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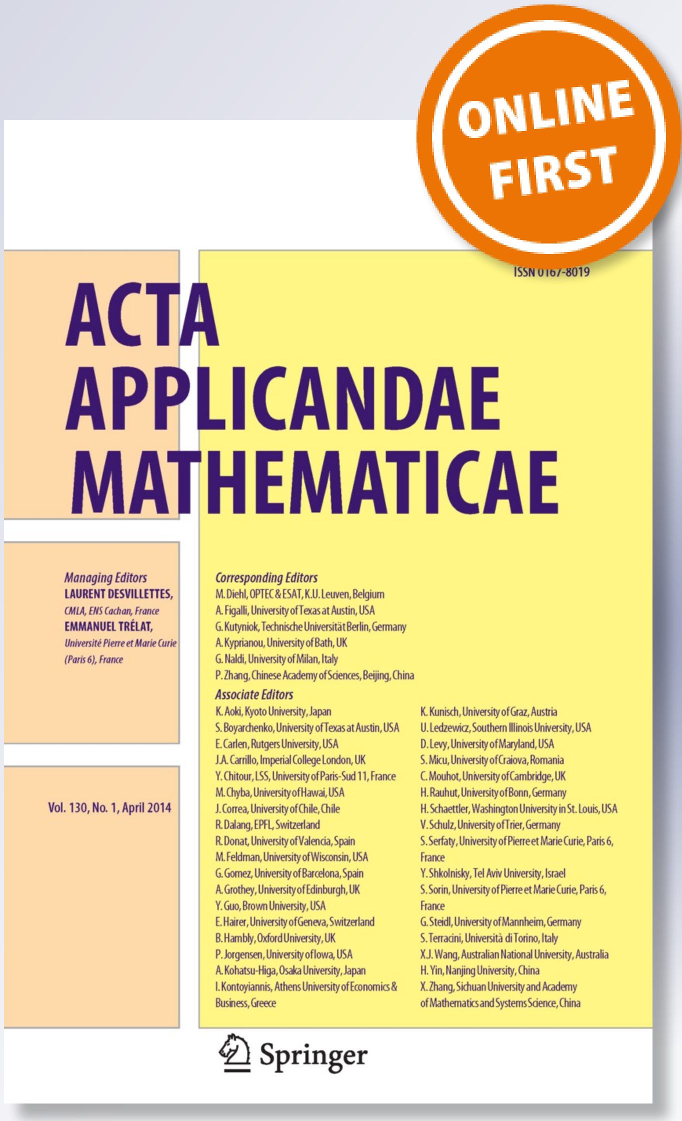

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Abstract These notes concern the study of sufficient conditions on the direction of the vorticity to guarantee regularity of solutions to the evolution Navier–Stokes equations. We emphasize here some thread lines of the research, taken as a whole. Interdependence among distinct results and techniques is discussed. We end by a sketch of the proof of a quite recent result.

Keywords Navier–Stokes equations · Slip boundary conditions · Direction of vorticity · Regularity

1 The Whole Space Results

We consider the 3D Navier–Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)u - \triangle u + \nabla p = 0, & \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where the velocity u and the pressure p are the unknowns. For brevity we assume that external force vanishes, and the kinematic viscosity is equal to one. Ω may denote the whole space R^3, the half space R^3_+, or an open, connected, bounded subset of R^3, with a smooth boundary $\Gamma = \partial\Omega$. In this last case, equations are mostly supplemented with the "stress-free" boundary conditions

$$\begin{cases} u \cdot n = 0, & \\ \omega \times n = 0 & \text{on } \Gamma \times (0, T], \end{cases} \tag{2}$$

A Salvatore, con profonda stima, in occasione del suo 80esimo compleanno.

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where $\omega = \nabla \times u = \mathrm{curl}\, u$ is the vorticity field, and n denotes the exterior unit normal vector to the boundary. In the case of flat boundaries, the above conditions coincide with the classical Navier boundary conditions without friction, see [28]. See also [30].

In the sequel $L^p := L^p(\Omega)$, $1 \le p \le \infty$ denotes the usual Lebesgue spaces equipped with norm $\|.\|_p$. Further, $H^k := H^k(\Omega)$, are the classical Sobolev spaces. We use the same symbol for both scalar and vector function spaces. Moreover

$$L^p_T(X) \stackrel{def}{=} L^p\big(0, T; X(\Omega)\big),$$

where $X = X(\Omega)$ is a generical Banach space, and $1 \le p \le \infty$. Arbitrary positive constants are denoted simply by c.

Solutions $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ are defined here in the well known Leray–Hopf weak sense, and are assumed to be weakly continuous from the right at time $t = 0$. We say that a Leray–Hopf weak solution u is strong if it belongs to $L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$. It is well known that strong solutions are smooth, if data and domain are also smooth.

We consider the problem of global existence of smooth solutions, under suitable hypotheses on the vorticity-direction. We set

$$\theta(x, y, t) \stackrel{def}{=} \angle\big(\omega(x, t), \omega(y, t)\big),$$

where the symbol "$\angle$" denotes the amplitude of the angle between two vectors. We are interested in sufficient conditions on $\sin\theta(x, y, t)$ to guarantee the regularity of the solutions.

It is right starting from the fundamental 1993 pioneering paper [16], by P. Constantin and Ch. Fefferman, where these authors prove that solutions to the evolution Navier–Stokes equations in the whole space are smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables. More precisely, they proved the following result.

Theorem 1 *Let be $\Omega = R^3$, and let u be a weak solution of (1) in $[0, T[$, with $u_0 \in H^1(R^3)$ and $\nabla \cdot u_0 = 0$. If*

$$\sin\theta(x, y, t) \le g(t)|x - y|$$

for some $g(t, x) \in L^{12}(0, T; L^\infty(\mathbb{R}^3))$, then the solution u is strong. Clearly, the result holds if g is a constant.

As in all the following theorems, conditions on $\sin\theta(x, y, t)$ are assumed for almost all x and y in Ω, and almost all t in $(0, T)$. Furthermore, these conditions are needed merely for points x and y such that $|x - y| < \delta$, for an arbitrary positive constant δ.

In Ref. [7], by following an approach similar to that introduced in [16], L.C. Berselli and the author show that regularity still holds in the whole space by replacing Lipschitz continuity by $\frac{1}{2}$-Hölder continuity. The following theorem was proved.

Theorem 2 *Let Ω, u, and u_0, be as in the previous theorem. Further, suppose that there exists $\beta \in [1/2, 1]$ and $g \in L^a(0, T; L^b(\Omega))$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2} \quad \text{with } a \in \left[\frac{4}{2\beta - 1}, \infty\right], \tag{3}$$

such that

$$\sin\theta(x, y, t) \le g(t, x)|x - y|^\beta \tag{4}$$

holds in $\Omega \times (0, T)$. *Then, the solution u is regular. In particular, the regularity result holds if*

$$\sin\theta(x, y, t) \le c|x - y|^{1/2}. \tag{5}$$

In a subsequent paper, see [1], we consider the case $\beta \le \frac{1}{2}$. This extension is useful since, as sketched below, it helps a better understanding of the whole set of results. However the proof is simply a quite obvious variant of the proof of Theorem 2 in Ref. [7], and could be considered as an appendix to this last reference. The result is the following.

Theorem 3 *Let u be a weak solution of* (1) *in* $[0, T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Let* $\beta \in [0, 1/2]$ *and assume that*

$$\sin\theta(x, y, t) \le c|x - y|^{\beta} \tag{6}$$

holds in $\Omega \times (0, T)$. *Moreover, suppose that*

$$\omega \in L^2\big(0, T; L^r\big), \tag{7}$$

where

$$r = \frac{3}{\beta + 1}. \tag{8}$$

Then the solution u is strong in $(0, T)$ *and, consequently, is regular. In particular* $\sin\theta(x, y, t) \le c|x - y|^{1/2}$ *is sufficient for regularity.*

The couple $(2, r)$ defined by Eq. (8) may be replaced by a larger family of couples of coefficients, by obvious modifications in the proof.

Next we made some general comments, which are independent from the presence or absence of initial or boundary conditions. In Ref. [7], the advantage of assuming $\beta > \frac{1}{2}$ is counterbalanced by replacing in (4) the constant c by a function $g \in L^a(0, T; L^b(\Omega))$. On the other hand, in [1], we mitigate the penalizing situation $\beta < \frac{1}{2}$ by assuming (7). So, the above statements are formally split into two families of sufficient conditions for regularity. However this distinction is not substantial. In fact, the two families perfectly glue at the intersection point $\beta = \frac{1}{2}$ since the conclusion (namely, "condition (5) implies regularity") is the same in both cases. Actually, we have just one family of strictly connected results.

Furthermore, all conditions (this means, for $\beta \in [0, 1]$) have, in some sense, an equivalent strength. This can be shown by a step by step analysis of the proofs, and also by appealing to scaling techniques. Further, this "strength" is "maximal", in the sense that it is equivalent to the strength of the Prodi–Serrin's sufficient conditions for regularity $u \in L^2(0, T; L^\infty(\Omega))$. This equivalence holds only in the presence of the minimal regularity required to the coefficients $g(t, x)$ by (3). This is the reason why this point is taken by us into considerable attention.

For a detailed discussion on the above argument, we refer the reader to [4], and also to Sect. 1 and Appendix, in Ref. [6].

Following the spirit of this presentation, we like to recall separately Refs. [5, 9, 10], and [11]. In particular, in Ref. [9], regularity is proved for the Cauchy problem, without any continuity assumption on $\sin\theta(x, y, t)$. This kind of condition is replaced by a smallness assumption. Essentially, it is proved that there is a sufficiently small constant C_1 (an explicit estimate for C_1 is given) such that regularity holds if

$$\sin\theta(x, y, t) \leq C_1.$$

Clearly, there are many very interesting papers related to our contributions. See, for instance, [12–15, 17–27, 29, 33], and references therein.

2 The Slip Boundary Condition

In Ref. [2] we consider the Navier–Stokes equations in $[0, T[\times\mathbb{R}^3_+$, endowed with the *slip boundary condition*, and prove the following result.

Theorem 4 *Assume that $u_0 \in H^1(\mathbb{R}^3_+)$ is divergence free, and tangent to the boundary. Let u be a weak solution of the Navier–Stokes equations* (1) *in* $[0, T) \times \mathbb{R}^3_+$, *endowed with the slip boundary condition* (2). *Let $\beta \in [0, 1/2]$ and assume that* (6), (7), *and* (8) *hold. Then the solution u is strong in* $(0, T]$ *and, consequently, is regular.*

In particular,

$$\sin\theta(x, y, t) \leq c|x - y|^{1/2}$$

is sufficient for regularity.

The last claim follows from the fact that weak solutions satisfy (7) for $r = 2$. Hence, if $\beta = 1/2$, assumption (7) is superfluous.

In Ref. [2] Theorem 4 is proved by appealing, separately, to the classical Dirichlet and Neumann *Green functions*, in the half space. This can be done for flat boundaries since the boundary conditions (2) can be written in the following separate form

$$\begin{cases} u_3 = 0, \\ \dfrac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \end{cases} \tag{9}$$

The third equation follows from $\omega_1 = \omega_2 = 0$ on Γ, plus differentiation of $u_3 = 0$ with respect to the first to independent variables.

The favorable circumstance given by Eqs. (9) is no more true if the boundary is not flat. This fact, lead Berselli and the author to try to extend the result to non-flat boundaries, again by appealing to Green's function theory. This non trivial achievement was done in [8]. In this last reference the authors extend Theorem 4 to the case in which $\Omega \subset R^3$ is an open, bounded set with a smooth boundary. Since the boundary is not flat, we can not appeal, separately, to the Dirichlet and the Neumann Green functions, as in the proof of Theorem 4. So the authors appeal to the representation formulas for Green's matrices derived in Solonnikov's fundamental work [31, 32]. With the aid of these tools, original local representation formulas for the velocity (in terms of the vorticity) were obtained. In this way, useful estimates for the vortex stretching terms were proved. The proof is quite involved. To contemplate, in the same proof, all the range of values of the parameter β, was undoubtedly feasible but, may be, not attractive to readers. So, in [8], the proof is presented only for the main case $\beta = \frac{1}{2}$. The above picture leads us to look for simpler proofs of the sharp results stated in Ref. [8] for non-flat boundaries. This aim was achieved in Ref. [4] for the particular case $\beta = 1$. This is just the original Constantin and Fefferman's Lipschitz condition, a very classical and mathematical significant assumption. In Sect. 4 below we give a sketch of the proof of the main result stated in [4]. Before treating this argument, we present, in Sect. 3, a comparison between some aspects of the proofs in the slip and the non-slip cases.

3 Slip and Non-slip Boundary Conditions. A Comparison

The situation concerning the non-slip boundary condition

$$u_{|\partial\Omega} = 0 \tag{10}$$

is, in a certain sense, opposite to that concerning the slip boundary condition. The starting point is, in both cases, Eq. (15) below. Obviously, the terms to be controlled in this last equation are the boundary and the non-linear integrals. In the slip boundary case, with flat boundary, one has

$$\begin{cases} \omega_1 = \omega_2 = 0, \\ \dfrac{\partial \omega_3}{\partial x_3} = 0, \end{cases} \tag{11}$$

on Γ. The third equation follows from tangential differentiation of the first two equations together with $\nabla \cdot \omega = 0$. Clearly, as for (9), Eq. (11) follow from (2) only on flat portions of the boundary. Equation (11) shows that the boundary integral in Eq. (15) vanishes in the case of flat boundaries. Actually, a smart appeal to the totally anti-symmetric Ricci tensor, see [8], leads to Lemma 1 below. This lemma allows the control of the boundary integral, even for non-flat boundaries. On the contrary, in the non-slip boundary case, the non zero viscosity prevent us from a suitable control of the boundary integral (see [3]) even in the presence of flat boundaries.

Concerning the control of the non-linear integral in Eq. (15), the situation is reversed. This control is easier under the non-slip boundary condition, even in the half-space case, since in this case appeal to the Green's function for the Dirichlet boundary value problem is clearly sufficient. The Green's function for the Neumann boundary value problem is not needed here. However, an even more crucial difference is that, in the non-slip case, the half-space simplified approach followed in the proof of Theorem 4 applies for any regular Ω. The reason is that the independence of the three boundary conditions

$$u_1 = u_2 = u_3 = 0, \tag{12}$$

holds also for non-flat boundaries. So, we may apply to the Green function for the Dirichlet problem exactly as for the half-space case.

4 The Lipschitz Continuous Case. A Sketch of the Proof

As already anticipated at the end of Sect. 2, the extension of Theorem 4 to non-flat boundaries is extremely delicate. In Ref. [4] we present a really straightforward proof of the desired result in the particular case $\beta = 1$, the original Constantin and Fefferman's Lipschitz condition, a very classical and mathematical significant assumption. In Ref. [4] the following result is proved.

Theorem 5 *Let $\Omega \subset R^3$ be an open, bounded set with a smooth boundary Γ. Assume that $u_0 \in H^1(\Omega)$ is divergence free and tangent to the boundary. Let u be a weak solution to (1)–(2) in $[0, T)$.*

In addition, assume that there exist $g \in L^a(0, T; L^b(\Omega))$, where

$$\frac{2}{a} + \frac{3}{b} = \frac{1}{2} \quad \text{with } a \in [4, \infty], \tag{13}$$

and also a positive $\delta(x, t)$, such that

$$\sin\theta(x, y, t) \leq g(t, x)|y - x|, \tag{14}$$

for $x, y \in \Omega$, satisfying $|y - x| < \delta(x, t)$. Then u is a strong solution in $(0, T)$, hence it is smooth in $\Omega \times (0, T]$.

Note that (3), for the parameter's value $\beta = 1$, coincides with (13).

Next we present a sketch of the proof of Theorem 5. By applying the curl operator to Eq. (1) we get the well-known equation

$$\omega_t + (u \cdot \nabla)\omega - \nu\Delta\omega = (\omega \cdot \nabla)u.$$

Scalar multiplication by ω, and integration in Ω followed by suitable integrations by parts, show that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu\|\nabla\omega\|_2^2 - \nu\int_\Gamma \frac{\partial\omega}{\partial n} \cdot \omega \, d\Gamma = \int_\Omega (\omega \cdot \nabla)u \cdot \omega \, dx. \tag{15}$$

Next, we appeal to the following result, proved in [8] (see Eq. (14) in this last reference).

Lemma 1 *Assume that u is divergence-free, that $u \cdot n = 0$, and that $\omega \times n = 0$ on Γ. Then there is a constant $c = c(\Omega) > 0$ such that*

$$\left|\frac{\partial\omega(x)}{\partial n} \cdot \omega(x)\right| \leq c\left|\omega(x)\right|^2, \quad \forall x \in \Gamma. \tag{16}$$

The proof is based on a smart appeal to the totally anti-symmetric Ricci tensor.

From (15) and (16) it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 \, dx + \frac{1}{2}\int_\Omega |\nabla\omega|^2 \, dx \leq c(\Omega)\int_\Gamma |\omega|^2 \, dS + \left|\int_\Omega (\omega \cdot \nabla)u \cdot \omega \, dx\right|. \tag{17}$$

This easily leads to the estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2 \, dx + \frac{1}{4}\int_\Omega |\nabla\omega|^2 \, dx \leq c(\Omega)\int_\Omega |\omega|^2 \, dx + \left|\int_\Omega (\omega \cdot \nabla)u \cdot \omega \, dx\right|. \tag{18}$$

Further, integration by parts leads to

$$\int_\Omega (\omega \cdot \nabla)u \cdot \omega \, dx = -\int_\Omega \big((\partial_j\omega_k)\omega_j - (\partial_j\omega_j)\omega_k\big)u_k \, dx + \int_\Gamma (\omega \cdot u)(\omega \cdot n) \, d\Gamma. \tag{19}$$

On the other hand, from assumption (14), by setting $y = x + h$, one shows that the estimate

$$\big|\omega(x) \times \omega(x + h)\big| \leq g(t, x)|h|\big|\omega(x)\big|\big|\omega(x + h)\big|$$

holds for sufficiently small h. So,

$$\left|\omega(x)\times\frac{\omega(x+h)-\omega(x)}{|h|}\right|\le g(t,x)\big|\omega(x)\big|\big|\omega(x+h)\big|. \tag{20}$$

In particular, by letting $h\to 0$, one gets

$$\big|\omega(x)\times\partial_j\omega(x)\big|\le g(t,x)\big|\omega(x)\big|^2, \tag{21}$$

for each $j=1,2,3$. It readily follows, by considering the expressions of the single components of $\omega(x)\times\partial_j\omega(x)$, that

$$\big|\omega_l\partial_j\omega_k-\omega_k\partial_j\omega_l\big|\le g(t,x)|\omega|^2, \tag{22}$$

for $k\neq l$. Furthermore, for $k=l$, inequality (22) is obvious. So (22) holds for each tern of indexes $\{i,j,k\}$. This estimate, together with Eq. (19), easily leads to the estimate

$$\left|\int_\Omega(\omega\cdot\nabla)u\cdot\omega\,dx\right|\le c\int_\Omega g(t,x)|u||\omega|^2\,dx.$$

To simplify the calculations, we set, in Eq. (13), $a=\infty$ and $b=6$. So,

$$g\in L^\infty(0,T;L^6(\Omega)).$$

By Hölder's inequality

$$\left|\int_\Omega(\omega\cdot\nabla)u\cdot\omega\,dx\right|\le\|g\|_6\|u\|_6\|\omega\|_2\|\omega\|_6.$$

It follows, by appealing to (18), and to the immersion $H^1(\Omega)\subset L^6(\Omega)$, that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2+\frac{1}{4}\|\nabla\omega\|_2^2\le c\big(1+\|g\|_6^2\|u\|_6^2\big)\|\omega\|_2^2. \tag{23}$$

Hence

$$\int_0^T\big(\|g(t)\|_6^2\|u(t)\|_6^2\big)\,dt\le\|g\|^2_{L^\infty(0,T;L^6(\Omega))}\|u\|^2_{L^2(0,T;L^6(\Omega))}. \tag{24}$$

This shows that

$$\big(1+\|g\|_6^2\|u\|_6^2\big)\in L^1(0,T).$$

The boundedness of u in $L^\infty(0,T;H^1(\Omega))$, hence its regularity follows from (23), by Gronwall's lemma.

Remark 1 It is worth noting that Eq. (16) can be full detailed, as remarked in [4], Lemma 3.2. We take this occasion to correct a quite evident mistake in this lemma. Equation (24) in [4] is correct. However, we have set in this equation $\omega_3=0$, which is clearly wrong, instead of setting $\omega_1=\omega_2=0$, as follows from the second boundary condition (2). It follows that Eq. (23) in [4] should be replaced by Eq. (25) below. Hence, the correct version of Lemma 3.2 in Ref. [4] is the following (numbering concerns the present paper, not [4]).

Lemma 2 *Under the assumptions of Lemma 1, one has*

$$-\frac{\partial\omega}{\partial n}\cdot\omega = 3(\kappa_1+\kappa_2)|\omega|^2 \tag{25}$$

on Γ. Here κ_j, $j=1,2$, denote the principal curvatures, and the ω_j are the coordinates of ω with respect to the τ_j, the unit tangent vectors to the principal directions. In particular, if Ω is convex, the boundary integral $-\int_\Gamma \frac{\partial\omega}{\partial n}\cdot\omega\, d\Gamma$ in Eq. (15) is greater or equal to zero.

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