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On some regularity results for the stationary Stokes system and the $2 - D$ Euler equations

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Dedicated to Professor João Paulo de Carvalho Dias on the occasion of his 70th birthday

Abstract. We revisit "minimal assumptions" on the data which guarantee that solutions to the $2 - D$ evolution Euler equations in a bounded domain are classical. Classical means here that all the derivatives appearing in the equations and boundary conditions are continuous up to the boundary. Following a well known device, the above problem led us to consider this same regularity problem for the Poisson equation under homogeneous Dirichlet boundary conditions. At this point, one was naturally led to consider the extension of this last problem to more general linear elliptic boundary value problems, and also to try to extend the results to more general data spaces. Pursuing and developing results that remained unpublished about thirty years, we survey the route followed in the study of these problems and we consider new results and open problems. In particular, we extend some minimal assumption results for the stationary Stokes system, and for and for the planar, evolution, Euler equations, to larger data spaces.

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1. Notation and some helping remarks

In this section, apart from basic notation, we make some *useful remarks* to help the reading of these notes. We start by notation. Ω is an open, bounded, connected set in $\mathbb{R}^n$, $n \geq 2$, locally situated on one side of its boundary Γ. We assume that Γ is of class $C^{2,\lambda}(\overline{\Omega})$, for some positive λ. By $C(\overline{\Omega})$ we denote the Banach space of all real continuous functions in $\overline{\Omega}$ endowed with the classical norm

$$\|f\| = \sup_{x \in \overline{\Omega}} |f(x)|.$$

The classical spaces $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$ are normalized by $\|u\|_1 = \|u\| + \|\nabla u\|$, and $\|u\|_2 = \|u\| + \|\nabla^2 u\|$, with clear notation. Further, for each $\lambda \in (0,1]$, we define the semi-norm

$$[f]_{0,\lambda} \equiv \sup_{x,y\in\Omega;\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\lambda}}, \tag{1.1}$$

and the Hőlder space $C^{0,\lambda}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_{0,\lambda} < \infty\}$, normalized by

$$\|f\|_{0,\lambda} = \|f\| + [f]_{0,\lambda}.$$

In particular, $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions in $\overline{\Omega}$. By $C^\infty(\overline{\Omega})$ we denote the set of all restrictions to $\overline{\Omega}$ of indefinitely differentiable functions in $\mathbb{R}^n$. Boldface symbols refer to vectors, vector spaces, and so on. Components of a generic vector $\boldsymbol{u}$ are indicated by u_i, with similar notation for tensors. Norms in function spaces, whose elements are vector fields, are defined in the usual way by means of the corresponding norms of the components.

The symbols $c, c_0, c_1, \ldots,$ denote positive constants depending at most on Ω and n. We may use the same symbol to denote different constants.

Next we make some useful remarks, preliminary to the reading of these notes. Concerning the space dimension n, it is worth noting, once and for all, that in all the results stated for the Euler equations the assumption $n = 2$ is strictly necessarily. On the contrary, all the results stated for elliptic equations hold for arbitrarily large values of n, at most with trivial modifications. Clearly, in reference [4], since the matter were the Euler equations, elliptic regularity results were of interest only for $n = 2$. Concerning this point, the following is an explanatory example. Theorems 2.2 and 2.3 hold independently of the space dimension. However, the resolution of problem (2.4) by appealing to that of (2.9), shown in [4] and recalled below, is strictly restricted to dimension $n = 2$. Concerning planar motions, in treating this particular situation we appeal to the distinct notation Curl to indicate the curl of a scalar. To readers which want to consult reference [4] we note that in this last reference we have appealed to the notation rot-Rot instead of notation curl-Curl, followed here.

In these notes, in treating Stokes and other elliptic problems, to simplify notation, and also due to the author's taste, we may assume that $n = 3$, as done, for instance, in the proof of Theorem 7.1. Proofs immediately apply to any dimension $n \geq 2$, with standard modifications. Compare (5.4), where n may be arbitrarily large, with (5.7), where $n = 3$. Extension to arbitrary large dimensions of (5.7) is well known (for $n = 2$, the first equation gets the typical logarithmic form).

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Center", University of Wisconsin-Madison, (October 1981–March 1982), and at the University of Minnesota–Minneapolis (March 1982–June 1982). The above paper appears first in the 1982 reference [3]. The author is grateful to Professor Robert E. L. Turner for the above invitation to Madison. The author would also like to take this occasion to thank Robert (Bob) for the continuous help in correcting the English of many papers (not the present version of this one), together with mathematical advice and remarks.

2. Preliminaries

To abbreviate, we say that solutions of stationary or evolution problems are *classical* if all derivatives appearing in equations and boundary conditions are continuous up to the boundary on their domain of definition. Furthermore, seeking for "minimal assumptions" on the data which guarantee that solutions to a specific stationary, or evolution, problem are classical, is called here *the minimal assumptions problem*.

This note originates from reference [4], where the main goal was looking for *minimal assumptions* on the data which guarantee *classical* solutions to the $2-D$ Euler equations in a bounded domain

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\pi = \boldsymbol{f} \quad \text{in } Q \equiv \mathbb{R}^+ \times \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } Q; \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \mathbb{R}\times\Gamma; \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \end{cases} \tag{2.1}$$

For simplicity, every time we are concerned with $2-D$ Euler equations, we assume that the domain Ω is simply connected.

The resolution of this problem appealed, in particular, to the minimal assumptions requirement for the elliptic boundary value problem (2.9) below. This opens the way to the study of the minimal assumptions problem for elliptic equations and systems (like Stokes, for instance). Here, pursuing and developing results that remain unpublished about thirty years, we survey the route followed in the study of these problems, and consider new results and open questions. We like to remark that the 1984 reference [4] was published in 1982, as a preprint of the well known Mathematics Research Center (MRC) series, see [3]. They totally coincide. They originate from a preliminary, hand written version (still conserved), denoted in the sequel by [UN]. At that time, some results and hints remained unpublished. Recently, we turned back to the above manuscript, and published part of the results and proofs. This origin will be sometimes recalled in the sequel.

After some attempts to individuate the best way to introduce this notes, we decided to begin by showing the development of the main lines followed in reference [4], simulating that we still are trying to solve the main open problem.

By considering $C(\overline{\Omega})$ as the curl's data space, one has the following result, proved in [4], Theorem 1.1. Here $n = 2$.

Theorem 2.1. *Let a divergence free vector field* $\boldsymbol{v}_0$*, tangent to the boundary, satisfy* $\operatorname{curl} \boldsymbol{v}_0 \in C(\overline{\Omega})$*, and let* $\operatorname{curl} \boldsymbol{f} \in L^1(\mathbb{R}^+; C(\overline{\Omega}))$*. Then, the problem* (2.1) *is uniquely solvable in the large,*

$$\operatorname{curl} \boldsymbol{v} \in C(\mathbb{R}^+; C(\overline{\Omega})), \tag{2.2}$$

and the estimate

$$\|\operatorname{curl} \boldsymbol{v}(t)\| \leq \|\operatorname{curl} \boldsymbol{v}_0\| + \int_0^t \|\operatorname{curl} \boldsymbol{f}(\tau)\| \, d\tau \tag{2.3}$$

holds. If $\operatorname{curl} \boldsymbol{f} = 0$*, then* $\|\operatorname{curl} \boldsymbol{v}(t)\| = \|\operatorname{curl} \boldsymbol{v}_0\|$.

The next step was to replace $\operatorname{curl} \boldsymbol{v}$ by $\nabla \boldsymbol{v}$ in the left hand side of estimate (2.3). Note that the two last equations in the following elliptic system

$$\begin{cases} \operatorname{curl} \boldsymbol{v} = \theta & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{2.4}$$

are still included in (2.1). Furthermore, it is well known that solutions $\boldsymbol{v}$ of problem (2.4) are completely determined by the *scalar* quantity θ (recall the assumption Ω simply-connected), so by $\operatorname{curl} \boldsymbol{v}$. Hence, if one shows that solutions $\boldsymbol{v}$ of problem (2.4) satisfy the estimate $\|\nabla \boldsymbol{v}\| \leq c\|\theta\|$, then we may replace $\operatorname{curl} \boldsymbol{v}$ by $\nabla \boldsymbol{v}$ in the left hand side of estimate (2.3). Unfortunately, this is known to be false. In other words, the data space $C(\overline{\Omega})$ is too wide. On the other hand, Hölder spaces are here too narrow. In fact, in this case (see [13], [15], and also [2], [12]), the above device works, since solutions of (2.4) satisfy the estimate

$$\|\nabla \boldsymbol{v}\|_{0,\lambda} \leq c\|\theta\|_{0,\lambda} \equiv c\|\operatorname{curl} \boldsymbol{v}\|_{0,\lambda}. \tag{2.5}$$

However this estimate is unnecessarily strong in the context of our "minimal assumptions problem". So, we looked for a functional space $C_*(\overline{\Omega})$, *as large as possible*, satisfying the embedding

$$C^{0,\lambda}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset C(\overline{\Omega}), \tag{2.6}$$

and for which the following result holds.

Theorem 2.2. *Let* $\theta \in C_*(\overline{\Omega})$, *and* $\boldsymbol{v}$ *be the solution of problem* (2.4). *Then* $\nabla \boldsymbol{v} \in C(\overline{\Omega})$, *and* $\|\nabla \boldsymbol{v}\| \le c_0 \|\theta\|_*$. *Hence for divergence free, tangent to the boundary, vector fields the estimate*

$$\|\nabla \boldsymbol{v}\| \le c_0 \|\operatorname{curl} \boldsymbol{v}\|_* \tag{2.7}$$

holds.

This result holds for arbitrary dimension. However, to go on, assume $n = 2$. In reference [4] the above result was claimed for a specific space $C_*(\overline{\Omega})$. From now on $C_*(\overline{\Omega})$ denotes this space. To avoid loosing the thread of the argument, definition will be shown later on, in Section 4, to which the reader is referred whenever necessary.

We remark that in Theorem 2.2 the loss of regularity going from the curl to the gradient is deliberately allowed since in the minimal assumptions problem nothing more than continuity should be required to $\nabla \boldsymbol{v}$.

To prove the Theorem 2.2 we showed how to confine the minimal regularity problem for the elliptic system (2.4), treated in Theorem 2.2, to a similar, simpler, regularity problem for equation (2.9) below. A classical argument shows that the solution $\boldsymbol{v}$ of the linear elliptic system (2.4) can be obtained by setting

$$\boldsymbol{v} = \operatorname{Curl} \psi, \tag{2.8}$$

where the scalar field ψ solves the problem

$$\begin{cases} -\Delta \psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{2.9}$$

We appeal here to a typical approach in studying planar motions. For a scalar function $\psi(x)$ (identified here with the third component of a vector field, normal to the plane of motion) we define the vector field $\operatorname{Curl} \psi = (\partial_2 \psi, -\partial_1 \psi)$. For a vector field $\boldsymbol{v} = (v_1, v_2)$ we define the scalar field $\operatorname{curl} \boldsymbol{v} = \partial_1 \boldsymbol{v}_2 - \partial_2 \boldsymbol{v}_1$ (the normal component of the curl). One has $-\Delta = \operatorname{curl} \operatorname{Curl}$. Note that $\operatorname{Curl} \psi$ is the rotation of the gradient $\nabla \psi$ by $\pi/2$ in the counterclockwise direction. Roughly speaking, in the usual three dimensional framework, we have work with three dimensional vectors, namely $(0, 0, \psi)$, and $(v_1, v_2, 0)$. By applying the classical three dimensional curl operator to these vector fields one easily understand the above, classical, simplification.

It follows that solutions $\boldsymbol{v}$ to the system (2.4) belong to $\mathbf{C}^1(\overline{\Omega})$ if the solutions ψ to the system (2.9) belong to $\mathbf{C}^2(\overline{\Omega})$. This situation led us to prove the following result.

Theorem 2.3. *Let* $\theta \in C_*(\overline{\Omega})$ *and let* ψ *be the solution to problem* (2.9). *Then* $\psi \in C^2(\overline{\Omega})$, *moreover,* $\|\psi\|_2 \le c_0 \|\theta\|_*$.

Theorem 2.2 follows immediately from this result, by appealing to the explicit expression (2.8) of the solution $\boldsymbol{v}$ of problem (2.4).

Theorem 2.3 was stated in [4] as Theorem 4.5. Actually, in this last reference the result was claimed for more general linear elliptic boundary value problems. However the proof remained unpublished. Recently, in reference [5], we followed the same lines to obtain a corresponding result for the Stokes system, see Theorem 5.2 bellow (in Section 7 we show a partial extension of this theorem to larger functional spaces).

With Theorem 2.2 in hands, we were still far from our goal since the estimate (2.7) does not fit the estimate (2.3). Roughly speaking, one has to extend this estimate from $C(\overline{\Omega})$ to $C_*(\overline{\Omega})$. This was the more difficult point to reach in reference [4], in particular due to the very weak assumption made in relation to the external forces. We have proved (see Lemma 4.4 in [4]) the following statement, may be the main result in that paper.

Theorem 2.4. *Let $C_*(\overline{\Omega})$ be the Banach space defined in Section 4. Assume that $\operatorname{curl} \boldsymbol{v}_0 \in C_*(\overline{\Omega})$ and $\operatorname{curl} \boldsymbol{f} \in L^1(\mathbb{R}^+; C_*(\overline{\Omega}))$. Then, the curl of the global solution $\boldsymbol{v}$ of problem (2.1) satisfies*

$$\operatorname{curl} \boldsymbol{v} \in C(\mathbb{R}^+; C_*(\overline{\Omega})).$$

Moreover

$$\|\operatorname{curl} \boldsymbol{v}(t)\|_* \leq e^{c_1 B t}(\|\operatorname{curl} \boldsymbol{v}_0\|_* + \|\operatorname{curl} \boldsymbol{f}\|_{L^1(0,t;C_*(\bar{\Omega}))}), \tag{2.10}$$

where

$$B = \|\operatorname{curl} \boldsymbol{v}_0\| + \|\operatorname{curl} \boldsymbol{f}\|_{L^1(0,t;C(\bar{\Omega}))}. \tag{2.11}$$

We advise readers which go back to [4] that in this reference they will find the notation $\zeta = \operatorname{curl} \boldsymbol{v}$, $\phi = \operatorname{curl} \boldsymbol{f}$, and $\zeta_0 = \operatorname{curl} \boldsymbol{v}_0$.

The above result, in the simpler case in which external forces vanish, has been rediscovered, later on, by other authors.

Theorems 2.4 and 2.2 yield the following statement (Theorem 1.4 in [4]).

Theorem 2.5. *Let $C_*(\overline{\Omega})$ be the Banach space defined in Section 4. Further, let $\operatorname{curl} \boldsymbol{v}_0 \in C_*(\overline{\Omega})$ and $\operatorname{curl} \boldsymbol{f} \in L^1(\mathbb{R}^+; C_*(\overline{\Omega}))$. Then, the global solution $\boldsymbol{v}$ to problem (2.1) is continuous in time with values in $\mathbf{C}^1(\overline{\Omega})$,*

$$\boldsymbol{v} \in C(\mathbb{R}^+; \mathbf{C}^1(\overline{\Omega})). \tag{2.12}$$

Furthermore, the estimate

$$\|\boldsymbol{v}(t)\|_{\mathbf{C}^1(\bar{\Omega})} \leq c e^{c_1 B t}\{\|\operatorname{curl} \boldsymbol{v}_0\|_{C_*(\bar{\Omega})} + \|\operatorname{curl} \boldsymbol{f}\|_{L^1(0,t;C_*(\bar{\Omega}))}\} \tag{2.13}$$

holds for all $t \in \mathbb{R}^+$, where B is given by (2.11).

Moreover, $\partial_t \boldsymbol{v}$ and $\nabla \pi$ are continuous in $\overline{Q}$ if both terms $\boldsymbol{f}_0$ and ∇F, in the canonical Helmholtz decomposition $\boldsymbol{f} = \boldsymbol{f}_0 + \nabla F$ satisfy, separately, this same continuity property. So, all derivatives that appear in equations (2.1) *are continuous in $\overline{Q}$* (*classical solution*).

If Ω is not simply connected the results still apply. See the appendix 1 in [4].

3. Further developments and open problems

Study and resolution of the above problems opens the way to new problems. First of all, problems related to the *minimal assumptions problem* for more general elliptic boundary value problems. In [4], the minimal regularity problem for the elliptic system (2.4) was confined to a similar regularity problem for equation (2.9) since Theorem 2.2 was sufficient for our purposes. However, at that time, as remarked in [4], we had proved an extension of this result to more general elliptic boundary value problems (the proof remained unpublished, even though we were not able to find it in the current literature). Further, in a recent paper, we extended the proof to the stationary Stokes system, see Theorem 5.2 below. Similar results hold for more general linear elliptic problems, as the reader may verify, since proofs depend only on the behavior of the associated Green's functions. See Section 5.

Another interesting research field is the extension of the results to larger data spaces. In fact, there may be other significant functional spaces, possibly larger then $C_*(\overline{\Omega})$, satisfying the required properties. An attempt in this direction was done in [UN], where a functional space $B_*(\overline{\Omega})$ was defined and studied. Below, we turn back to this space, and to the even larger space $D_*(\overline{\Omega})$. Unfortunately, we merely obtained partial extensions of the results. As an example of this situation, compare Theorem 5.2 with Theorem 7.1.

Partial extensions of Theorems 2.4 and 2.5 to initial data in the functional space $B_*(\overline{\Omega})$ are shown in Section 8, see Theorems 8.1 and 8.2.

Plan of the paper:

In Section 4 we recall definition and properties of the space $C_*(\overline{\Omega})$ and introduce the new spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$, which satisfy the inclusions

$$C_*(\overline{\Omega}) \subset B_*(\overline{\Omega}) \subset D_*(\overline{\Omega}).$$

In Section 5 we consider Stokes and other elliptic problems with data in $C_*(\overline{\Omega})$.

In Section 6 we consider, in particular, a family of Banach spaces $D^{0,\alpha}(\overline{\Omega})$, a kind of weak extension of the classical Hölder spaces.

In Section 7 the full aim would be to extend the results with data in $C_*(\overline{\Omega})$ to data in the new spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$. Some partial extension results are shown for solutions to the Stokes equations. Second order derivatives of solutions are bounded but, possibly, not continuous. The proofs depend essentially on the properties of the related Green's functions.

In Section 8 we extend the results proved in reference [4] for the Euler equations with data in $C_*(\overline{\Omega})$, to data in $B_*(\overline{\Omega})$. The extension obtained is partial, since continuity is replaced by boundedness, and external forces vanish.

4. The spaces $C_*(\overline{\Omega})$, $B_*(\overline{\Omega})$, and $D_*(\overline{\Omega})$

Spaces $B_*(\overline{\Omega})$, and $D_*(\overline{\Omega})$ will be used later on, however it looks better to introduce these spaces together with $C_*(\overline{\Omega})$ for comparision reasons. The space $C_*(\overline{\Omega})$ was introduced in [4]. Main properties were refered in this reference, however proofs were not included in the paper. For complete proofs see [5].

Set

$$I(x;r) = \{y : |y - x| \le r\}, \qquad \Omega(x;r) = \Omega \cap I(x;r), \qquad \Omega_c(x;r) = \Omega - \Omega(x;r).$$

For $f \in C(\overline{\Omega})$ define

$$\omega_f(x;r) = \sup_{y \in \Omega(x;r)} |f(x) - f(y)|, \qquad \omega_f(r) \equiv \sup_{x,y \in \Omega(x;r)} |f(x) - f(y)|, \tag{4.1}$$

and introduce the semi-norm

$$[f]_{*,\delta} = \int_0^\delta \sup_{x \in \overline{\Omega}} \omega_f(x;r) \frac{dr}{r} = \int_0^\delta \omega_f(r) \frac{dr}{r}. \tag{4.2}$$

The finiteness of the above integral is known as *Dini's continuity condition*, see [11], equation (4.47). In this reference, problem 4.2, it is remarked that if f satisfies Dini's condition in the whole space $\mathbb{R}^n$, then its Newtonian potential is a C^2 function in $\mathbb{R}^n$.

We define

$$C_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_* < \infty\}. \tag{4.3}$$

A norm is introduced by setting

$$\|f\|_{*,\delta} \equiv [f]_{*,\delta} + \|f\|.$$

Since

$$[f]_{*,\delta_1} \le [f]_{*,\delta_2} \le [f]_{*,\delta_1} + 2\left(\log \frac{\delta_2}{\delta_1}\right) \|f\|, \tag{4.4}$$

for $0 < \delta_1 < \delta_2$, norms are essentially independent of δ.

To fix ideas, we chose $\delta = R$, where R denotes the diameter of Ω, and set

$$[f]_* = [f]_{*,R}.$$

The results obtained in the framework of $C_*(\overline{\Omega})$ spaces led to consider their possible extension to larger functional spaces of continuous functions. We consider functional spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$, for which $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega}) \subset D_*(\overline{\Omega})$. The space $B_*(\overline{\Omega})$ was considered in [UN] as follows. For each $f \in C(\overline{\Omega})$, we define the semi-norm

$$\langle f \rangle_* = \sup_{x \in \overline{\Omega}} \int_0^R \omega_f(x; r) \frac{dr}{r}, \tag{4.5}$$

and the related functional space

$$B_*(\overline{\Omega}) \equiv \{ f \in C(\overline{\Omega}) : \langle f \rangle_* < +\infty \} \tag{4.6}$$

endowed with the norm

$$\|f\|^* \equiv \langle f \rangle_* + \|f\|. \tag{4.7}$$

The reader should compare (4.5) with (4.2). Obviously, $\langle f \rangle_* \le [f]_*$. Actually, the $B_*(\overline{\Omega})$ norm is "much weaker". In [UN] we have shown that the inclusion $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega})$ is proper, by constructing oscillating functions which belong to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$. This construction was recently published in reference [6]. Further, we may show that $B_*(\overline{\Omega})$ is compactly embedded in $C(\overline{\Omega})$, and that (4.4) still holds for the $B_*(\overline{\Omega})$ semi-norm.

The space $D_*(\overline{\Omega})$ is defined as follows. Set

$$S(x; r) = \{ y \in \Omega : |y - x| = r \}$$

and define, for $f \in C(\overline{\Omega})$, $x \in \overline{\Omega}$, and $r > 0$, the quantity

$$\tilde{\omega}_f(x_0; r) \equiv \sup_{y \in S(x; r)} |f(y) - f(x)|. \tag{4.8}$$

Further, we define the semi-norm

$$(f)_* \equiv \sup_{x \in \overline{\Omega}} \int_0^R \tilde{\omega}_f(x; r) \frac{dr}{r}, \tag{4.9}$$

and the related functional space

$$D_*(\overline{\Omega}) \equiv \{ f \in C(\overline{\Omega}) : (f)_* < \infty \}.$$

A norm in $D_*(\overline{\Omega})$ is introduced by setting

$$\| |f| \|_* = (f)_* + \|f\|. \tag{4.10}$$

We remark that (4.4) still holds for the $D_*(\overline{\Omega})$ semi-norm.

Note that, compared to $B_*(\overline{\Omega})$, the space $D_*(\overline{\Omega})$ is defined by replacing in (4.5) the expression of $\omega_f(x;r)$ shown in (4.2) by the expression given by (4.8). Sets $\Omega(x;r)$ are replaced by sets $S(x;r)$. Note that $S(x;r)$ has no boundary points even when the distance of x to the boundary is less than r.

It is worth noting that the above substitution in the definition of $C_*(\overline{\Omega})$ is irrelevant sinve it would leave this space invariant.

Main properties of $C_*(\overline{\Omega})$ are the following.

Theorem 4.1. *$C_*(\overline{\Omega})$ is a Banach space.*

Theorem 4.2. *The embedding $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact.*

Theorem 4.3. *The set $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$.*

It is worth noting that the crucial property required for the space $C_*(\overline{\Omega})$ in the proofs of Theorems 2.3 and 5.2, is Theorem 4.3. This theorem is proved, see [5], by appealing to the well known mollification technique. This density result *up to the boundary* requires a previous, suitable extension, of the functions outside $\overline{\Omega}$. The following result holds.

Theorem 4.4. *Set $\Omega_\tau \equiv \{x : \mathrm{dist}(x, \Omega) < \tau\}$. There is a $\tau > 0$ such that the following statement holds. There is a linear continuous map T from $C(\overline{\Omega})$ to $C(\overline{\Omega}_\tau)$, such that its restriction to $C_*(\overline{\Omega})$ is continuous from $C_*(\overline{\Omega})$ to $C_*(\overline{\Omega}_\tau)$, and Tf, restricted to $\overline{\Omega}$, coincides with f.*

5. Stokes, and other elliptic problems, in $C_*(\overline{\Omega})$

In [4] it was remarked that, at that time, we have proved the Theorem 2.3 for solutions to more general linear elliptic boundary value problems. In fact, in [UN] we have obtained the following regularity result.

Theorem 5.1. *For every $f \in C_*(\overline{\Omega})$ the solution u to the problem*

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega, \\ \mathscr{B}u = 0 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

belongs to $C^2(\overline{\Omega})$. *Moreover, there is a constant* c_0 *such that the estimate*

$$\|u\|_2 \le c_0 \|f\|_*, \qquad \forall f \in C_*(\overline{\Omega}). \tag{5.2}$$

holds.

Here, $\mathscr{L}$ is a second order partial differential elliptic operator with smooth coefficients, and $\mathscr{B}$ is a linear differential operator, of order less or equal to one, acting on the boundary Γ. We assumed that $\mathscr{L}$, $\mathscr{B}$, and Ω are such that, for each $f \in C(\overline{\Omega})$, the problem (5.5) has a unique solution $u \in C^1(\overline{\Omega})$, given by

$$u(x) = \int_\Omega g(x, y) f(y)\, dy, \tag{5.3}$$

where g is the Green function associated with he above boundary value problem. Our hypotheses on $\mathscr{L}$, $\mathscr{B}$, and Ω are given by assuming the following two requirements:

- For each $f \in C(\overline{\Omega})$ the solution u of problem (5.5) is unique, belongs to $C^1(\overline{\Omega})$, and is given by (5.3). Furthermore, if $f \in C^\infty(\overline{\Omega})$ then $u \in C^2(\overline{\Omega})$.
- The above Green's function $g(x, y)$ satisfies the estimates

$$\left|\frac{\partial g}{\partial x_i}\right| \le \frac{k}{|x-y|^{n-1}}, \qquad \left|\frac{\partial^2 g}{\partial x_i \partial x_j}\right| \le \frac{k}{|x-y|^{n}}, \tag{5.4}$$

where $i, j = 1, \dots, n$.

These estimates are well known for long time, for a large class of problems. The above setup, and related comments and proofs, may be shown in reference [6].

In particular, the proof of Theorem 5.1 may be extended to a larger class of problems, like non-homogeneous boundary value problems, elliptic systems, in particular the Stokes system, higher order problems, etc. The main point is that solutions u are given by equations like (5.3), where the Green functions g satisfy suitable estimates, which extend that shown in equation (5.4). Recently, we have adapted the unpublished proof of Theorem 5.2 to show a similar regularity result for solutions to the Stokes system (see, for instance, [10], [14], [20])

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma. \end{cases} \tag{5.5}$$

If $\boldsymbol{f} \in \boldsymbol{C}(\overline{\Omega})$, this problem has a unique generalized solution $(\boldsymbol{u}, p) \in \boldsymbol{C}^1(\overline{\Omega}) \times C(\overline{\Omega})$, where p is defined up to a constant. The solution is given by

$$u_i(x) = \int_\Omega G_{ij}(x, y) f_j(y)\, dy, \qquad p(x) = \int_\Omega g_j(x, y) f_j(y)\, dy, \tag{5.6}$$

where $\boldsymbol{G}$ and $\boldsymbol{g}$ are respectively the Green's tensor and vector associated with the above boundary value problem. Furthermore, the following estimates hold.

$$
\begin{aligned}
&|G_{ij}(x,y)| \le \frac{C}{|x-y|}, \\
&\left|\frac{\partial G_{ij}(x,y)}{\partial x_k}\right| \le \frac{C}{|x-y|^2}, \qquad |g_j(x,y)| \le \frac{C}{|x-y|^2}, \\
&\left|\frac{\partial^2 G_{ij}(x,y)}{\partial x_k \partial x_l}\right| \le \frac{C}{|x-y|^3}, \qquad \left|\frac{\partial g_j(x,y)}{\partial x_k}\right| \le \frac{C}{|x-y|^3},
\end{aligned}
\tag{5.7}
$$

where the positive constant C depends only on Ω. For an overview on the classical theory of hydrodynamical potentials, and the construction of the Green functions $\boldsymbol{G}$ and $\boldsymbol{g}$, we refer to Chapter 3 of the classical treatise [14]. The estimates (5.7) are contained in equations (46) and (47) in this last reference. They may also be found in [16]; see also [9] and [21]. The estimates (5.7) are a particular case of a set of much more general results, due to many authors. See, for instance, [1], [16], [17], [18] and [19].

It is well known, see [9], that for every $\boldsymbol{f} \in \boldsymbol{C}^{0,\lambda}(\overline{\Omega})$ the solution $(\boldsymbol{v}, p)$ to the Stokes system (5.5) belongs to $\boldsymbol{C}^{2,\lambda}(\overline{\Omega}) \times C^{1,\lambda}(\overline{\Omega})$. Hence, as above, Hölder spaces look too strong as data spaces for getting classical solutions. On the other hand, as above, it is well known that $\boldsymbol{f} \in \boldsymbol{C}(\overline{\Omega})$ does not guarantee classical solutions. In reference [5], we proved the following result.

Theorem 5.2. *For every $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$ the solution $(\boldsymbol{u}, p)$ to the Stokes system* (5.5) *belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$. Moreover, there is a constant c_0, depending only on Ω, such that the estimate*

$$
\|\boldsymbol{u}\|_2 + \|\nabla p\| \le c_0 \|\boldsymbol{f}\|_*, \qquad \forall \boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega}),
\tag{5.8}
$$

holds.

A partial generalization of the above theorem to data in a larger space is presented in Section 7, see Theorem 7.1.

6. Elliptic problems in $\boldsymbol{B}_*(\overline{\Omega})$, $\boldsymbol{D}_*(\overline{\Omega})$, and $\boldsymbol{D}^{0,\alpha}(\overline{\Omega})$

In [UN] we considered the problem (2.9) with data in $B_*(\overline{\Omega})$, and proved that the first order derivatives of ψ are Lipschitz continuous in $\overline{\Omega}$. Hence, second order derivatives are bounded. However we were (and are) not able to prove the continuity of these derivatives. Continuity would hold if the density Theorem 4.3 were to hold with $C_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$; an interesting open problem. This led us, at that time, to replace in the published work [4] the space $B_*(\overline{\Omega})$ by the more handy space $C_*(\overline{\Omega})$. Actually, in [UN], the result was proved for linear elliptic boundary value problem whose solutions are given by

$$u(x) = \int_\Omega G(x, y) f(y)\, dy,$$

where the *scalar* Green function $G(x, y)$ satisfies the estimates stated in (5.7). Recently, in reference [6], we have published the proof of this result in a more general form, since $B_*(\overline{\Omega})$ was replaced by the larger space $D_*(\overline{\Omega})$, see below. In Section 7 we appeal to the same ideas to extend the result proved in [6] to the Stokes problem (5.5), see Theorem 7.1. Obviously, all the results proved for data in $D_*(\overline{\Omega})$ hold for data in $B_*(\overline{\Omega})$.

The results obtained in the framework of $C_*(\overline{\Omega})$ spaces also lead us to consider the problem of their *restriction* to smaller functional spaces, instead of extension to larger spaces. The main motivation, within the realm of solutions to second order linear elliptic boundary value problems, can be illustrated as follows. If $f \in C^{0,\lambda}(\overline{\Omega})$, the second order derivatives of the solution satisfy $D^2u \in C^{0,\lambda}(\overline{\Omega})$. Let's say, for brevity, that they fully "remember" their origin. On the other hand, if the data f is in $C_*(\overline{\Omega})$, then the second order derivatives of the solution are merely continuous. Roughly speaking, they completely "forget" that f produces a finite the integral on the right hand side of (4.2). This situation leads us to look for data spaces, between Hölder and $C_*(\overline{\Omega})$ spaces, for which solutions "remember", at least partially, their origin. The following is a significant example of a functional space of "intermediate type". Define, for each $\alpha > 0$, the semi-norm

$$[f]_{0;\alpha} \equiv \sup_{x, y \in \overline{\Omega}\ 0<|x-y|<1} \frac{|f(x) - f(y)|}{(-\log|x-y|)^{-\alpha}}, \tag{6.1}$$

and the related norm $\|f\|_{0;\alpha} \equiv [f]_{0;\alpha} + \|f\|$. Next, define functional spaces $D^{0,\alpha}(\overline{\Omega})$ in the obvious way. Roughly speaking, we have replaced in the definition of Hölder spaces the quantity

$$\frac{1}{|x-y|} \quad \text{by} \quad \log\frac{1}{|x-y|}.$$

This similitude leads us to call these spaces H-log (Hölder-logarithmic) spaces. The family of H-log spaces enjoys some typical, significant property. For instance, $D^{0,\alpha}(\overline{\Omega})$ is a Banach space, and $C^\infty(\overline{\Omega})$ is a dense subspace. Furthermore, for $0 < \beta < 1 < \alpha$, and $0 < \lambda \leq 1$, the following strict embeddings

$$C^{0,\lambda}(\overline{\Omega}) \subset D^{0,\alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset D^{0,\beta}(\overline{\Omega}) \subset C(\overline{\Omega}) \tag{6.2}$$

hold. Note that $D^{0,1}(\overline{\Omega}) \subset C_*(\overline{\Omega})$ is false. The embeddings $D^{0,\alpha}(\overline{\Omega}) \subset D^{0,\beta}(\overline{\Omega}) \subset C(\overline{\Omega})$, for $\alpha > \beta > 0$, and the embeddings $D^{0,\alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega})$, for $\alpha > 1$, are compact.

In reference [7] we have considered boundary value problems with data in $D^{0,\alpha}(\overline{\Omega})$. For a second order linear elliptic problem we show that if $f \in D^{0,\alpha}(\overline{\Omega})$, for some $\alpha > 1$, then $D^2\boldsymbol{u} \in D^{0,(\alpha-1)}(\overline{\Omega})$. Furthermore, this result is optimal.

In a forthcoming paper, see [8], we set the above distinct situations in a unique framework by considering a more general family of data spaces $D_\omega(\overline{\Omega})$, satisfying the inclusions $C^{0,1}(\overline{\Omega}) \subset D_\omega(\overline{\Omega}) \subset C_*(\overline{\Omega})$. Hölder and H-log spaces, and related results, turn out to be particular cases.

7. The Stokes equations with data in $D_*(\overline{\Omega})$. Uniform boundedness of $\nabla^2 u$ and ∇p

In this section we consider the Stokes system and show that the first order derivatives of the velocity $\boldsymbol{u}$, and the pressure p, are Lipschitz continuous in $\overline{\Omega}$ for given external forces in $D_*(\overline{\Omega})$ (so, in particular, in $B_*(\overline{\Omega})$). We prove the following result.

Theorem 7.1. *Let* $\boldsymbol{f} \in \boldsymbol{D}_*(\overline{\Omega})$, *and let* $(\boldsymbol{u}, p)$ *be the solution to problem* (5.5). *There is a constant* C, *which depends only on* Ω, *such that*

$$\|\boldsymbol{u}\|_{1,1} + \|p\|_{0,1} \le C \| \, |\boldsymbol{f}| \, \|_*. \tag{7.1}$$

So $\nabla^2 \boldsymbol{u}, \nabla p \in L^\infty(\Omega)$.

Proof. To fix ideas, we assume that $n = 3$. Extension to space dimensions $n \neq 3$ is obvious. The point is merely writing (5.7) for the n-dimensional case. In the following we merely consider the velocity, since the pressure is treated similarly (see also [5]). Let $\boldsymbol{e}_i(x)$, $i = 1, 2, 3$, denote three constant vector fields in $\mathbb{R}^3$, everywhere equal to the corresponding cartesian coordinate unit vector $\boldsymbol{e}_i$. Define the auxiliary systems

$$\begin{cases} -\Delta \boldsymbol{v}_i(x) + \nabla q_i(x) = \boldsymbol{e}_i(x) & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v}_i = 0 & \text{in } \Omega, \\ \boldsymbol{v}_i = 0 & \text{on } \Gamma. \end{cases} \tag{7.2}$$

Clearly, $\boldsymbol{v}_i$ and q_i are smooth. Fix a constant $K(\Omega)$ such that

$$\|\boldsymbol{v}_i\|_{1,1} + \|q_i\|_{0,1} \le K(\Omega), \tag{7.3}$$

for $i = 1, 2, 3$. Next, in correspondence to each point $x_0 \in \Omega$, define the auxiliary system (a kind of "tangent problem" at point x_0)

$$\begin{cases} -\Delta \boldsymbol{v}(x_0, x) + \nabla q(x_0, x) = \boldsymbol{f}(x_0, x) & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = 0 \quad \text{on } \Gamma, \end{cases} \tag{7.4}$$

where $\boldsymbol{f}(x_0, x) \equiv \boldsymbol{f}(x_0)$, $\forall x \in \Omega$, is a constant vector in Ω. Since

$$\boldsymbol{f}(x_0, x) = \sum_i f_i(x_0)\boldsymbol{e}_i(x),$$

the functions $\boldsymbol{v}(x_0, x)$ and $q(x_0, x)$ are smooth for each fixed x_0. Moreover,

$$\|\boldsymbol{v}(x_0, \cdot)\|_{1,1} + \|q(x_0, \cdot)\|_{0,1} \le K|\boldsymbol{f}(x_0)| \le K\|\boldsymbol{f}\|. \tag{7.5}$$

Recall that K is independent of x_0. For convenience set $\boldsymbol{v}(x) = \boldsymbol{v}(x_0, x)$, and so on. By setting

$$\boldsymbol{w}(x) \equiv \boldsymbol{u}(x) - \boldsymbol{v}(x),$$

one has

$$w_i(x) = \int_\Omega G_{ij}(x, y)\big(f_j(y) - f_j(x_0)\big)\, dy.$$

Furthermore,

$$\partial_k w_i(x) - \partial_k w_i(x_0) = \int_\Omega \big(\partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y)\big)\big(f_j(y) - f_j(x_0)\big)\, dy,$$

where ∂_k stands for differentiation with respect to x_k, and $\partial_k w_i(x_0)$ means the value of $\partial_k w_i(x)$ at the particular point $x = x_0$.

Clearly

$$|\partial_k w_i(x) - \partial_k w_i(x_0)| \le \int_\Omega |\partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y)|\, |f_j(y) - f_j(x_0)|\, dy.$$

By setting $\rho = |x - x_0|$ one gets

$$
\begin{aligned}
&|\partial_k w_i(x) - \partial_k w_i(x_0)| \\
&\quad \le \int_{\Omega(x_0;2\rho)} |\partial_k G_{ij}(x,y) - \partial_k G_{ij}(x_0,y)|\,|f_j(y) - f_j(x_0)|\,dy \\
&\qquad + \int_{\Omega_c(x_0;2\rho)} |\partial_k G_{ij}(x,y) - \partial_k G_{ij}(x_0,y)|\,|f_j(y) - f_j(x_0)|\,dy \\
&\quad \equiv I_1(x_0,x,\rho) + I_2(x_0,x,\rho).
\end{aligned} \tag{7.6}
$$

By appealing to (5.7) (to be adapted, if $n \neq 3$), we show that

$$
\begin{aligned}
I_1(x_0,x,\rho) \le C\Big(&\int_{\Omega(x_0;2\rho)} \frac{C}{|x_0-y|^2}|f_j(y) - f_j(x_0)|\,dy \\
&+ \int_{\Omega(x;3\rho)} \frac{C}{|x-y|^2}|f_j(y) - f_j(x_0)|\,dy\Big) \\
\equiv{}& J_1(x_0,x,\rho) + J_2(x_0,x,\rho).
\end{aligned} \tag{7.7}
$$

By setting $r = |x_0 - y|$, and by appealing to polar-spherical coordinates centered in x_0, one easily shows that $J_1(x_0,x,\rho) \le C\rho, \tilde{\omega}_{\boldsymbol{f}}\big(\Omega(x_0;2\rho)\big)$, where C depends only on Ω. Similarly, $J_2(x_0,x,\rho) \le C\rho\tilde{\omega}_{\boldsymbol{f}}\big(\Omega(x;3\rho)\big)$. It follows that (recall definition (4.8))

$$
I_1(x_0,x,\rho) \le C\rho\tilde{\omega}_{\boldsymbol{f}}(3\rho),
$$

where C does not depend on the particular points $x_0, x \in \overline{\Omega}$, and $\rho = |x - x_0|$.

On the other hand, by appealing to the mean-value theorem and to (5.7), we get

$$
|\partial_k G_{ij}(x,y) - \partial_k G_{ij}(x_0,y)| \le C\rho|x' - y|^{-3} \le C\rho 2^3 |x_0 - y|^{-3},
$$

for each $y \in \Omega_c(x_0;2\rho)$, where the point x' belongs to the straight segment joining x_0 to x or, if necessary, to a smooth path $\gamma = \gamma(x_0,x)$, contained in Ω, joining x_0 and x, and such that its lenght is bounded by $C\rho$, where the constant $C \ge 1$ does not depend on the particular points x_0 and x. Consequently,

$$
I_2(x_0,x,\rho) \le c\rho \int_{\Omega_c(x_0;2\rho)} |f_j(y) - f_j(x_0)| \frac{dy}{|x_0-y|^3} \le c\rho \int_{2\rho}^{R} \tilde{\omega}_{\boldsymbol{f}}(r)\frac{dr}{r}.
$$

Hence,

$$
I_2(x_0,x,\rho) \le c\rho \int_0^R \tilde{\omega}_{\boldsymbol{f}}(r)\frac{dr}{r} \le C\rho \|\,|\boldsymbol{f}|\,\|_*.
$$

Next, by appealing to equation (7.6), and to the estimates proved above for I_1 and I_2, we show that

$$|\nabla \boldsymbol{w}(x) - \nabla \boldsymbol{w}(x_0)| \leq C\rho\big(\| \, |\boldsymbol{f}| \, \|_* + \tilde{\omega}_{\boldsymbol{f}}(3\rho)\big).$$

Consequently,

$$\begin{aligned} |\nabla \boldsymbol{u}(x) - \nabla \boldsymbol{u}(x_0)| &\leq |\nabla \boldsymbol{w}(x) - \nabla \boldsymbol{w}(x_0)| + |\nabla \boldsymbol{v}(x) - \nabla \boldsymbol{v}(x_0)| \\ &\leq C\rho\big(\| \, |\boldsymbol{f}| \, \|_* + \tilde{\omega}_{\boldsymbol{f}}(3\rho) + K\|\boldsymbol{f}\|\big). \end{aligned}$$

So,

$$\frac{|\nabla \boldsymbol{u}(x) - \nabla \boldsymbol{u}(x_0)|}{|x - x_0|} \leq C\| \, |\boldsymbol{f}| \, \|_*, \qquad \forall x, x_0 \in \Omega, \; x \neq x_0. \tag{7.8}$$

This proves (7.1) for the velocity $\boldsymbol{u}$. Similar calculations lead to the corresponding result for the pressure. □

It is worth noting that the above proof depends only on having suitable estimates for the Green's functions. For instance, the argument applied in the above proof to study the system (5.5) with data in $D_*(\overline{\Omega})$ can be applied to the system (2.9) with data in $D_*(\overline{\Omega})$, since the scalar Green's function $G(x, y)$ related to this last problem satisfies exactly the estimates claimed in equation (5.7) for the components $G_{ij}(x, y)$. It follows that Theorem 2.3 holds, in a similar "weak form", for data in $D_*(\overline{\Omega})$. Continuity of the second order derivatives should be replaced by boundedness. This immediately leads to the following "weak form" of Theorem 2.2.

Theorem 7.2. *Let $\theta \in D_*(\overline{\Omega})$, and let $\boldsymbol{v}$ be the solution of problem* (2.4). *Then $\nabla \boldsymbol{v} \in L^\infty(\overline{\Omega})$, and $\|\nabla \boldsymbol{v}\|_{L^\infty(\overline{\Omega})} \leq c_0 \| \, |\theta| \, \|_*$. So, for divergence free vector fields, tangent to the boundary, the estimate*

$$\|\nabla \boldsymbol{v}\|_{L^\infty(\overline{\Omega})} \leq c_0 \| \, |\mathrm{curl}\, \boldsymbol{v}| \, \|_* \tag{7.9}$$

holds.

We state this specific case since it will be useful in considering the Euler equations with data in $B_*(\overline{\Omega})$. For more results and comments on the above subject we refer to [6].

8. The space $B_*(\overline{\Omega})$ and the Euler equations

Concerning possible extensions of the results obtained for the $2 - D$ evolution Euler equations, from $\mathbf{C}_*(\overline{\Omega})$ to $B_*(\overline{\Omega})$, we show here a partial result in this direction.

We clearly pay the price of the loss of regularity for solutions to the auxiliary elliptic system (2.4). This leads us to replace continuity in time by boundedness in time.

Furthermore, we simplify our task, in a *quite substantial way*, by assuming that external forces vanish, instead of assuming the very stringent condition $\operatorname{curl} \boldsymbol{f} \in L^1\big(R^+; B_*(\overline{\Omega})\big)$.

Below, we prove the following weak extension of Theorem 2.4.

Theorem 8.1. *Let $\boldsymbol{v}$ be the solution to the Euler equations* (2.1), *where the initial data $\boldsymbol{v}_0$ is divergence free, tangential to the boundary, and satisfies* $\operatorname{curl} \boldsymbol{v}_0 \in B_*(\overline{\Omega})$. *Furthermore, suppose $\boldsymbol{f} = 0$. Then* $\operatorname{curl} \boldsymbol{v} \in L^\infty\big(0, T; B_*(\overline{\Omega})\big)$, *and there is a constant C_T* (*an explicit expression can be easily obtained*) *such that*

$$\|\operatorname{curl} \boldsymbol{v}(t)\|^* \leq C_T \|\operatorname{curl} \boldsymbol{v}_0\|^*, \tag{8.1}$$

for a.a. $t \in (0, T)$.

A weak extension of Theorem 2.5 follows immediately from Theorem 8.1 together with Theorem 7.2. One has the following result.

Theorem 8.2. *Under the assumptions of Theorem* 8.1 *the estimate*

$$\|\nabla \boldsymbol{v}\|_{L^\infty(Q_T)} \leq C_T \|\operatorname{curl} \boldsymbol{v}_0\|^* \tag{8.2}$$

holds almost everywhere in Q_T.

To prove Theorem 8.1, we appeal to some estimates previously obtained in a more general form in reference [4]. For clarity, instead of stating these estimates in the weakest form, strictly necessary to prove the Theorem 8.1 below, we rather prefer to show some more general formulations of the estimates. This allows us to present a short overview on the structure of the proof of Theorem 2.13, suitable for readers interested in a deeper examination of reference [4]. In order to make an easier link with this last reference, we appeal here to the notation used in [4] (compare, for instance, (8.3) and (8.5) below with (2.4) and (2.9), respectively).

As already remarked, the velocity $\boldsymbol{v}(t)$, at each time t, can be obtained from the vorticity $\zeta(t) \equiv \operatorname{curl} \boldsymbol{v}(t)$, by setting, for each fixed t, $\theta = \zeta(t)$ in the elliptic system

$$\begin{cases} \operatorname{curl} \boldsymbol{v} = \theta & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases} \tag{8.3}$$

On the other hand, the solution to this system is given by

$$\boldsymbol{v} = \operatorname{Curl} \psi, \tag{8.4}$$

where ψ is the solution of the elliptic problem

$$\begin{cases} -\Delta\psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases} \tag{8.5}$$

So, at least in principle, we may obtain the velocity from the vorticity. However, since the vorticity is not a priori known, we start from a "fictitious vorticity" $\theta(x,t)$, and look for a fixed point $\theta = \zeta$. In the sequel we replace "fictitious vorticity" simply by "vorticity", and so on for other quantities. From each suitable "vorticity" we obtain a "velocity", by appealing to (8.5) and (8.4). From this "velocity" we construct streamlines $\boldsymbol{U}(s,t,x)$, by appealing to Lagrangian coordinates. Finally, a well know technique (here dimension 2 is crucial) gives a correspondent fictitious "vorticity" ζ. So, a map $\theta \to \zeta$ is, formally, well defined. A rigorous fixed point was obtained in reference [4] in the framework of $C(\overline{\Omega})$ spaces, as follows:

Fix an arbitrary positive time T, an initial data $\boldsymbol{v}_0$, and an external force $\boldsymbol{f}$. Set $\zeta_0 \equiv \operatorname{curl} \boldsymbol{v}_0$, $\phi \equiv \operatorname{curl} \boldsymbol{f}$, and define (see (2.11))

$$B = \|\zeta_0\| + \int_0^T \|\phi(\tau)\|\, d\tau. \tag{8.6}$$

Further, define the convex, bounded, closed subset of $C(\overline{Q}_T)$,

$$\mathbf{K} = \{\theta \in C(\overline{Q}_T) : \|\theta\|_T \le B\}. \tag{8.7}$$

From now on, the symbol $\theta = \theta(x,t)$ denotes an arbitrary element of $\mathbf{K}$. As already explained, the idea is to prove the existence and uniqueness of a fixed point in $\mathbf{K}$, for a suitable map Φ, such that to this fixed point there corresponds a solution of the Euler equation (2.1) with the above given data. The map $\Phi[\theta] = \zeta$ is defined as the following composition of single maps:

$$\Phi : \theta \to \psi \to \boldsymbol{v} \to \boldsymbol{U} \to \zeta. \tag{8.8}$$

Given $\theta = \theta(x,t) \in \mathbf{K}$ we get $\psi = \psi(x,t)$ by solving the elliptic system (8.5), where t is treated as a parameter. The crucial estimates for $\psi(x)$ follow from

$$\psi(x) = \int_\Omega g(x,y)\, dy,$$

where g is the Green function associated to problem (8.5). Knowing ψ, the velocity $\boldsymbol{v}$ is obtained by setting $\boldsymbol{v}(x,t) = \operatorname{Curl} \psi(x,t)$.

The next step is to get ζ, from $\boldsymbol{v}$. We introduce the streamlines $\boldsymbol{U}$ associated with the "velocity" $\boldsymbol{v}(x,t)$ obtained in the previous step. The streamlines $\boldsymbol{U}(s,t,x)$ are the solution to the system of ordinary differential equations

$$\begin{cases} \frac{d}{ds}\boldsymbol{U}(s,t,x) = \boldsymbol{v}\big(s,\boldsymbol{U}(s,t,x)\big), & \text{for } s \in [0,T], \\ \boldsymbol{U}(t,t,x) = x. \end{cases} \tag{8.9}$$

$\boldsymbol{U}(s,t,x)$ denotes the position at time s of the physical particle which occupies the position x at time t. A main tool is here the following estimate (see equation (2.6) in [4]).

$$\begin{aligned} &|\boldsymbol{U}(s,t,x) - \boldsymbol{U}(s_1,t_1,x_1)| \\ &\quad \le c_1 B|s-s_1| + c_2(1+c_1 B)(|x-x_1|^\rho + |t-t_1|^\rho), \end{aligned} \tag{8.10}$$

where c_1 depends only on Ω, $\rho \equiv e^{-c_1 BT}$, and $c_2 = \max\{1, eR\}$, where R denotes the diameter of Ω. Knowing $\boldsymbol{U}$, we set ([4], equation (2.8))

$$\begin{aligned} \zeta(t,x) &= \zeta_0\big(\boldsymbol{U}(0,t,x)\big) + \int_0^t \phi\big(s,\boldsymbol{U}(s,t,x)\big)\,ds \\ &\equiv \zeta_1(t,x) + \zeta_2(t,x), \end{aligned} \tag{8.11}$$

where, as already remarked, $\zeta_0 \equiv \operatorname{curl} \boldsymbol{v}_0$, and $\phi \equiv \operatorname{curl} \boldsymbol{f}$. The curl of the solution is here expressed separately in terms of the curls of the initial data and of the external forces. The main estimates for these two terms were proved in [4], respectively in Lemmas 4.3 and 4.2. The reader may verify that the control of the external forces term is much more involved than that of the initial data term.

The composition map $\Phi[\theta] = \zeta$ turns out to be well defined over $\mathbf{K}$, by appealing to (8.8). In the proof of Theorem 2.1 in [4], we close the above scheme by showing that $\Phi(\mathbf{K}) \subset \mathbf{K}$, and that there is a (unique) fixed point in $\mathbf{K}$. Finally, it was proved that this fixed point is the curl of the solution to the Euler equations (2.1). The velocity follows from the curl by appealing to (8.3).

After this flying visit to the proof of Theorem 2.1, we prove the Theorem 8.1.

Proof of Theorem 8.1. A main tool in proving the regularity Theorem 2.4 for data in $C_*(\overline{\Omega})$ was the following result, see the Lemma 4.1 in [4].

Lemma 8.3. *Let* $\boldsymbol{a} \in \mathbf{C}_*(\overline{\Omega})$ *and* $\boldsymbol{U} \in \mathbf{C}^{0,R}(\overline{\Omega};\overline{\Omega})$, $0 < R \le 1$. *Then* $\boldsymbol{a} \circ \boldsymbol{U} \in C_*(\overline{\Omega})$*; moreover*

$$[\boldsymbol{a} \circ \boldsymbol{U}]_* \le \frac{1}{R}[\boldsymbol{a}]_*. \tag{8.12}$$

Note that the need for property (8.12) narrows the possible choice of spaces, candidate to replace $\mathbf{C}_*(\overline{\Omega})$.

Actually, in our preliminary version [UN] the above lemma is written in terms of metric spaces. However, at that time, it seemed to us a little "out of place" to present a so simple result in an abstract form.

The absence of external forces $\boldsymbol{f}$ lead us to revive below, directly, the simple idea used in the proof of Lemma 8.3, without appealing to the original statement itself.

We deal with solutions whose existence is already guaranteed by Theorem 2.1. We merely want to show the additional regularity claimed in Theorem 8.1. Since in this theorem the external forces vanish, the following very simplified form of (8.10) holds.

$$|\boldsymbol{U}(0,t,x) - \boldsymbol{U}(0,t,y)| \leq K|x-y|^\rho, \tag{8.13}$$

where $B = \|\zeta_0\|$. Following (8.11), and taking into account that ζ_2 vanishes, one has $\zeta = \zeta_1$. So the curl of the solution $\boldsymbol{v}$ to the Euler equation (2.1) is simply given by

$$\zeta(t,x) = \zeta_0\big(\boldsymbol{U}(0,t,x)\big).$$

It follows that

$$\begin{aligned} \omega_{\zeta(t)}(x;r) &= \sup_{y\in\overline{\Omega}(x;r)} |\zeta(t,x) - \zeta_1(t,y)| \\ &= \sup_{y\in\overline{\Omega}(x;r)} \big|\zeta_0\big(\boldsymbol{U}(0,t,x)\big) - \zeta_0\big(\boldsymbol{U}(0,t,y)\big)\big|. \end{aligned} \tag{8.14}$$

Further, by appealing to (8.13), one gets

$$\omega_{\zeta(t)}(x;r) \leq \omega_{\zeta_0}\big(\boldsymbol{U}(0,t,x); Kr^\rho\big). \tag{8.15}$$

So, by recalling definition (4.5), one has

$$\langle\zeta(t)\rangle_* \equiv \sup_{x\in\overline{\Omega}} \int_0^R \omega_{\zeta(t)}(x;r)\frac{dr}{r} \leq \sup_{x\in\overline{\Omega}} \int_0^R \omega_{\zeta_0}\big(\boldsymbol{U}(0,t,x); Kr^\rho\big)\frac{dr}{r}. \tag{8.16}$$

Since

$$\{\boldsymbol{U}(0,t,x) : x \in \overline{\Omega}\} = \overline{\Omega},$$

it follows, by appealing to the change of variables $\tau = Kr^\rho$, that

$$
\begin{aligned}
\langle\zeta(t)\rangle_* &\leq \sup_{\bar{x}\in\bar{\Omega}} \int_0^R \omega_{\zeta_0}(\bar{x}; Kr^\rho)\frac{dr}{r} \\
&= \frac{1}{\rho} \sup_{x\in\bar{\Omega}} \int_0^{KR^\rho} \omega_{\zeta_0}(x;\tau)\frac{d\tau}{\tau} \equiv \frac{1}{\rho}\langle\zeta_0\rangle_{*,KR^\rho}, \qquad (8.17)
\end{aligned}
$$

with obvious notation. On the other hand, $\|\zeta(t)\| = \|\zeta_0\|$, for all t. Since (4.4) also applies for B_* semi-norms, one shows that

$$
\|\zeta(t)\|^* \leq C_T \|\zeta_0\|^*,
$$

for all $t \in [0, T]$. Theorem 8.1 is proved. Theorem 8.2 follows by appealing to Theorem 7.2.

It would be interesting to prove Theorem 8.1 in the presence of external forces, even in a simplified version, for instance, curl $\boldsymbol{f} \in C\big(\mathbb{R}^+; B_*(\bar{\Omega})\big)$. We believe that a (possibly modified) version of this result holds by appealing to the measure preserving properties of the streamlines, together with the control of the linear dimensions of figures in finite time.

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