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On the global regularity for singular p-systems under non-homogeneous Dirichlet boundary conditions

H. Beirão da Veiga

Dipartimento di Matematica Applicata "U. Dini", Università di Pisa, Via Buonarroti 1/C, 56127 Pisa, Italy

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ABSTRACT

We consider the non-homogeneous Dirichlet boundary value problem for elliptic, singular, p-systems of N equations in n space variables, $1 < p \leq 2$. We prove $W^{2,q}$ and $C^{1,\alpha}$ regularity, up to the boundary, under suitable assumptions on the couple p, q. In particular, there is a constant K_0, independent of p and q, such that our regularity results hold at least for $q \leq \frac{K_0}{p-2}$. The singular case $\mu = 0$ is covered. We extend earlier results of the author and F. Crispo, from the homogeneous to the non-homogeneous case.
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1. Introduction

We study the following non-homogeneous Dirichlet boundary value problem in n space variables

$$\begin{cases} -\nabla \cdot \big((\mu + |\nabla w|)^{p-2} \nabla w \big) = f & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth, open and bounded, subset of $\mathbb{R}^n$, $n \geq 3$. The vector fields w, ψ, and f take values in $\mathbb{R}^N$ (N and n may be distinct, and arbitrarily large). The singular case $\mu = 0$ is covered.

The method followed here may be applied to more general equations of the same type. It would be interesting to investigate in this direction. However, the twin problem

$$\begin{cases} -\nabla \cdot \big((\mu + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w \big) = f & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

is completely covered by the proof below.

We extend the proofs obtained with Crispo, [1], from the homogeneous to the non-homogeneous boundary value problem. We prove, under suitable hypothesis, $W^{2,q}(\Omega)$ regularity up to the boundary, for arbitrarily large values of q. This implies, in particular, $C^{1,\alpha}(\overline{\Omega})$ regularity. Even this weaker result seems new, in the non-homogeneous case.

The following is our main result.

Theorem 1.1. *Assume $\mu \geq 0$. Further, let $p \in (1, 2]$ and $q \geq 2$, $q \neq n$, satisfy the condition (see (1.5) and (1.6))*

$$C_2(q)(2-p) < 1. \tag{1.3}$$

E-mail address: bveiga@dma.unipi.it.

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Let $\psi \in W^{2-\frac{1}{q},q}(\partial\Omega)$ and $f \in L^{r(q)}(\Omega)$; see (2.6). Then, the (unique) weak solution $w \in W^{1,p}(\Omega)$ of problem (1.1) (see (2.1)), belongs to $W^{2,q}(\Omega)$. Moreover,

$$\|D^2 w\|_q \le C\big(\|D^2\psi\|_q + \mu^{2-p}\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}}\big). \tag{1.4}$$

We assume, following standard practice, that ψ is the trace on the boundary of a suitable vector field $\psi \in W^{2,q}(\Omega)$.

The positive constant $C_2(q)$ is defined by

$$\|D^2 v\|_q \le C_2(q)\|\Delta v\|_q, \tag{1.5}$$

for all $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. There is a constant K, independent of p and q, such that

$$C_2(q) \le Kq, \tag{1.6}$$

at least for $q > \frac{2n}{n+2}$ (see [2,3]). So, our results hold, at least for all q satisfying

$$q < \frac{1}{K(2-p)}. \tag{1.7}$$

Corollary 1.1. *Let μ, p, q, f, and ψ be as in Theorem 1.1. Then, if $q > n$, the weak solution of problem (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$, for $\alpha = 1 - \frac{n}{q}$.*

In particular, when $q = 2$, one has the following corollary, where the constant $C_1 = C_1(\Omega)$ is defined by

$$\|D^2 v\| \le C_1\|\Delta v\|, \tag{1.8}$$

for all $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Note that, if Ω is convex, $C_1 = 1$. For details we refer the reader to [4] (Chapter I, estimate (20)).

Corollary 1.2. *Let $p \in (1, 2]$ satisfy $(2-p)C_1 < 1$, where C_1 is defined by (1.8). Assume that $\psi \in W^{\frac{3}{2},2}(\partial\Omega)$, and that $f \in L^{r(2)}(\Omega)$, where*

$$r(2) = \frac{2n}{(n-2)p + (4-n)}. \tag{1.9}$$

Let w be the unique weak solution of problem (1.1). Then w belongs to $W^{2,2}(\Omega)$. Furthermore, if Ω is convex, the result holds for any $1 < p \le 2$.

Note that $r(2) < n$. Further, in the limit case $p = 2$, the system (1.1) reduces to the Poisson equation, and we recover exactly the classical result

$$\|w\|_{2,q} \le C(\|f\|_q + \|\psi\|_{2,q}),$$

since $r(q) = q$ for $p = 2$.

Note that, roughly speaking, the interval of values of the parameter p for which we prove q-regularity may shrink, as q increases.

In the homogeneous case, $\psi = 0$, the above results were proved in [1]. We assume the reader is acquainted with this last reference since in the sequel we will often appeal to it, in order to avoid unnecessary repetitions.

Next we reproduce some comments, presented in [1], concerning the case $p < 2$. For $N = 1$, in [5] the author proves $W^{2,p}_{loc}$-regularity for any $p < 2$. In [6] the authors show $W^{2,2} \cap C^{1,\alpha}$-regularity up to the boundary, in $\Omega \subset \mathbb{R}^2$.

For N-dimensional systems, we recall [7,8]. The techniques used in these references, sometimes quite involved, seem to be not easily applicable if external forces are introduced. The results proved in [7], are local. However, as far as we know, this is the only paper in which the L^2_{loc}-regularity of second derivatives is considered. Actually, the results proved in Refs. [9,1] seem to be, in the non-scalar case, the first regularity results up to the boundary, for the second derivatives of solutions.

For related results we also refer the reader to papers [10–21], and references therein. Last but not least, we recall the famous classical treatise by Ladyzhenskaya and Ural'tseva [22], where related results and deep methods are developed.

For a wide-ranging review on many related results, concerning in particular nonlinear potential theory, non-linear barriers and capacities, and boundary regularity, we recommend the volume [23]. See also [24]. In this context, we recall here the 1972 paper [25], where many pioneering results in all the above fields were established ([25] is full of misprints, luckily very easy to correct). Furthermore, in Ref. [26], by appealing to [25] (and to work previously developed in [27], see also [28]), the following quite explicit geometrical sufficient condition is proved. A boundary point y is regular, with respect to the problem (1.1), if a $(n-1)$-dimensional external cone property is satisfied at the point y (a Lipschitz image of such a cone is sufficient). We recall that a boundary point y is said to be regular if to each continuous boundary data ψ the corresponding solution is continuous in y.

2. Some preliminaries

We start by the following standard definition.

Definition 2.1. Assume that $f \in W^{-1,p'}(\Omega)$. We say that w is a weak solution of problem (1.1) if $w \in W^{1,p}(\Omega)$ satisfies $w - \psi \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega (\mu + |\nabla w|)^{p-2} \nabla w \cdot \nabla \varphi dx = \int_\Omega f \cdot \varphi dx, \tag{2.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

We recall that the existence and uniqueness of a weak solution $w \in W^{1,p}(\Omega)$ can be obtained by appealing to the theory of monotone operators; see Lions volume [29, Chap. 2, sec. 2]. For the reader's convenience, we show in Appendix a $W_0^{1,p}(\Omega)$ estimate, independent of $\mu \geq 0$. See (A.1).

We use the summation convention on repeated indexes. For clearness we remark that the αth component of the $N-$ vector field on the left hand side of the first equation (1.1) is defined by

$$-\partial_j\big((\mu + |\nabla w|)^{p-2}\partial_j u_\alpha\big).$$

In Ref. [30] a crucial device was to write, and to appeal directly, to the very explicit form of all differential equations and boundary conditions in question. This reveals to be quite fruitful, in many subsequent papers (see, for instance, [31] and references therein). Here, this merely leads to the following form for the first equation (1.1)

$$-\Delta w = (p-2)\frac{\nabla w \cdot \nabla^2 w \cdot \nabla w}{(\mu + |\nabla w|)|\nabla w|} + f\,(\mu + |\nabla w|)^{2-p}. \tag{2.2}$$

We use the notation $\nabla w \cdot \nabla^2 w \cdot \nabla w$ to indicate the vector whose αth component is $\nabla w \cdot \big(\partial_j \nabla w\big)\, \partial_j w_\alpha = (\partial_l w_\beta)(\partial^2_{jl} w_\beta)(\partial_j w_\alpha)$.

If $\mu > 0$, sufficiently regular solutions to (2.2) satisfy the first equation (1.1), and inversely. Moreover, they satisfy (2.1).

We introduce the typical change of variables

$$w = \psi + u. \tag{2.3}$$

Obviously, Eqs. (2.2) and

$$-\Delta u = \Delta\psi + (p-2)\frac{\nabla w \cdot \nabla^2 w \cdot \nabla w}{(\mu + |\nabla w|)|\nabla w|} + f\,(\mu + |\nabla w|)^{2-p} \tag{2.4}$$

are strictly equivalent. Note that in (2.4) we appeal to (2.3) only for the linear Laplacian term. This is crucial to controlling the nonlinear term. In the case (1.2) replace $(\mu + |\nabla w|)|\nabla w|$ by $\mu + |\nabla w|^2$.

In the sequel we start by proving the existence of a (unique) strong solution

$$u \in W_0^{1,p}(\Omega) \cap W^{2,q}(\Omega),$$

of problem (2.4) in the case $\mu > 0$ (see Theorem 3.1). Clearly, $w = \psi + u$ solves (2.2) (hence (2.1)). Furthermore, we prove that the $W^{2,q}(\Omega)$ norms of the above strong solutions are uniformly bounded with respect to μ. This allows us, by passing to the limit as $\mu \to 0$, to extend the $W^{2,q}(\Omega)$ regularity result to weak solutions of problem (1.1) in the case $\mu = 0$.

By capital letters, C, C_1, C_2, etc., we denote positive constants independent of $\mu \geq 0$ (some estimates may depend on a fixed, arbitrarily large, upper bound for the values of μ).

We set $\partial_i u = \frac{\partial u}{\partial x_i}$, $\partial^2_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. We denote by $D^2 u$ the set of all the second partial derivatives of u. Moreover we set

$$|D^2 u|^2 := \sum_{\alpha=1}^{N}\sum_{j,h=1}^{n}\big|\partial^2_{jh} u_\alpha\big|^2. \tag{2.5}$$

As in [1], we assume that $q \geq 2$, and set

$$r(q) = \begin{cases} \dfrac{nq}{n(p-1) + q(2-p)} & \text{if } q \in [2, n], \\ q & \text{if } q \geq n. \end{cases} \tag{2.6}$$

Note that $r(q)$ is a strictly increasing function. Moreover, $r(q) > q$ for any $q < n$. Clearly, in (2.6), $r(n) = n$ in both cases.

3. The case $\mu > 0$

In this section we assume that $\mu > 0$. Following [1], we start by proving Theorem 1.1 under this last assumption. Actually, in the sequel we restrict ourselves to showing the indispensable manipulations needed to easily reconstruct a complete proof, with the help of Ref. [1].

Theorem 3.1. *Let $\mu > 0$, and let p, q, and f be as in Theorem 1.1. Then, there is a unique strong solution $u \in W_0^{1,p}(\Omega)\cap W^{2,q}(\Omega)$ of problem (2.4). Moreover, the following estimate holds*

$$\|\Delta u\|_q \leq \frac{2}{\alpha}B + \left(\frac{2C_3^{2-p}}{\alpha}\right)^{\frac{1}{p-1}} \|f\|_{r(q)}^{\frac{1}{p-1}}, \tag{3.1}$$

where

$$\alpha = 1 - C_2(q)(2-p),$$

and $B = B(\psi, f)$ is given by

$$B(\psi, f) = \|\Delta\psi\|_q + \mu^{2-p}\|f\|_q + (2-p)\|D^2\psi\|_q + C_4^{2-p}\|\Delta\psi\|_q^{2-p}\|f\|_{r(q)}. \tag{3.2}$$

The following result is an immediate corollary of Theorem 3.1.

Theorem 3.2. *Let $\mu > 0$, and let p, q, and f be as in Theorem 1.1. Then, there is a unique strong solution $w \in W^{1,p}(\Omega)\cap W^{2,q}(\Omega)$, of problem (2.2), for which $w = \psi$ on $\partial\Omega$. Moreover, the estimate (1.4) holds.*

As in [1], we prove Theorem 3.1 by appealing to the following fixed point theorem.

Theorem 3.3. *Let X be a reflexive Banach space and $\mathbb{K}$ a non-empty, convex, bounded, closed subset of X. Let F be a map defined in $\mathbb{K}$, such that $F(\mathbb{K}) \subset \mathbb{K}$.*

Assume that there is a Banach space Y such that the following hold.

(i) $X \subset Y$, with compact (completely continuous) immersion.

(ii) If $v^n \in \mathbb{K}$ converges weakly in X to some $v \in \mathbb{K}$ then there is a subsequence v^m such that $F(v^m) \to F(v)$ in Y.

Under the above hypotheses the map F has a fixed point in $\mathbb{K}$.

We appeal to this theorem with $X = W^{2,q}$ and $Y = L^q$. Clearly, point (i) in Theorem 3.3 holds immediately.

Proof of Theorem 3.1. Define

$$\mathbb{K} = \mathbb{K}(R) = \{v \in W^{2,q}(\Omega) : \|\Delta v\|_q \leq R,\ v = 0 \text{ on } \partial\Omega\}.$$

Let $f \in L^{r(q)}$ be given. For each $v \in W^{2,q} \cap W_0^{1,q}$ we define $u = F(v)$ as the solution to the linear problem

$$\begin{cases} -\Delta u = \Delta\psi + (p-2)\dfrac{\nabla\widetilde{w}\cdot\nabla^2\widetilde{w}\cdot\nabla\widetilde{w}}{(\mu + |\nabla\widetilde{w}|)|\nabla\widetilde{w}|} + f\,(\mu + |\nabla\widetilde{w}|)^{2-p}, & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where $\widetilde{w} = \psi + v$. We look for a fixed point $u = v$ in $\mathbb{K}$.

The main point in the proof is showing that $F(\mathbb{K}) \subset \mathbb{K}$ for some R which, in fact, is essentially equivalent to proving an a priori estimate. In the sequel we merely show this point, since the remainder of the proof follows as in [1].

For each $v \in W^{2,q} \cap W_0^{1,q}$ we define $C_3 = C_3(q)$ by

$$\begin{aligned} &\|\nabla v\|_{q^*} \leq C_3\|\Delta v\|_q, \quad \text{if } q \in (1, n), \\ &\|\nabla v\|_\infty \leq C_3\|\Delta v\|_q, \quad \text{if } q \in (n, +\infty). \end{aligned} \tag{3.4}$$

Moreover, for each $\psi \in W^{2,q}$ one has

$$\begin{aligned} &\|\nabla\psi\|_{q^*} \leq C_4\|\psi\|_{2,q}, \quad \text{if } q \in (1, n), \\ &\|\nabla\psi\|_\infty \leq C_4\|\psi\|_{2,q}, \quad \text{if } q \in (n, +\infty), \end{aligned} \tag{3.5}$$

where $C_4 = C_4(q)$.

For $q \in (1, n)$, by using $(3.4)_1$ and $(3.5)_1$, we have

$$\begin{aligned} \||\nabla\widetilde{w}|^{2-p}f\|_q &\leq \|\nabla\widetilde{w}\|_{q^*}^{2-p}\|f\|_{r(q)} \leq (\|\nabla\psi\|_{q^*}^{2-p} + \|\nabla v\|_{q^*}^{2-p})\|f\|_{r(q)} \\ &\leq C_4^{2-p}\|\Delta\psi\|_q^{2-p}\|f\|_{r(q)} + C_3^{2-p}\|\Delta v\|_q^{2-p}\|f\|_{r(q)}. \end{aligned}$$

Analogously, for $q > n$, by using $(3.4)_2$, $(3.5)_2$, and by recalling that $r(q) = q$ if $q > n$, we show that the last inequality holds by replacing the exponent q^* by ∞. So, in both cases,

$$\| |\nabla \widetilde{w}|^{2-p} f \|_q \le C_4^{2-p} \|\Delta \psi\|_q^{2-p} \|f\|_{r(q)} + C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_{r(q)}. \tag{3.6}$$

Hence, by appealing to the inequality $(\mu + |\nabla \widetilde{w}|)^{2-p} \le \mu^{2-p} + |\nabla \widetilde{w}|^{2-p}$, one shows that

$$\|(\mu + |\nabla \widetilde{w}|)^{2-p} f\|_q \le \mu^{2-p} \|f\|_q + C_4^{2-p} \|\Delta \psi\|_q^{2-p} \|f\|_{r(q)} + C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_{r(q)}. \tag{3.7}$$

Next, we estimate the second term on the right hand side of (3.3) by appealing to the following algebraic lemma.

Lemma 3.4. *One has, pointwisely in Ω,*

$$|\nabla v \cdot \nabla^2 v \cdot \nabla v| \le |\nabla v|^2 |D^2 v|. \tag{3.8}$$

For the proof, see Section 5. This shows that

$$\left\| \frac{\nabla \widetilde{w} \cdot \nabla^2 \widetilde{w} \cdot \nabla \widetilde{w}}{(\mu + |\nabla \widetilde{w}|)|\nabla \widetilde{w}|} \right\|_q \le \|D^2 \widetilde{w}\|_q \le \|D^2 \psi\|_q + C_2(q) \|\Delta v\|_q. \tag{3.9}$$

The estimates (3.7) and (3.9) show that the right-hand side of (3.3) belongs to $L^q(\Omega)$, for each $v \in W^{2,q} \cap W_0^{1,q}$. More precisely,

$$\begin{aligned} \|\Delta u\|_q \le\; & \|\Delta \psi\|_q + \mu^{2-p} \|f\|_q + (2-p) \|D^2 \psi\|_q + C_4^{2-p} \|\Delta \psi\|_q^{2-p} \|f\|_{r(q)} + (2-p) C_2(q) \|\Delta v\|_q \\ & + C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_{r(q)}. \end{aligned} \tag{3.10}$$

So, by a classical regularity result for the second order derivatives of solutions to the Laplace equation under Dirichlet boundary conditions, we show from Eq. (3.3) that there is a unique $u \in W^{2,q}(\Omega)$, solution to the Dirichlet problem (3.3), where $\widetilde{w} = \psi + v$.

It remains to show that if $\|\Delta v\|_q \le R$ then the corresponding solution $u = F(v)$ satisfies the same estimate, namely $\|\Delta u\|_q \le R$.

Since $v \in \mathbb{K}$ it follows that

$$\|\Delta u\|_q \le B + (2-p) C_2(q) R + C_3^{2-p} \|f\|_{r(q)} R^{2-p}, \tag{3.11}$$

where $B = B(\psi, f)$ is given by (3.2). So $u \in \mathbb{K}(R)$ if

$$[1 - (2-p) C_2(q)] R \ge B + C_3^{2-p} \|f\|_{r(q)} R^{2-p}.$$

This inequality is satisfied if, for instance, its left hand side is equal to two times the sum of the two terms on the right hand side. This holds for

$$R = \frac{2}{\alpha} B + \left(\frac{2 C_3^{2-p}}{\alpha} \right)^{\frac{1}{p-1}} \|f\|_{r(q)}^{\frac{1}{p-1}}. \tag{3.12}$$

So, $\|\Delta u\|_q \le R$. Hence $F(\mathbb{K}) \subset \mathbb{K}$, and the estimate (3.1) is satisfied. □

Theorem 3.2 follows immediately. In particular, the estimate (1.4) follows easily from (3.1) by appealing to (1.5) and (2.3), and by noting that B is bounded by the right hand side of (1.4).

4. The case $\mu = 0$

We start by recalling the definition of weak solution w^μ of problem (1.1), for $\mu \ge 0$. One has $w^\mu - \psi \in W_0^{1,p}(\Omega)$, moreover

$$\int_\Omega (\mu + |\nabla w^\mu|)^{p-2} \nabla w^\mu \cdot \nabla \varphi dx = \int_\Omega f \cdot \varphi dx, \tag{4.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. These conditions are satisfied by the strong solutions w^μ, for $\mu > 0$, constructed in the previous section. These solutions are uniformly bounded in $W^{2,q}(\Omega)$, since the right hand side of (1.4) does not depend on the parameter μ (assume, for instance, $\mu \le 1$). So, suitable sub-sequences, which we continue to denote by w^μ, weakly converge in $W^{2,q}(\Omega)$ to some w. The argument followed in [1] shows that we may pass to the limit in (4.1) to prove that

$$\int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi dx = \int_\Omega f \cdot \varphi dx, \tag{4.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. So, $w \in W^{2,q}(\Omega)$ is the solution (known to be unique), corresponding to $\mu = 0$. Clearly (1.4) holds. To prove the above claim, we have to show that, for each fixed $\varphi \in W_0^{1,p}(\Omega)$, the left hand side of Eq. (4.1) converges to the left hand side of (4.2). The manipulations followed in Section 4 of [1] apply here, by simply replacing the symbols u and u^μ used in [1], by w and w^μ, respectively. For the reader's convenience, we repeat here the proof given in the above reference.

Since $w^\mu \rightharpoonup w$ weakly in $W^{2,q}(\Omega)$, one has strong convergence (of suitable subsequences) in $W^{1,s}(\Omega)$, for any s if $q > n$, and for $s < q^*$ if $q < n$. So, strong convergence in $W^{1,p}(\Omega)$ holds.

We write the integral on the left-hand side of (4.1) as

$$\int_\Omega \left[(\mu + |\nabla w^\mu|)^{p-2}\, \nabla w^\mu - (\mu + |\nabla w|)^{p-2}\, \nabla w\right] \cdot \nabla\varphi dx + \int_\Omega (\mu + |\nabla w|)^{p-2}\, \nabla w \cdot \nabla\varphi dx, \tag{4.3}$$

and show that the first integral tends to zero, and that the second integral tends to the left hand side of (4.2). The inequality

$$|(\mu + |A|)^{p-2}A - (\mu + |B|)^{p-2}B| \le C\frac{|A - B|}{(\mu + |A| + |B|)^{2-p}}, \tag{4.4}$$

see [32], where C is independent of μ, shows that the absolute value of the first integral in Eq. (4.3) is bounded by

$$C\int_\Omega (\mu + |\nabla w| + |\nabla w^\mu|)^{p-2}\, |\nabla w - \nabla w^\mu||\nabla\varphi|dx.$$

Since

$$(\mu + |\nabla w| + |\nabla w^\mu|)^{p-2}\, |\nabla w - \nabla w^\mu| \le |\nabla w - \nabla w^\mu|^{p-1}\,,$$

the absolute value of the first integral in Eq. (4.3) is bounded by

$$C\|\nabla w^\mu - \nabla w\|_p^{p-1}\|\nabla\varphi\|_p,$$

which tends to zero with μ.

Finally,

$$\lim_{\mu\to 0^+}\int_\Omega (\mu + |\nabla w|)^{p-2}\, \nabla w \cdot \nabla\varphi dx = \int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi dx,$$

by Lebesgue's dominated convergence theorem.

5. Proof of Lemma 3.4

We prove here the *algebraic* estimate (3.8). Note that the differential structure is not significant here. Set $I = \nabla w \cdot \nabla\nabla w \cdot \nabla w$. One has, for an arbitrary $\xi \in \mathbb{R}^N$,

$$|I \cdot \xi| = \left|\sum_{j,l}\left(\sum_\alpha \xi_\alpha(\partial_j w_\alpha)\right)\left(\sum_\beta (\partial_l w_\beta)(\partial^2_{jl} w_\beta)\right)\right|.$$

By appealing to the Cauchy–Schwartz inequality one shows that

$$\begin{aligned} |I \cdot \xi| &\le \sum_{j,l}(|\xi||\partial_j w|)(|\partial_l w||\partial^2_{jl} w|) \\ &\le |\xi|\sum_{j,l}(|\partial_j w||\partial_l w|)|\partial^2_{jl} w| \\ &\le |\xi|\left(\sum_{j,l}|\partial_j w|^2|\partial_l w|^2\right)^{\frac{1}{2}}|D^2 w|. \end{aligned}$$

Eq. (3.8) follows.

Appendix. $W^{1,p}$ estimate

Let $f \in W^{1,p'}(\Omega)$. From (2.1) one gets, where $u = w - \psi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (\mu + |\nabla w|)^{p-2}\, \nabla w \cdot \nabla u dx = \langle f, u\rangle.$$

Hence,

$$\int_\Omega (\mu + |\nabla w|)^{p-2}\, \nabla w \cdot \nabla w dx = \int_\Omega (\mu + |\nabla w|)^{p-2}\, \nabla w \cdot \nabla\psi dx + \langle f, u\rangle.$$

Since $p-2 \leq 0$, it readily follows that

$$\int_{\Omega} (\mu + |\nabla w|)^{p-2} |\nabla w|^2 dx \leq \|\nabla w\|_p^{p-1} \|\nabla \psi\|_p + \|f\|_{-1,p'} (\|\nabla w\|_p + \|\nabla \psi\|_p).$$

So, if $\mu = 0$, straight forward calculations lead to an a priori bound. For instance,

$$\|\nabla w\|_p \leq 2\|\nabla \psi\|_p + 3^{\frac{1}{p-1}} \|f\|_{-1,p'}^{\frac{1}{p-1}}.$$

If $\mu > 0$, we may argue as follows. The integral on the left hand side is greater than or equal to the integral restricted to the set where $|\nabla u| \geq \mu$. In particular it is greater than or equal to

$$2^{p-2}(\|\nabla w\|_p^p - \mu^p |\Omega|).$$

Hence (for convenience we drop the lower labels in norms notation),

$$\|\nabla w\|^p \leq 2^{2-p} \|\nabla \psi\| \|\nabla w\|^{p-1} + 2^{2-p} \|f\| \|\nabla w\| + 2^{2-p} \|f\| \|\nabla \psi\| + \mu^p |\Omega|.$$

It readily follows that

$$\|\nabla w\|_p \leq c_p \left(\|\nabla \psi\|_p + \|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu |\Omega|^{\frac{1}{p}} \right), \tag{A.1}$$

where c_p is a suitable positive constant.

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