p-2}}\,.$$

Furthermore, by interpolation and Sobolev's embedding theorem,

$$\|\, \nabla u\, \|_{\frac{2p}{p-2}} \leq \|\, \nabla u\, \|_2^{1-\frac{n}{p}}\, \|\, \nabla u\, \|_{2^*}^{\frac{n}{p}} \leq c\, \|\, \nabla u\, \|_2^{1-\frac{n}{p}}\, \|\, \nabla^2 u\, \|_2^{\frac{n}{p}}\,,$$

since $(p-2)/(2p) = (1 - n/p)/2 + (n/p)/2^*$. Here $2^* = 2n/(n-2)$ is a well known Sobolev's embedding exponent (note that each single component of the tensor ∇u satisfies an homogeneous, Dirichlet or Neumann, boundary condition on Γ). Consequently,

$$\|\, |\overline{u}\,|\, \nabla u\, \|_2\, \|\, \nabla^2 u\, \|_2 \leq c\, \|\, \overline{u}\, \|_p\, \|\, \nabla u\, \|_2^{1-\frac{n}{p}}\, \|\, \nabla^2 u\, \|_2^{1+\frac{n}{p}}\,.$$

Hence, by Young's inequality,

$$\|\, |\overline{u}\,|\, \nabla u\, \|_2\, \|\, \nabla^2 u\, \|_2 \leq c \leq c\, \|\, \overline{u}\, \|_p^q\, \|\, \nabla u\, \|_2^2 + (\mu/2)\, \|\, \nabla^2 u\, \|_2^2\,. \tag{20}$$

From (19) and (20) we get, for $t \in (0, T]$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \frac{\mu}{2}\|\nabla^2 u\|_2^2 \leq c\,\|\,\overline{u}\,\|_p^q\,\|\,\nabla u\,\|_2^2\,. \tag{21}$$

This estimate immediately leads to (5) since, by the Prodi–Serrin's assumption,

$$\|\,\overline{u}\,\|_p^q \in L^1(0, T)\,.$$

3. Some remarks on the limit case $(q, p) = (\infty, n)$

The Prodi–Serrin's condition for $(q, p) = (\infty, n)$, namely

$$u \in L^\infty(0, T; L^n(\Omega)\,)\,, \tag{22}$$

always deserves a separate treatment. Before referring a couple of known regularity results merely under the above assumption for $n-1$ components

$$\overline{u} \in L^\infty(0, T; L^n(\Omega)\,)\,, \tag{23}$$

which is the aim of this paper, it looks necessary to say some words about results under the full (22). For long time authors tried to prove that assumption (22) by itself was sufficient to guarantee regularity of solutions. Only very recently, in the famous article [12], the authors succeed in proving that, in the whole space case, the assumption (22) guarantees smoothness of solutions. Further, extension to the boundary has been obtained, see [17] for the half-space case, and [15] for curved smooth boundaries. This problem was, for a long time, one of the most challenging, and difficult, open problems in the mathematical theory of Navier–Stokes equations. This situation led to many unsuccessful attempts to solve it and, consequently, to an extremely wide literature on results under related, but stronger, assumptions. It is completely out of our aim here to go inside this literature. We just refer the classical references [19] where uniqueness of solutions was proved under assumption (22), and [13] and [20] where strong regularity was proved by assuming left time-continuity in $L^n(\Omega)$.

After the above digression, we turn back to references where (23) is assumed, namely [6] and [8]. In both cases $\Omega = \mathbb{R}^n$. In reference [6] it was shown that sufficiently small left-discontinuities on the norm $\|\overline{u}(t)\,\|_n$ do not obstruct the regularity of solutions (in other words, they can not exist). Basically, it was proved that there is a positive constant $C(n)$ such that if (23) holds in $(\tau - \epsilon_0, \tau)$, and

$$\limsup_{t\to\,\tau-\,0}\|\overline{u}(t)\|_n^n - \|\overline{u}(\tau)\|_n^n \leq C(n)\,\mu^n\,, \tag{24}$$

then the solution u is smooth in $(\tau - \epsilon, \tau + \epsilon)$, for some $\epsilon > 0$. Note that the left hand side is necessarily larger or equal to zero.

Essentially, the above statement is equivalently to saying that the solution is smooth in $(0, T]$ if

$$\sup_{\tau\in(0,\,T]}\Big(\big(\limsup_{t\to\,\tau-\,0}\|\overline{u}(t)\|_n^n\big) - \|\overline{u}(\tau)\|_n^n\Big) \leq C(n)\,\mu^n\,. \tag{25}$$

Since

$$\|\overline{u}(t)\|_n^n - \|\overline{u}(\tau)\|_n^n \leq n\,\|\overline{u}\|_{L^\infty(\,L^n(\Omega)\,)}\left(\|\overline{u}(t)\|_n - \|\overline{u}(\tau)\|_n\right),$$

we may replace in the above inequalities (with obvious adaptations)

$$\limsup_{t\to\,\tau-\,0}\|\overline{u}(t)\|_n^n - \|\overline{u}(\tau)\|_n^n$$

by

$$\limsup_{t\to\,\tau-\,0}\|\overline{u}(t)\|_n - \|\overline{u}(\tau)\|_n\,.$$

Note that $\|u - v\|$ may even vanish for arbitrarily large values of $\big|\,\|u\| - \|v\|\,\big|$.

In reference [8] the author replaced the space $L^n(\mathbb{R}^n)$ by the weak-L^n space $L^n_w(\mathbb{R}^n)$ (also called a Marcinkiewicz space), endowed with the canonical quasi-norm

$$[\,v\,]_n \equiv \sup_{\tau>\,0}\,|\,\{x\in\,\Omega\,:\,|v(x)|\geq\,\tau\,\}\,|^{\frac{1}{n}} <\,\infty\,,$$

and essentially proved that there is a positive constant C such that a weak solution u is smooth in $(0,\,T]$ if it satisfies

$$\|\overline{u}\|_{L^\infty(0,\,T;\,L^n_w(\mathbb{R}^n)\,)}\,\leq\,C\,.$$

It would be of interest to extend to the Marcinkiewicz space also a sufficient condition of type (24).

Proofs in reference [6] follow [5], where (23) was replaced by the full condition (22), but solutions live in a bounded domain Ω, under *non-slip* boundary conditions (this result was also proved at that time in [14], by a completely different approach). We recall that in references [5] and [6] all results hold under a quite weak condition, called Assumption A, see below. The proofs under this assumption are elementary. Conditions (22) and (23) are a simple consequence of this more general condition. Assumption A also holds if $\overline{u}\in\,BV(0,\,T;\,L^n)$. We believe that the simple ideas introduced in the context of Assumption A may be technically improved, to obtain stronger results.

We say that a vector field $v(t,\,x)$, satisfies the hypothesis A at time τ with respect to a positive constant Λ if, for some positive constant ϵ_0, there is a real non-negative function $k(t)$, square integrable in $(\tau-\,\epsilon_0,\,\tau\,)$, such that

$$\int_{A(t,\,k(t)\,)}|\,v(t,\,x)\,|^n\,dx\leq\,\Lambda^n \tag{26}$$

for almost all $t\in\,(\tau\,-\,\epsilon_0,\,\tau)$, where

$$A(t\,,k) = \left\{\,x\in\Omega:\,|v(t,x)|\,\geq\,k\,\right\}.$$

It looks superfluous to “inform” readers that many regularity results under assumptions similar to the full condition (22), but with L^n replaced by larger functional spaces, are nowadays well known. Clearly, these results are not contained in [12]. On the other hand it seems quite difficult to extend these results to boundary value problems, like the above non-slip boundary condition. In this direction, very significant results are proved in references [10] and [11].

To end this section we remark that it would be of great interest to extend to the “two components case”, even in the whole space, the main result proved in reference [12].

4. Non-flat boundaries

It is of basic interest to understand how crucial is in the Theorem 1.1 the flat-boundary hypothesis. Does the result hold in the neighborhood of non-flat boundary points? In a previous ArXiv's note we have proposed to start from the following particular case.

Problem 4.1. Let (r, θ, z) be the canonical cylindrical coordinates in the three dimensional space, and consider the subset defined by imposing to r the constraint $\rho_0 < r < \rho_1$, where ρ_0 and ρ_1 are positive constants. Assume the slip boundary condition on the two lateral cylindrical surfaces, and space periodicity with respect to the axial z-direction. To *prove* or *disprove* that Prodi–Serrin's condition on the two "tangential" components u_θ and u_z implies smoothness.

In a forthcoming paper, together with Josef Bemelmans and Johannes Brand, we will give a positive reply to the above problem.

References

[1] H.-O. Bae, H.J. Choe, L^∞-bound of weak solutions to Navier–Stokes equations, in: Proceedings of the Korea–Japan Partial Differential Equations Conference, Taejon, 1996, in: Lecture Notes Ser., vol. 39, Seoul Nat. Univ, Seoul, 1997, 13 pp.

[2] H.-O. Bae, H.J. Choe, A regularity criterium for the Navier–Stokes equations, Comm. Partial Differential Equations 32 (2007) 1173–1187.

[3] H.-O. Bae, H.J. Choe, B.-J. Jin, Pressure representation and boundary regularity of the Navier–Stokes equations with slip boundary condition, J. Differential Equations 244 (2008) 2741–2763.

[4] H.-O. Bae, J. Wolf, A local regularity condition involving two velocity components of Serrin-type for the Navier–Stokes equations, C. R. Acad. Sci. Paris, Ser. I 354 (2016) 167–174.

[5] H. Beirão da Veiga, Remarks on the smoothness of the $L^\infty(0, T; L^3)$ solutions to the $3-D$ Navier–Stokes equations, Port. Math. 54 (1997) 381–391.

[6] H. Beirão da Veiga, On the smoothness of a class of weak solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 2 (2000) 315–323.

[7] H. Beirão da Veiga, F. Crispo, C.R. Grisanti, Reducing slip boundary value problems from the half to the whole space. Applications to inviscid limits and to non-Newtonian fluids, J. Math. Anal. Appl. 377 (2011) 216–227.

[8] L.C. Berselli, A note on regularity of weak solutions of the Navier–Stokes equations in $\mathbb{R}^n$, Jpn. J. Math. 28 (2002) 51–60.

[9] D. Chae, H.-J. Choe, Regularity of solutions to the Navier–Stokes equations, Electron. J. Differential Equations 05 (1999) 1–7.

[10] A. Cheskidov, R. Shvydkoy, The regularity of weak solutions of the $3D$ Navier–Stokes equations in $B^{-1}_{\infty,\infty}$, Arch. Ration. Mech. Anal. 195 (2010) 159–169.

[11] A. Cheskidov, R. Shvydkoy, A unified approach to regularity problems for the $3D$ Navier–Stokes and Euler equations: the use of Kolmogorov's dissipation range, J. Math. Fluid Mech. 16 (2014) 263–273.

[12] L. Escauriaza, G. Seregin, V. Sverak, $L_{3,\infty}$ solutions to the Navier–Stokes equations and backward uniqueness, Russian Math. Surveys 58 (2003) 211–250.

[13] Y. Giga, Solutions for semi-linear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations 62 (1986) 186–212.

[14] H. Kozono, H. Sohr, Regularity criterium on weak solutions to the Navier–Stokes equations, Adv. Differential Equations 2 (2007) 2924–2935.

[15] A.S. Mikhailov, T.N. Shilkin, $L_{3,\infty}$-solutions to the $3-D$-Navier–Stokes system in a domain with curved boundary, J. Math. Sci. 143 (2003) 211–250.

[16] C.L.M.H. Navier, Memoire sur les lois du mouvement des fluides, Mém. Acad. Sci. Inst. Fr. (2) 6 (1823) 389–440.

[17] G. Seregin, On smoothness of $L_{3,\infty}$-solutions to the Navier–Stokes equations up to the boundary, Math. Ann. 332 (2005) 219–238.

[18] J. Serrin, Mathematical principles of classical fluid mechanics, in: C. Truesdell (Ed.), Strömungsmechanik I, in: S. von Flügge (Ed.), Handbuch der Physik, vol. 8/1, Springer-Verlag, Berlin, 1959, pp. 125–263.

[19] H. Sohr, Zur Regularitätstheorie der instationären Gleichungen von Navier–Stokes, Math. Z. 184 (1983) 359–375.

[20] W. von Wahl, Regularity of weak solutions of the Navier–Stokes equations, in: F.F. Browder (Ed.), Proc. Symposia in Pure Mathematics, vol. 45, Amer. Math. Soc., Providence, R.I., 1986, pp. 497–503.