Concerning the $W^{k,p}$-inviscid limit for 3-D flows under a slip boundary condition

by H. Beirão da Veiga and F. Crispo

April 30, 2013

Abstract

We consider the $3-D$ evolutionary Navier-Stokes equations with a Navier slip-type boundary condition, see (1.2), and study the problem of the strong convergence of the solutions, as the viscosity goes to zero, to the solution of the Euler equations under the zero-flux boundary condition. We prove here, in the flat boundary case, convergence in Sobolev spaces $W^{k,\,p}(\Omega)$, for arbitrarily large k and p (for previous results see [42] and [9]). However this problem is still open for non-flat, arbitrarily smooth, boundaries. The main obstacle consists in some boundary integrals, which vanish on flat portions of the boundary. However, if we drop the convective terms (Stokes problem), the inviscid, strong limit result holds, as shown below. The cause of this different behavior is quite subtle.

As a by-product, we set up a very elementary approach to the regularity theory, in L^p-spaces, for solutions to the Navier-Stokes equations under slip type boundary conditions.

Mathematics Subject Classification 35Q30, 76D05, 76D09.

Keywords. Stokes and Navier-Stokes equations, inviscid limit, slip boundary conditions.

1 Introduction and results.

We investigate strong convergence up to the boundary, as $\nu \to 0$, of the solutions u^ν of the Navier-Stokes equations

$$(1.1)\qquad \begin{cases} \partial_t\, u^\nu + (u^\nu\cdot\nabla)\, u^\nu - \nu\,\Delta\, u^\nu + \nabla\,\pi = 0, \\ div\, u^\nu = 0\,, \\ u^\nu(0) = u_0\,, \end{cases}$$

under the boundary condition

$$(1.2)\qquad \begin{cases} (u^\nu\cdot \underline{n})_{|\Gamma} = 0, \\ \omega^\nu \times \underline{n} = 0\,, \end{cases}$$

to the solution u of the Euler equations

$$(1.3)\qquad \begin{cases} \partial_t\, u + (u\cdot\nabla)\, u + \nabla\,\pi = 0, \\ div\, u = 0\,, \\ u(0) = u_0\,, \end{cases}$$

under the zero-flux boundary condition

$$u \cdot \underline{n} = 0 \,. \tag{1.4}$$

Here $\omega = curl\, u$, and Ω is an open bounded set in $\mathbb{R}^3$ locally situated on one side of its boundary Γ. For convenience, we assume that Ω is simply-connected. We denote by $\underline{n} = (n_1, n_2, n_3)$ the unit outward normal to Γ. We recall that application of the operator *curl* to the first equation (1.1) leads to the well known equation

$$\partial_t \omega - \nu \Delta \omega + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = 0 \,. \tag{1.5}$$

The problem is studied in a suitable time-interval $[0, T]$, independent of ν. We recall that, on flat portions of the boundary, the slip boundary conditions (1.2) and (5.1) coincide. In the general case they differ by lower order terms. The literature on this type of conditions is very wide. Navier was the first to propose these conditions, see [34]. We also refer to [37], and to [1], [3], [5], [6], [11], [12], [13], [17], [18], [22], [23], [28], [33] and [41].

Concerning the vanishing viscosity limit in bounded domains, with slip boundary conditions, the problem is mainly studied in the $2-D$ case. Note that, in this case, the assumption $\omega \times \underline{n} = 0$ on Γ is simply replaced by $\omega = 0$. We refer to the classical papers, [2], [24], [36]. See also the more recent papers [9], [14], and [31].

Vanishing viscosity limit results in $3-D$ domains, without boundary conditions, have been studied by many authors. See, for instance, [15], [25], [26], [27], [30], [38], and the more recent papers [8], [32] (in [8], [25] and [32] results are proved in *strong topology*). Concerning this last problem in bounded domains, under slip boundary conditions, we refer to [9], [21], [42] and references therein.

We point out that our present work should be seen in the framework of *strong* inviscid limit results in three-dimensional domains. This means here to obtain a priori estimates in $L^\infty(0, T; W^{k,p}(\Omega))$, independent of $\nu > 0$, with $k > 1 + \frac{3}{p}$ (in this context, concerning the existence of strong, unique solutions to the Euler equations (1.3), (1.4) we refer to [39]). To our knowledge the $3-D$ strong inviscid limit has been studied by few authors. In [42] the following result ([42], Theorem 8.1) is stated.

Theorem 1.1. *Assume that the initial data u_0 belongs to $W^{3,2}(\Omega)$, is divergence free, and satisfies the boundary conditions (1.2). Then*

$$\begin{cases} u^\nu \to u \quad in \quad L^p(0, T_0; W^{3,2}(\Omega))\,, & for\ each\ p \in [1, \infty[\,, \\ u^\nu \to u \quad in \quad C([0, T_0]; W^{2,2}(\Omega))\,. \end{cases} \tag{1.6}$$

In reference [9] the results are obtained in the context of $W^{2,p}$ and $W^{3,p}$ spaces, for any finite p. Ω is a cubic domain, and the boundary condition (1.2) is imposed only on two opposite faces. On the other faces periodicity is assumed, as a device to avoid unessential technical difficulties. The following result holds (see [9], Theorem 1.1).

Theorem 1.2. *Let Ω be a cubic domain, and impose the boundary condition (1.2) only on two opposite faces (on the other faces assume periodicity). Assume* *that* $p > \frac{3}{2}$ *and that the initial data* u_0 *belongs to* $W^{3,\,p}(\Omega)$ *, is divergence free, and satisfies the boundary conditions* (1.2)*. Then*

$$\tag{1.7} \begin{cases} u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\,T_0;\,W^{3,\,p}(\Omega)) \quad weak-*\,, \\ \\ u^\nu \to u \quad in \quad C([0,\,T_0];\,W^{s,\,p}(\Omega))\,, \quad for\ each\ s < 3\,. \end{cases}$$

Further,

$$\tag{1.8} \partial_t\, u^\nu \to \partial_t\, u \quad in \quad L^\infty(0,\,T_0;\,W^{1,\,p}(\Omega))$$

and, if $p \geq 2$ *,*

$$\tag{1.9} \partial_t\, u^\nu \to \partial_t\, u \quad in \quad L^p(0,\,T_0;\,W^{1,\,3\,p}(\Omega))\,.$$

We note that, by arguing as in [9], the estimates proved in reference [42] may lead to the more stringent results stated in Theorem 1.2 in the particular case $p = 2$.

The proofs presented in both the above references seem to require flat-boundaries. Hence the problem remains open in the presence of smooth boundaries. The main obstacle consists of some boundary integrals, resulting from an integration by parts related to the viscous term (these integrals vanish on flat portions of the boundary).

In this paper we consider the following problems:

a) *Extension of Theorem 1.2 to arbitrary* $W^{k,\,p}$ *spaces.* We prove the following result:

Theorem 1.3. *Let* Ω *be a cubic domain, and impose the boundary condition* (1.2) *only on two opposite faces (on the other faces assume periodicity). Assume that* $p \geq 2$ *. The initial data* u_0 *belongs to* $W^{k_0,\,p}(\Omega)$ *, for some* $k_0 \geq 3$ *, is divergence free, satisfies the boundary condition* (1.2) *and the necessary compatibility conditions on* Γ *, at time* $t = 0$*. Then*

$$\tag{1.10} \begin{cases} u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\,T_0;\,W^{k_0,\,p}(\Omega)) \quad weak-*\,, \\ \\ u^\nu \to u \quad in \quad C([0,\,T_0];\,W^{s,\,p}(\Omega))\,, \quad for\ each\ s < k_0\,. \end{cases}$$

Further,

$$\tag{1.11} \partial_t\, u^\nu \to \partial_t\, u \quad in \quad L^\infty(0,\,T_0;\,W^{k_0-2,\,p}(\Omega))\,,$$

and

$$\tag{1.12} \partial_t\, u^\nu \to \partial_t\, u \quad in \quad L^p(0,\,T_0;\,W^{k_0-2,\,3\,p}(\Omega))\,.$$

We remark that the above theorem holds for $p > \frac{3}{2}$ (except (1.12)). However, for simplicity, we everywhere assume that $p \geq 2$. Moreover, in the following we just prove the a priori estimates that lead to the existence of the solutions to the problem (1.1), (1.2), for fixed $\nu > 0$. The effective construction of the solution may be obtained, for instance, via the classical Faedo-Galerkin procedure (see, for instance, [15], [16], [29], [40]).

Observe that the previous theorem gives in particular the $W^{k_0,p}$-regularity, for each $k_0 \geq 3$, for solutions of the Navier-Stokes boundary value problem (1.1), (1.2) with fixed viscosity in the flat-boundary case. See the next point c).

b) *To develop a suitable strategy for extending the inviscid limit results to smooth, arbitrary, domains.* We partially succeed in this attempt, insofar as our approach works well for Stokes problems. The reason for these two distinct behaviors is quite subtle, as shown below. It suffices to say that the obstacle resulting from the addition of the convective term is not due to the related volume integral (the classical "trilinear form", when $p = 2$), but to the destabilizing effect of the convective term on boundary integrals. The reader should note that exactly the same boundary integrals already occur in the Stokes framework. In other words, the main obstacle is due to the combination, near the boundary, of viscosity with convection. In fact, if we drop the convective terms (Stokes problem) the inviscid limit result holds. Actually, we show the following result.

Theorem 1.4. *Assume that Ω is of class C^3. For each $\nu > 0$ denote by u^ν the solution to the initial boundary value Stokes problem*

$$
\begin{cases}
\partial_t u^\nu - \nu \Delta u^\nu + \nabla \pi = 0, \\
div\, u^\nu = 0\,, \\
u^\nu(0) = u_0\,,
\end{cases}
\tag{1.13}
$$

under the boundary condition (1.2). *Assume that the initial data u_0 belongs to $W^{2,p}(\Omega)$, is divergence free in Ω, and satisfies the boundary conditions* (1.2). *Further, consider the solution u to the problem*

$$
\begin{cases}
\partial_t u + \nabla \pi = 0, \\
div\, u = 0\,, \\
u(0) = u_0\,,
\end{cases}
\tag{1.14}
$$

under the zero-flux boundary condition (1.4). *Then*

$$
\begin{cases}
u^\nu \rightharpoonup u \quad in \quad L^\infty(0,\, T_0;\, W^{2,p}(\Omega)) \quad weak - *\,, \\
\\
u^\nu \to u \quad in \quad C([0,\, T_0];\, W^{s,p}(\Omega))\,, \quad for\ each\ s < 2\,.
\end{cases}
\tag{1.15}
$$

Further

$$
\partial_t u^\nu \to \partial_t u \quad in \quad L^\infty(0,\, T_0;\, L^p(\Omega))\,, \tag{1.16}
$$

and

$$
\partial_t u^\nu \to \partial_t u \quad in \quad L^p(0,\, T_0;\, L^{3p}(\Omega))\,. \tag{1.17}
$$

Since the problem (1.13) is linear, the above results hold for arbitrarily large values of T_0. Note that, from (1.14), it follows that $\partial_t u = 0$, hence $u(t,\, x) = u_0(x)$ for all t.

Our main interest in the above result is theoretical: Just a better understanding of the real obstacle due to the introduction of the convective term. So, we did not try to extend the Theorem 1.4 to higher order derivatives.

Remark 1.1. *It is worth noting that, in the presence of the convective term, we are able to lower the order of the derivatives occurring in the above troublesome boundary integrals, see* (3.16). *However this seems insufficient to overcome the main obstacle. This main challenging point will be discussed in section 3 below.*

c) *Regularity of the solutions.* As a by-product of our estimates we also touch the regularity problem for solutions to the Navier-Stokes problem (1.1), (1.2), for fixed viscosity ν. It is worth noting that in reference [19] G. Grubb proves very general, strong, and complete, regularity results for solutions to the Navier-Stokes equations under different, even nonhomogeneous, boundary conditions. The above reference follows previous work in collaboration with V.A. Solonnikov (see, for instance, [20], and references in [19]). Actually, the proofs in reference [19] are particularly involved. Hence, simpler approaches are desirable, even if the results obtained are much less general. We give only just a contribution in this direction. Further developments seem possible. We consider the case in which an external force f is present in the right hand side of the first equation (1.1). To simplify the presentation, we assume that

$$f \in L^p(0,\, T;\, W^{k,\,p}(\Omega))\,, \tag{1.18}$$

for a suitable integer k. However, more general (or different) choices are possible.

Theorem 1.5. *Let Ω be a regular open, bounded, set in $\mathbb{R}^3$ and let $\nu > 0$ be given. Assume that $p \geq 2$ and that an external force f, satisfying* (1.18) *for $k = 1$, has been added to the right hand side of the first equation* (1.1). *Let $u_0 \in W^{1,\,p}(\Omega)$ be a given divergence free vector field satisfying* (1.4). *Then there is a $T > 0$ such that the solution $u = u^\nu$ to the problem* (1.1), (1.2) *in* $[0,\, T]$ *satisfies*

$$u \in L^\infty(0,\, T;\, W^{1,\,p}(\Omega)) \tag{1.19}$$

and

$$\begin{cases} |\,\omega\,|^{\frac{p-2}{2}}\,|\,\nabla\,\omega\,| \in L^2(0,\, T;\, L^2(\Omega)) \\ \\ |\,\omega\,|^{\frac{p}{2}} \in L^2(0,\, T;\, W^{1,\,2}(\Omega))\,, \end{cases} \tag{1.20}$$

where $\omega = curl\, u$. In particular, by setting $p = 2$,

$$u \in L^2(0,\, T;\, W^{2,\,2}(\Omega))\,. \tag{1.21}$$

Theorem 1.6. *Let Ω be a regular open, bounded, set in $\mathbb{R}^3$ and let $\nu > 0$ be given. Assume that $p \geq 2$ and that an external force f, satisfying* (1.18) *for $k = 2$, has been added to the right hand side of the first equation* (1.1). *Let $u_0 \in W^{2,\,p}(\Omega)$ be a given divergence free vector field satisfying* (1.2). *Then there is a $T > 0$ such that the solution $u = u^\nu$ to the problem* (1.1), (1.2) *in* $[0,\, T]$ *satisfies*

$$u \in L^\infty(0,\, T;\, W^{2,\,p}(\Omega)) \tag{1.22}$$

and

$$(1.23) \qquad \begin{cases} |\,\zeta\,|^{\frac{p-2}{2}}\,|\,\nabla\,\zeta\,| \in\, L^2(0,\,T;\,L^2(\Omega)) \\ \\ |\,\zeta\,|^{\frac{p}{2}}\,\in\, L^2(0,\,T;\,W^{1,\,2}(\Omega))\,, \end{cases}$$

where $\zeta = \,-\,\Delta\,u\,.$ *In particular, by setting* $p=\,2,$ *one shows that*

$$(1.24) \qquad u \in\, L^2(0,\,T;\,W^{3,\,2}(\Omega))\,.$$

By directly appealing to the equations (1.1) and to the results stated above, we may easily obtain regularity results for time derivatives and pressure, and also an estimate for T in terms of initial data and viscosity.

2 Sharp $W^{k,p}$- inviscid limit results in the flat boundary case

From now on we denote the solutions u^ν by u. We use the following notation:

$$\|u\|_p = \,\|u\|_{L^p(\Omega)}\,, \quad \|u\|_{k,\,p} = \,\|u\|_{W^{k,\,p}(\Omega)}\,, \quad \|u\|_{p;\,\Gamma} = \,\|u\|_{L^p(\Gamma)}\,.$$

Moreover, the Einstein's convention of summation over repeated indexes is assumed.

We start with the following auxiliary result (see [35] and [4]).

Lemma 2.1. *Let* Ω *be a regular open, bounded, set in* $\mathbb{R}^3$ *,* Γ *its boundary and let* v *be a sufficiently regular vector field. Then,*

$$(2.1) \qquad \begin{aligned} &-\int_\Omega \Delta\,v\,\cdot\,(\,|v|^{p-\,2}\,v)\,dx = \,\tfrac{1}{2}\,\int\,|v|^{p-\,2}\,|\,\nabla\,v|^2\,dx \\ &+4\,\tfrac{p-\,2}{p^2}\,\int\,\big|\,\nabla\,|v|^{\frac{p}{2}}\,\big|^2\,dx -\,\int_\Gamma\,|\,v\,|^{p-\,2}\,(\partial_i\,v_j)\,n_i\,v_j\,d\Gamma\,. \end{aligned}$$

Note that

$$(2.2) \qquad \big|\,\nabla\,|v|^{\frac{p}{2}}\,\big|^2 \leq\, \big(\,\frac{p}{2}\big)^2\,|v|^{p-\,2}\,|\nabla\,v|^2\,.$$

For the proof of the next lemma see, for instance, [10].

Lemma 2.2. *Assume, in addition to the hypotheses of Lemma 2.1, that* v *is a vector field satisfying* $v \times \underline{n} =\, 0$ *on* Γ *. Then* $curl\,v$ *is tangential to* Γ*. In particular,*

$$(2.3) \qquad \partial_j\,(\,(\,curl\,v)\,\cdot\,\underline{n})\,(\,curl\,v)_j =\, 0\,.$$

Lemma 2.3. *If* v *and* $\underline{n}$ *are two arbitrary, sufficiently regular, vector fields then*

$$(2.4) \qquad \begin{aligned} (\partial_i\,v_j)\,n_i\,v_j &=\,(curl\,v)\,\times\,\underline{n}\cdot\,v +\,(\partial_j\,v_i)\,n_i\,v_j \\ &=\,(curl\,v)\,\times\,\underline{n}\cdot\,v +\,\partial_j\,(v\,\cdot\,\underline{n})\,v_j -\,(\partial_j\,n_i)\,v_i\,v_j\,. \end{aligned}$$

Throughout the remaining part of this section we consider a cubic domain $\Omega = (]0,1[)^3$ and set

$$\Gamma = \{x : 0 \leq x_1,\, x_2 \leq 1\,, \quad \text{and} \quad x_3 = 0 \quad \text{or} \quad x_3 = 1\}\,.$$

The boundary condition (1.2) is imposed on Γ. The problem is assumed to be periodic, with period equal to 1, both in the x_1 and x_2 directions.

In the sequel k, l, m denote non-negative integers and k_0 denotes an arbitrary, but fixed, integer such that $k_0 \geq 3$. For any k we formally set

$$\omega^{k+1} = curl\, \zeta^k \quad \text{and} \quad \zeta^{k+1} = curl\, \omega^{k+1}\,, \tag{2.5}$$

with $\zeta^0 = u$. Further, for any suitably smooth function g by $\Delta^k g$ we mean the Laplace operator applied k times to g, with $\Delta^0 g = g$.

In the flat boundary case, the boundary condition (1.2) is simply

$$u_3 = \omega^1_1 = \omega^1_2 = 0 \quad \text{on} \quad \Gamma\,. \tag{2.6}$$

Further, from $\omega^1_1 = \omega^1_2 = 0$ on Γ and $div\, \omega^1 = 0$ it follows that

$$\partial_3\, \omega^1_3 = 0 \quad \text{on} \quad \Gamma\,. \tag{2.7}$$

Proposition 2.1. *Assume that $\omega^1_1 = \omega^1_2 = 0$ on Γ and $div\, \omega^1 = 0$ in Ω. Then*

$$(\partial_i\, \omega^1_j)\, n_i\, \omega^1_j = 0 \quad on \quad \Gamma\,. \tag{2.8}$$

The result follows by appealing to (2.7) and to $n_1 = n_2 = 0$.

The following lemma was inspired by [42].

Lemma 2.4. *Let u be a vector field in Ω, and $\omega^1 = curl\, u$. Assume that $u_3 = \omega^1_1 = \omega^1_2 = 0$ on Γ. Then the vector fields $(u \cdot \nabla)\, \omega^1$ and $(\omega^1 \cdot \nabla)\, u$ are normal to Γ.*

The proof is left to the reader. Note that $\partial_3\, u_1 = \omega^1_2 + \partial_1\, u_3 = 0$ on Γ, and similarly for $\partial_3\, u_2$.

Lemma 2.5. *Assume, in addition to the hypotheses of Lemma 2.4, that ω^1 satisfies the equation* (1.5). *Then*

$$(\, curl\, \zeta^1) \times \underline{n} = 0$$

on Γ, where $\zeta^1 = curl\, \omega^1$.

Since ω^1 is normal to the boundary, so is $\partial_t\, \omega^1$. From (1.5), by appealing to Lemma 2.4, it follows that $curl\, \zeta^1 = -\,\Delta\, \omega^1$ is normal to the boundary.

For an integer $k \geq 1$, let us consider the following equations, obtained by formally taking $2(k-1)$ and $2k-1$ times the curl of equations (1.5), respectively:

$$\partial_t\, \omega^k - \nu\, \Delta\, \omega^k = (-1)^{k-1} \Delta^{k-1} \left((\omega^1 \cdot \nabla)\, u - (u \cdot \nabla)\, \omega^1 \right)\,, \tag{2.9}$$

$$\partial_t\, \zeta^k - \nu\, \Delta\, \zeta^k = (-1)^{k-1}\, curl\, \Delta^{k-1} \left((\omega^1 \cdot \nabla)\, u - (u \cdot \nabla)\, \omega^1 \right)\,. \tag{2.10}$$

Our next aim is to use an induction argument to prove that the previous properties, stated for ω^1 and ζ^1, hold as well for ω^k, for any $k \in \left[1, \frac{k_0+1}{2}\right]$, and for ζ^k, for any $k \in \left[1, \frac{k_0}{2}\right]$.

We will prove the following result.

Proposition 2.2. *Assume that the hypotheses of Lemma 2.4 hold. Further assume that ω^k satisfies* (2.9) *for any $k \in \left[1, \frac{k_0+1}{2}\right]$. Then one has*

$$\omega^k \times \underline{n} \,=\, 0\,, \quad \zeta^k \cdot \underline{n} \,=\, 0 \quad \text{on } \Gamma\,, \quad \forall\, k \in \left[1, \frac{k_0+1}{2}\right]. \tag{2.11}$$

Moreover the right hand side of (2.9) *is normal to* Γ.

We introduce some further notation. If f and g are suitably regular scalar functions defined on Ω, we set

$$\left(D^{|\gamma|} f\right) \cdot \left(D^{|\gamma|} g\right) = \sum_{i_1, \cdots, i_{|\gamma|}=1}^{3} \left(\partial^{|\gamma|}_{\partial x_{i_1} \cdots \partial x_{i_{|\gamma|}}} f(x)\right) \left(\partial^{|\gamma|}_{\partial x_{i_1} \cdots \partial x_{i_{|\gamma|}}} g(x)\right). \tag{2.12}$$

However, we are not interested in the number of times each summand appears in the above sum, since we will show that, under our hypotheses (flat boundary), each single term is equal to zero. So, we simply write

$$\left(D^{|\gamma|} f\right) \cdot \left(D^{|\gamma|} g\right) = \sum_{\gamma_1+\gamma_2+\gamma_3=|\gamma|} c(\gamma_1, \gamma_2, \gamma_3) \left(\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} f(x)\right) \left(\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} g(x)\right), \tag{2.13}$$

where, for each i, $\gamma_i \leq |\gamma|$ is a non-negative integer, $c(\gamma_1, \gamma_2, \gamma_3)$ is a non-negative constant and $\partial_i^{\gamma_i} f(x) = \frac{\partial^{\gamma_i}}{\partial x_i^{\gamma_i}} f(x)$.

Lemma 2.6. *For each non-negative integer m such that $m \leq \frac{k_0-1}{2}$, the following identities hold:*

$$[\Delta^m\left(\left(\omega^1 \cdot \nabla\right) u\right)]_i = \sum_{|\gamma|=0}^{m} \sum_{l=0}^{m-|\gamma|} C_m(|\gamma|, l) \left(D^{|\gamma|} \omega_j^{m-|\gamma|+1-l}\right) \cdot \left(D^{|\gamma|} \partial_j\, \zeta_i^l\right), \tag{2.14}$$

$$[\Delta^m\left(\left(u \cdot \nabla\right) \omega^1\right)]_i = \sum_{|\gamma|=0}^{m} \sum_{l=0}^{m-|\gamma|} C_m(|\gamma|, l) \left(D^{|\gamma|} \zeta_j^l\right) \cdot \left(D^{|\gamma|} \partial_j\, \omega_i^{m-|\gamma|+1-l}\right). \tag{2.15}$$

where $C_m(|\gamma|,\, l)$ are positive constants.

Proof. We just show the first identity. The second can be proved in the same way. Since $\zeta^0 = u$, the identity is trivial for $m = 0$. Assuming its validity for some $m \leq \frac{k_0-3}{2}$, we prove it for $m+1$. By using the Leibniz rule formula we have

$$[\Delta^{m+1}\left(\left(\omega^1 \cdot \nabla\right) u\right)]_i = \Delta \sum_{|\gamma|=0}^{m} \sum_{l=0}^{m-|\gamma|} C_m(|\gamma|, l) \left(D^{|\gamma|} \omega_j^{m-|\gamma|+1-l}\right) \cdot \left(D^{|\gamma|} \partial_j\, \zeta_i^l\right)$$

$$= \sum_{|\gamma|=0}^{m} \sum_{l=0}^{m-|\gamma|} C_m(|\gamma|, l) \sum_{h=0}^{2} \binom{2}{h} \partial_s^h \left(D^{|\gamma|} \omega_j^{m-|\gamma|+1-l}\right) \cdot \partial_s^{2-h} \left(D^{|\gamma|} \partial_j\, \zeta_i^l\right).$$

Since $\sum_{s=1}^{3} \partial_s^2\, \omega_j^l = -\,\omega_j^{l+1}$ and $\sum_{s=1}^{3} \partial_s^2\, \zeta_j^l = -\,\zeta_j^{l+1}$, the previous sum is equal to

$$
\begin{aligned}
&-\sum_{|\gamma|=0}^{m}\sum_{l=0}^{m-|\gamma|} C_m(|\gamma|,l)\left(D^{|\gamma|}\omega_j^{m-|\gamma|+2-l}\right)\cdot\left(D^{|\gamma|}\partial_j\,\zeta_i^l\right)\\
&-\sum_{|\gamma|=0}^{m}\sum_{l=0}^{m-|\gamma|} C_m(|\gamma|,l)\left(D^{|\gamma|}\omega_j^{m-|\gamma|+1-l}\right)\cdot\left(D^{|\gamma|}\partial_j\,\zeta_i^{l+1}\right)\\
&+2\sum_{|\gamma|=0}^{m}\sum_{l=0}^{m-|\gamma|} C_m(|\gamma|,l)\,\partial_s\left(D^{|\gamma|}\omega_j^{m-|\gamma|+1-l}\right)\cdot\partial_s\left(D^{|\gamma|}\partial_j\,\zeta_i^l\right).
\end{aligned}
$$

Finally, by setting $l+1=r$ in the second term, $|\gamma|+1=|\beta|$ in the third term, and denoting by $C_{m+1}(|\gamma|,l)$ the constants obtained by collecting equal summands, we get

$$
\begin{aligned}
[\Delta^{m+1}(\,(\omega^1\cdot\nabla)\,u\,)]_i = &-\sum_{|\gamma|=0}^{m}\sum_{l=0}^{m-|\gamma|} C_m(|\gamma|,l)\left(D^{|\gamma|}\omega_j^{m+1-|\gamma|+1-l}\right)\cdot\left(D^{|\gamma|}\partial_j\,\zeta_i^l\right)\\
&-\sum_{|\gamma|=0}^{m}\sum_{r=1}^{m+1-|\gamma|} C_m(|\gamma|,r-1)\left(D^{|\gamma|}\omega_j^{m+1-|\gamma|+1-r}\right)\cdot\left(D^{|\gamma|}\partial_j\,\zeta_i^r\right)\\
&+\sum_{|\beta|=1}^{m+1}\sum_{l=0}^{m+1-|\beta|} C_m(|\beta|-1,l)\left(D^{|\beta|}\omega_j^{m+1-|\beta|+1-l}\right)\cdot\left(D^{|\beta|}\partial_j\,\zeta_i^l\right)\\
&=\sum_{|\gamma|=0}^{m+1}\sum_{l=0}^{m+1-|\gamma|} C_{m+1}(|\gamma|,l)\left(D^{|\gamma|}\omega_j^{m+1-|\gamma|+1-l}\right)\cdot\left(D^{|\gamma|}\partial_j\,\zeta_i^l\right).
\end{aligned}
$$

□

Lemma 2.7. *Let l and α be non-negative integers such that*

$$2l+2\alpha\le k_0\,, \tag{2.16}$$

with $l\in\left[0,\frac{k_0-1}{2}\right]$. Assume that $\zeta_3^l=0$ on Γ for each $l\in\left[0,\frac{k_0-1}{2}\right]$. Then

$$\partial_3^{2\alpha}\,\zeta_3^l\;=\;0\quad on\quad\Gamma\,,$$

for each l and α in $\left[0,\frac{k_0-1}{2}\right]$ satisfying (2.16).

Proof. We proceed by induction on α. If $\alpha=0$, the claim follows directly from the hypotheses. Assuming that the property holds for some $\alpha\in\left(0,\frac{k_0-1}{2}-1\right]$ and for each l, we prove it for $\alpha+1$. We have

$$\partial_3^{2(\alpha+1)}\zeta_3^l\;=\;-\,\partial_3^{2\alpha}\left(\zeta_3^{l+1}+\partial_1^2\zeta_3^l+\partial_2^2\zeta_3^l\right)\;=\;-\,\partial_3^{2\alpha}\zeta_3^{l+1}-\partial_1^2\partial_3^{2\alpha}\zeta_3^l-\partial_2^2\partial_3^{2\alpha}\zeta_3^l$$

where we have used $-\Delta\zeta_3^l=\zeta_3^{l+1}$. The proof is completed if one observes that, from the induction hypothesis, each term on the right hand side vanishes on Γ. □

Lemma 2.8. *Let ω^k and ζ^k be as in (2.5). Assume that ω^m satisfies equation (2.9) with $k = m$, for some $m \in \left[1, \frac{k_0-1}{2}\right]$. Further assume that $\omega^k \times \underline{n} = 0$ on Γ for each $k \in \{1, \cdots, m\}$ and that $\zeta^k \cdot \underline{n} = 0$ on Γ for each $k \in \{0, \cdots, m\}$. Then, on Γ,*

$$\omega^{m+1} \times \underline{n} = 0 \,, \quad \zeta^{m+1} \cdot \underline{n} = 0 \,.$$

Proof. The second claim immediately follows from the first claim and from Lemma 2.2, since $\zeta^{m+1} = curl\, \omega^{m+1}$. As far as the first claim is concerned, observing that $\omega^{m+1} = -\Delta\, \omega^m$ and that $\partial_t \omega^m$ is normal to the boundary, it is enough to show that the right hand side of (2.9), with $k = m$, is normal to the boundary. This means

$$\left[\Delta^{m-1} \left((\omega^1 \cdot \nabla)\, u - (u \cdot \nabla)\, \omega^1 \right)\right]_i = 0 \,, \quad i = 1, 2 \,. \tag{2.17}$$

Fix $i = 1$. The same arguments hold for $i = 2$. By using Lemma 2.6 we prove that for each $|\gamma| \in \{0, \cdots, m-1\}$, for each $l \in \{0, m-1-|\gamma|\}$ the product $\left(D^{|\gamma|} \omega_j^{m-|\gamma|-l} \right) \cdot \left(D^{|\gamma|} \partial_j\, \zeta_i^l \right)$ and the product $\left(D^{|\gamma|} \zeta_j^l \right) \cdot \left(D^{|\gamma|} \partial_j\, \omega_i^{m-|\gamma|-l} \right)$ are equal to zero on Γ. For convenience here we use the notation $m - |\gamma| - l = h$.

Let us consider the first product.

Case a) Let $j = 3$ and $\gamma_3 = 2\alpha + 1$, for any admissible $\alpha \geq 0$. We show that $\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \omega_3^h = 0$ on Γ, for each h. Note that it is enough to show that $\partial_3^{\gamma_3} \omega_3^h = 0$ on Γ (a similar argument holds in the next cases).

For $\alpha = 0$ there holds $\partial_3^1 \omega_3^h = 0$ on Γ, from $\omega_1^h = \omega_2^h = 0$ on Γ and $div\, \omega^h = 0$. If the property holds for $\gamma_3 = 2\alpha + 1$ then, for $\gamma_3 = 2\alpha + 3$ we have

$$\partial_3^{2\alpha+1} \partial_3^2 \omega_3^h = -\, \partial_3^{2\alpha+1} \left(\omega_3^{h+1} + \partial_1^2 \omega_3^h + \partial_2^2 \omega_3^h \right) = 0$$

on Γ where we have used $-\Delta\, \omega_3^h = \omega_3^{h+1}$ and the induction hypothesis.

Case b) Let $j \neq 3$ and $\gamma_3 = 2\alpha$, for any admissible $\alpha \geq 0$. We show that $\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \omega_j^h = 0$ on Γ, for each h.

For $\alpha = 0$ the claim follows from the hypotheses $\omega_1^h = \omega_2^h = 0$ on Γ. If the property holds for $\gamma_3 = 2\alpha$ then, for $\gamma_3 = 2\alpha + 2$ we have

$$\partial_3^{2\alpha} \partial_3^2 \omega_j^h = -\, \partial_3^{2\alpha} \left(\omega_j^{h+1} + \partial_1^2 \omega_j^h + \partial_2^2 \omega_j^h \right) = 0$$

on Γ, where we have used $-\Delta\, \omega_j^h = \omega_j^{h+1}$ and the induction hypothesis.

Case c) Let $j = 3$ and $\gamma_3 = 2\alpha$, for any admissible $\alpha \geq 0$. We show that $\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_3 \zeta_1^l = 0$ on Γ, for any l.

Observing that $\partial_3 \zeta_1^l = \omega_2^{l+1} + \partial_1 \zeta_3^l = 0$ on Γ, the property follows from *Case b)* and Lemma 2.7.

Case d) Let $j \neq 3$ and $\gamma_3 = 2\alpha + 1$, for any admissible $\alpha \geq 0$. We want to show that $\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_j\, \zeta_1^l = 0$ on Γ, for any l. Observing that

$$\partial_3^{2\alpha+1} \partial_j\, \zeta_1^l = \partial_j\, \partial_3^{2\alpha} \partial_3 \zeta_1^l \,,$$

the property follows from *Case c)*.

As far as the terms $\left(D^{|\gamma|} \zeta_j^l \right) \cdot \left(D^{|\gamma|} \partial_j\, \omega_i^{m-|\gamma|-l} \right)$ are concerned, it is easy to recognize that one falls into the previous cases. Indeed

Case a') If $j = 3$ and $\gamma_3 = 2\alpha + 1$, for any admissible $\alpha \geq 0$, one shows that $D^{|\gamma|}\partial_3\, \omega_i^{m-|\gamma|+1-l} = 0$ on Γ using *Case b).*

Case b') If $j \neq 3$ and $\gamma_3 = 2\alpha$, for any admissible $\alpha \geq 0$, one shows that $D^{|\gamma|}\partial_j\, \omega_i^{m-|\gamma|+1-l} = 0$ on Γ using *Case b).*

Case c') If $j = 3$ and $\gamma_3 = 2\alpha$, for any admissible $\alpha \geq 0$, one shows that $D^{|\gamma|}\zeta_3^l = 0$ on Γ using Lemma 2.7.

Case d') If $j \neq 3$ and $\gamma_3 = 2\alpha + 1$, for any admissible $\alpha \geq 0$, one shows that $D^{|\gamma|}\zeta_j^l = 0$ on Γ using *Case c).* □

Proof of Proposition 2.2. The identities (2.11) follow from Lemma 2.8. The last claim in the Proposition follows from this same lemma, for each $k \leq \frac{k_0-1}{2}$. It remains only to show that if $k \in \left(\frac{k_0-1}{2}, \frac{k_0+1}{2}\right]$ the right hand side of (2.9) is normal to the boundary (this case is not included in Lemma 2.8). The result can be easily shown by following the proof of (2.17). □

Proposition 2.3. *Under the assumptions of Proposition 2.2 one has*

$$(\partial_i\, \omega_j^k)\, n_i\, \omega_j^k = 0 \quad on \quad \Gamma\,. \tag{2.18}$$

The result follows from $\omega_1^k = \omega_2^k = 0$ on Γ and $div\, \omega^k = 0$.

Proposition 2.4. *Assume that the hypotheses of Lemma 2.4 hold. Further assume that ζ^k satisfies equation (2.10), for some $k \in \left[1, \frac{k_0}{2}\right]$. Then*

$$(\partial_i\, \zeta_j^k)\, n_i\, \zeta_j^k = 0 \quad on \quad \Gamma\,. \tag{2.19}$$

Recall that $\zeta_3^k = 0$ on Γ and $\partial_3\, \zeta_1^k = \omega_2^{k+1} + \partial_1\, \zeta_3^k$, $\partial_3\, \zeta_2^k = \partial_2\, \zeta_3^k - \omega_1^{k+1}$. The result follows from Proposition 2.2, since $\omega^{k+1} = -\Delta\omega^k$, and $\omega_1^{k+1} = \omega_2^{k+1} = 0$ since the right hand side of (2.9) normal to Γ.

The following norm-equivalence results hold.

Lemma 2.9. *For each non-negative integer l one has*

$$\|\omega\|_{l,\,p} \simeq \|u\|_{l+1,\,p}\,, \quad \|\zeta\|_{l,\,p} \simeq \|u\|_{l+2,\,p}\,,$$

where $\omega = \omega^1$ and $\zeta = \zeta^1$.

The first claim follows from $curl\, u = \omega$ and $div\, u = 0$ in Ω, together with the boundary condition $u \cdot \underline{n} = 0$. The second claim follows from $curl\, \omega = \zeta$ and $div\, \omega = 0$ in Ω, together with the boundary condition $\omega \times \underline{n} = 0$.

From Lemma 2.9, by appealing to an induction argument on k one has the following result.

Lemma 2.10. *For each non-negative integer l one has*

$$\|\omega^k\|_{l,\,p} \simeq \|u\|_{2k-1+l,\,p}\,, \quad \forall\, k \in \left[1, \frac{k_0+1}{2}\right]\;;$$

$$\|\zeta^k\|_{l,\,p} \simeq \|u\|_{2k+l,\,p}\,, \;\forall\, k \in \left[1, \frac{k_0}{2}\right]\,.$$

Recall that $\omega = \omega^1$, $\zeta = \zeta^1$ and that, from (2.5), $\omega^{k+1} = -\Delta\,\omega^k$, $\zeta^{k+1} = -\Delta\,\zeta^k$.

In the following we assume that the hypotheses of Proposition 2.2 hold. By multiplying both sides of equation (2.9) by $|\,\omega^k\,|^{p-2}\,\omega^k$, by integrating in Ω and by taking into account Lemma 2.1, it follows that

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega^k\|_p^p + \tfrac{\nu}{2}\int_\Omega |\,\omega^k\,|^{p-2}\,|\,\nabla\,\omega^k|^2\,dx + 4\,\nu\,\tfrac{p-2}{p^2}\int_\Omega \left|\,\nabla\,|\,\omega^k\,|^{\frac{p}{2}}\,\right|^2 dx \\
&\quad = (-1)^{k-1}\int_\Omega |\,\omega^k\,|^{p-2}\,\omega^k\cdot\Delta^{k-1}\left((\omega^1\cdot\nabla)\,u - (u\cdot\nabla)\,\omega^1\right)\,dx \\
&\quad + \nu\int_\Gamma |\,\omega^k\,|^{p-2}\,(\partial_i\,\omega_j^k)\,n_i\,\omega_j^k\,d\Gamma\,.
\end{aligned}
\tag{2.20}
$$

Observe that the last integral vanishes due to (2.18). Further, the highest order term in the volume integral on the right hand side vanishes, i.e.

$$
(-1)^{k-1}\int_\Omega |\,\omega^k\,|^{p-2}\,\omega^k\cdot(u\cdot\nabla)(\Delta^{k-1}\omega^1)\,dx = \frac{1}{p}\int_\Omega (u\cdot\nabla)|\omega^k|^p\,dx = 0\,.
\tag{2.21}
$$

This is due to $div\,u = 0$ in Ω and $u\cdot\underline{n} = 0$ on Γ. However, for convenience we preserve the above notation, even if the term $(-1)^{k-1}\nabla\,(\,\Delta^{k-1}\omega^1) \equiv \nabla\omega^k$ is assumed to be absent.

Moreover, by appealing to (2.2) the sum of the first two integrals on the left hand side can be bounded from below by $\nu\,\frac{2(\,2\,p-3)}{p^2}\int_\Omega \left|\,\nabla\,|\,\omega\,|^{\frac{p}{2}}\,\right|^2 dx$, for $p > \frac{3}{2}$. Hence, one has

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega^k\|_p^p + \nu\,\tfrac{2(\,2\,p-3)}{p^2}\int_\Omega \left|\,\nabla\,|\,\omega\,|^{\frac{p}{2}}\,\right|^2 dx \\
&\quad = (-1)^{k-1}\int_\Omega |\,\omega^k\,|^{p-2}\,\omega^k\cdot\Delta^{k-1}\left((\omega^1\cdot\nabla)\,u - (u\cdot\nabla)\,\omega^1\right)\,dx\,.
\end{aligned}
\tag{2.22}
$$

From the continuous immersion of $W^{1,2}$ in L^6 it follows that

$$
\|\omega^k\,\|_{3\,p}^p \leq c\,(\,\|\,\nabla\,|\omega^k|^{\frac{p}{2}}\,\|_2^2 + \|\omega^k\|_p^p\,)\,.
\tag{2.23}
$$

So, from (2.22), one gets

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega^k\|_p^p + c\,\nu\,\|\,\omega^k\,\|_{3\,p}^p \leq c\,\nu\,\|\,\omega^k\,\|_p^p \\
&\quad + \|\,\omega^k\,\|_p^{p-1}\|\,\Delta^{k-1}\left((\omega^1\cdot\nabla)\,u - (u\cdot\nabla)\,\omega^1\right)\|_p\,.
\end{aligned}
\tag{2.24}
$$

By a suitable estimate of the last L^p-norm we get the following result (see the appendix for details).

Proposition 2.5. *Let be $k \leq (k_0+1)/2$ and assume that $p > \frac{3}{2}$. Then*

$$
\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega^k\|_p^p + c\,\nu\,\|\,\omega^k\,\|_{3\,p}^p \leq c\,\|\,\omega^k\,\|_p^{p+1} + c\,\nu\,\|\,\omega^k\,\|_p^p\,.
\tag{2.25}
$$

By arguing exactly as for ω^k, by starting from equation (2.10) instead of (2.9), and by using (2.19) instead of (2.18) one obtains

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\zeta^k\|_p^p + c\,\nu\,\|\,\zeta^k\,\|_{3\,p}^p \leq c\,\nu\,\|\,\zeta^k\,\|_p^p \\
&\quad + \|\,\zeta^k\,\|_p^{p-1}\,\|\,curl\,\Delta^{k-1}\left((\omega^1\cdot\nabla)\,u - (u\cdot\nabla)\,\omega^1\right)\|_p\,,
\end{aligned}
\tag{2.26}
$$

where, as in the ω^k-case, the highest order term $(-1)^{k-1}\nabla\,(\Delta^{k-1}\, curl\,\omega^1)\ \equiv \nabla\,\zeta^k$, in the last L^p-norm, is assumed to be absent. A suitable estimate of the last L^p-norm gives the following result.

Proposition 2.6. *Let be $k_0 \geq 4$, $k \in \big(1, \frac{k_0}{2}\big]$ and $p > \frac{3}{2}$. Then*

$$\frac{1}{p}\,\frac{d}{d\,t}\|\,\zeta^k\|_p^p + c\,\nu\,\|\,\zeta^k\,\|_{3\,p}^p \,\leq\, c\,\|\,\zeta^k\,\|_p^{p+1} + c\,\nu\,\|\,\zeta^k\,\|_p^p\,. \tag{2.27}$$

A careful exploitation of the estimates (2.25) and (2.27) allows us to prove the Theorem 1.3, as done for the Theorem 1.1 in [9]. See Section 3 of [9] for a detailed proof. Actually, estimate (2.25) leads to the Theorem 1.3 if k_0 is odd and estimate (2.27) leads to the Theorem 1.3 if k_0 is even.

For the reader's convenience in the appendix we sketch the proof of Proposition 2.5. The proof of Proposition 2.6 is similar.

3 A challenging open problem: The control of the boundary integrals.

We start by proving some estimates already obtained in the previous section for higher order derivatives and flat boundaries.

We set

$$\omega = curl\,u\,,\quad \zeta = curl\,\omega\,,\quad \chi = curl\,\zeta\,. \tag{3.1}$$

By multiplying both sides of equation (1.5) by $|\,\omega\,|^{p-2}\,\omega$, by integrating in Ω, and by taking into account Lemma 2.1, it follows that

$$\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega\|_p^p + \tfrac{\nu}{2}\,\int_\Omega\,|\,\omega\,|^{p-2}\,|\,\nabla\,\omega|^2\,dx + 4\,\nu\,\tfrac{p-2}{p^2}\,\int_\Omega\,\Big|\,\nabla\,|\,\omega\,|^{\frac{p}{2}}\,\Big|^2\,dx\\
&+\tfrac{1}{p}\,\int_\Omega\,(\,u\cdot\,\nabla\,)\,|\,\omega\,|^p\,dx\,-\,\int_\Omega\,|\,\omega\,|^{p-2}\,\big(\,(\,\omega\,\cdot\,\nabla)\,u\,\big)\,\cdot\,\omega\,dx\\
&=\,\nu\,\int_\Gamma\,|\,\omega\,|^{p-2}\,(\partial_i\,\omega_j)\,n_i\,\omega_j\,d\Gamma\,,
\end{aligned} \tag{3.2}$$

where, in fact, the third integral on the left hand side vanishes. Hence, one gets

$$\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\omega\|_p^p + \tfrac{\nu}{2}\,\int_\Omega\,|\,\omega\,|^{p-2}\,|\,\nabla\,\omega|^2\,dx + 4\,\nu\,\tfrac{p-2}{p^2}\,\int_\Omega\,\Big|\,\nabla\,|\,\omega\,|^{\frac{p}{2}}\,\Big|^2\,dx\\
&\leq\,2\,\int_\Omega\,|\,\nabla\,u\,|\,|\,\omega\,|^p\,dx\,+\,2\,\nu\,\Big|\,\int_\Gamma\,|\,\omega\,|^{p-2}\,(\,\partial_j\,(\omega\,\cdot\,\underline{n})\,\omega_j - (\partial_j\,n_i)\,\omega_i\,\omega_j\,)\,d\Gamma\,\Big|\,.
\end{aligned} \tag{3.3}$$

Remark 3.1. The role played here by the $\nu-$terms occurring in the left hand side of equation (3.3) is to counterbalance the $\nu-$boundary integrals that appear in the right hand side. In [9] we can ignore the $\nu-$terms since in the flat boundary case the boundary integrals vanish.

Next in the above argument we replace ω by ζ. By applying the operator *curl* to both sides of equation (1.5) we show, with obvious notation, that

$$\partial_t\,\zeta - \,\nu\,\Delta\,\zeta + \,(u\,\cdot\,\nabla)\,\zeta + \sum c\,(\,D\,u\,)\,(\,D\,\omega\,) = 0\,. \tag{3.4}$$

By multiplying both sides of the above equation by $|\,\zeta\,|^{p-2}\,\zeta$, by integrating in Ω, and by taking into account Lemma 2.1, we show that

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\zeta\|_p^p + \tfrac{\nu}{2}\int_\Omega |\,\zeta\,|^{p-2}\,|\,\nabla\,\zeta|^2\,dx + 4\,\nu\,\tfrac{p-2}{p^2}\int_\Omega \left|\nabla\,|\,\zeta\,|^{\frac{p}{2}}\right|^2 dx\\
&\le c\int_\Omega |\,\nabla\,u\,|\,|\,\nabla\,\omega\,|\,|\,\zeta\,|^{p-1}\,dx + c\,\nu\,\Big|\int_\Gamma |\,\zeta\,|^{p-2}\,(curl\zeta)\,\times\,\underline{n}\cdot\zeta\,d\Gamma\,\Big|\\
&+c\,\nu\,\Big|\int_\Gamma |\,\zeta\,|^{p-2}\,(\partial_j\,n_i)\,\zeta_i\,\zeta_j\,d\Gamma\,\Big|\,.
\end{aligned}
\tag{3.5}
$$

Recall (2.4) and (2.3). Note that, by appealing to (2.23), written for ζ, we may include in the left hand side of (3.5) the term $c\,\nu\,\|\,\zeta\,\|_{3\,p}^p$ provided that we add $c\,\nu\,\|\,\zeta\,\|_p^p$ to the right hand side.

Finally, by applying the operator *curl* to the equation (3.4), and by following devices similar to those used in obtaining (3.5), we get the estimate

$$
\begin{aligned}
&\tfrac{1}{p}\,\tfrac{d}{d\,t}\|\,\chi\|_p^p + \tfrac{\nu}{2}\int_\Omega |\,\chi\,|^{p-2}\,|\,\nabla\,\chi|^2\,dx + 4\,\nu\,\tfrac{p-2}{p^2}\int_\Omega \left|\nabla\,|\,\chi\,|^{\frac{p}{2}}\right|^2 dx\\
&c\int_\Omega (\,|\,D\,u\,|\,|\,D^2\,\omega\,| + |\,D^2\,u\,|\,|\,D\,\omega\,|\,)\,|\,\chi\,|^{p-1}\,dx\\
&+c\,\nu\,\Big|\int_\Gamma |\,\chi\,|^{p-2}\,(\partial_i\,\chi_j)\,n_i\,\chi_j\,d\Gamma\,\Big|\,.
\end{aligned}
\tag{3.6}
$$

We call *convective integrals* the volume integrals occurring in the right hand sides of equations (3.5) and (3.6). The convective integrals are controlled by the terms on the left hand sides of the above estimates exactly as in reference [9]. Thus, in the following, there remains solely the problem of the control of the boundary integrals, as $\nu \to 0$.

In the sequel we concentrate on the $\zeta-$approach since, even in this case, we do not succeed in proving the desired estimates.

The second boundary integral on the right hand side of (3.5) can be estimated by terms in the left hand side, *uniformly* with respect to ν, by appealing to the following result.

Lemma 3.1. *To each $\alpha \in (1,\,2]$ there corresponds a positive constant $c = c_\alpha$ such that*

$$
\|g\|_{2;\,\Gamma}^2 \le c_\alpha\,\|\nabla\,g\|_2^\alpha\,\|g\|_2^{2-\alpha} + c_\alpha\,\|g\|_2^2. \tag{3.7}
$$

In particular, to each $\epsilon > 0$ there corresponds a positive C_ϵ such that

$$
\|\zeta\|_{p;\,\Gamma}^p \le \epsilon\,\|\nabla\,|\zeta|^{\frac{p}{2}}\|_2^2 + C_\epsilon\,\|\zeta\|_p^p\,. \tag{3.8}
$$

Proof. Equation (3.7) follows from

$$
\|g\|_{2;\,\Gamma}^2 \le \|\,g\|_{\frac{1}{2}+\delta,\,2}^2 \le c_\alpha\,\|\,g\|_{1,\,2}^\alpha\,\|g\|_2^{2-\alpha}\,,
$$

for each $\delta,\ 0 < \delta \le \frac{1}{2}$, where α depends on δ. Further, by applying (3.7) to $g = |\zeta|^{\frac{p}{2}}$ one proves (3.8). Actually, $C_\epsilon = c_\alpha^{\frac{2}{2-\alpha}}\,\epsilon^{-\frac{\alpha}{2-\alpha}} + c_\alpha$. □

By appealing to (3.5) and by fixing a sufficient small, positive, ϵ in (3.8), we obtain the following result.

Proposition 3.1. *One has*

$$
\begin{aligned}
&\frac{1}{p}\ \frac{d}{dt}\|\,\zeta\|_p^p + c\,\nu\int_\Omega |\zeta|^{p-2}\,|\,\nabla\,\zeta|^2\,dx \\
(3.9)\qquad &+c\,\nu\int_\Omega \big|\,\nabla\,|\,\zeta\,|^{\frac{p}{2}}\,\big|^2\,dx \le \int_\Omega |\,\nabla\,u\,|\,|\,\nabla\,\omega\,|\,|\,\zeta\,|^{p-1}\,dx \\
&+\nu\,\Big|\int_\Gamma |\,\zeta\,|^{p-2}\,(curl\,\zeta)\,\times\,\underline{n}\cdot\zeta\,d\Gamma\,\Big| + c\,\nu\,\|\,\zeta\,\|_p^p\,,
\end{aligned}
$$

where $\zeta = -\Delta\,u\,.$

In passing to the limit as $\nu \to 0$, the last term on the right hand side goes to zero. However, even without the coefficient ν, we may appeal to the time-derivative term to control terms of the form $c\,\|\,\zeta\,\|_p^p$. So, the last obstacle to overcome, in the case of a non flat boundary, is just the boundary integral on the right hand side of equation (3.9). Before going on, we point out that, in the case of Stokes problems, the above boundary integral vanishes since equation (3.19) below holds. This leads to Theorem 1.4. However, in the case of the Navier-Stokes equations, (3.19) does not hold in general. In this case we have the following result (see below for notation).

Theorem 3.2. *Let the boundary* Γ *be a surface of class* C^k, $k \ge 2$. *Then*

$$
(3.10)\qquad \big|\,curl\,(u\times\,\omega)\,\times\,\underline{n}\,\big|\ \le\ H\,|u|\,|\,\omega\,|\,.
$$

on Γ.

Proof. Let us consider a point x_0 on Γ and a system of orthogonal curvilinear coordinates ξ_j in a neighborhood of x_0. We assume that the surface Γ is locally described by the equation $\xi_3 = 0$. Moreover the surfaces $\xi_3 =$ *constant* are parallel to Γ in the usual sense, and the coordinate ξ_3 increase outside Ω. On each parallel surface, the lines $\xi_j =$ *constant*, $j = 1,\,2$, are lines of curvature.

We denote by $\underline{i}_j$ the unit vector, tangent to the ξ_j line, and pointing in the direction of increasing ξ_j. If $s(\xi_j)$ denotes the arc length along a ξ_j-line, the (positive) h_j scale functions are defined by

$$
h_j = \frac{d\,s(\xi_j)}{d\,\xi_j}\,.
$$

For convenience, we introduce the following simplified notation. H denotes any bounded function that depends, pointwise, at most on the scale functions h_j, and on their first and second order derivatives. Further, we denote by R any function which is bounded, pointwisely, by expressions of the form $H\,|u|\,|\,\omega\,|$.

We recall the following expression for the *curl* of a vector field v in curvilinear, orthogonal, coordinates.

$$
\begin{aligned}
&curl\,v = \frac{1}{h_2\,h_3}\Big[\frac{\partial\,(h_3\,v_3)}{\partial\,\xi_2} - \frac{\partial\,(h_2\,v_2)}{\partial\,\xi_3}\Big]\,\underline{i}_1 + \\
(3.11)\qquad &\frac{1}{h_3\,h_1}\Big[\frac{\partial\,(h_1\,v_1)}{\partial\,\xi_3} - \frac{\partial\,(h_3\,v_3)}{\partial\,\xi_1}\Big]\,\underline{i}_2 + \frac{1}{h_1\,h_2}\Big[\frac{\partial\,(h_2\,v_2)}{\partial\,\xi_1} - \frac{\partial\,(h_1\,v_1)}{\partial\,\xi_2}\Big]\,\underline{i}_3\,.
\end{aligned}
$$

We set $v = u\times\,\omega$ in (3.11). Note that the left hand side of (3.10) is equal to the absolute value of the tangential component of $curl\,v$, which consists of the two first terms on the left hand side of (3.11). Due to the similarity of these two terms it is sufficient to treat one of them. We consider the first one; moreover we drop the $H-$ type term $\frac{1}{h_1\, h_3}$. This leads to

$$\frac{\partial}{\partial\, \xi_2}\,[\, h_3\,(u_1\,\omega_2 -\; u_2\,\omega_1\,)\,] - \frac{\partial}{\partial\, \xi_3}\,[\, h_2\,(u_3\,\omega_1 -\; u_1\,\omega_3\,)\,]\,. \tag{3.12}$$

For convenience, throughout this proof, we denote the components of the vectors u and ω with respect to the new basis $\underline{i}_j$ by the same symbols used before in terms of the Cartesian coordinates. For instance,

$$u = \sum_{1}^{3} u_j\, \underline{i}_j\,.$$

Note that

$$u_3 = \omega_1 = \omega_2 = 0$$

for $\xi_3 = 0$, hence

$$\frac{\partial\, u_3}{\partial\, \xi_j} = \frac{\partial\, \omega_i}{\partial\, \xi_j} = 0\,,$$

for $i,\, j \;=\; 1,\, 2\,.$ It follows that the first term in equation (3.12) and the "first half" of the second term vanish on Γ. So, we take into account only $\frac{\partial}{\partial\, \xi_3}\,(\, h_2\, u_1\, \omega_3\,)$. Since $\omega_2 \;=\; 0$ on Γ, it follows from (3.11) that $\frac{\partial\, u_1}{\partial\, \xi_3} = -\frac{u_1}{h_1}\,\frac{\partial\, h_1}{\partial\, \xi_3}$ on Γ. It remains to show that $|\,\frac{\partial\, \omega_3}{\partial\, \xi_3}\,| \leq H\,|\,\omega\,|$. This follows easily by taking into account that ω is divergence free. In fact

$$div\, \omega = \frac{1}{h_1\, h_2\, h_3}\sum_{i=1}^{3}\frac{\partial}{\partial\, \xi_i}\,(h_j\, h_k\, \omega_i\,) = 0\,, \tag{3.13}$$

in Ω, where $i,\, j,\, k$ take distinct values. □

Equation (1.5) can be written in the form

$$\partial_t\, \omega + \nu\, curl \zeta + curl\,(u \times\, \omega) \;=\; 0\,. \tag{3.14}$$

Since $(\partial_t\, \omega) \times\, \underline{n} = 0$ on Γ, it follows that

$$-\,\nu\,(\, curl\, \zeta\,) \times\, \underline{n} = \; curl\,(u \times\, \omega)\, \times\, \underline{n} \tag{3.15}$$

on Γ. Hence we have

Corollary 3.1. *The estimate*

$$\nu \left|\,(\, curl\, \zeta\,) \times\, \underline{n}\,\right| \;\leq\; H\,|u|\,|\omega| \tag{3.16}$$

holds on Γ. *In particular*

$$\nu \Big|\, \int_\Gamma |\,\zeta\,|^{p-\,2}\,(curl\, \zeta) \,\times\, \underline{n} \cdot \zeta\, d\Gamma\, \Big| \leq c\, \int_\Gamma |\,\zeta\,|^{p-\,1}\,|\, u\,|\,|\,\omega\,|\, d\Gamma\,. \tag{3.17}$$

Roughly speaking, equation (3.17) allows us to lower by two the order of the higher derivative occurring in the boundary integral, however, at the *high* cost of losing multiplication by ν. From equations (3.9) and (3.17) the following result holds.

Proposition 3.2. *One has*

$$\begin{aligned} &\frac{1}{p}\ \frac{d}{dt}\|\,\zeta\|_p^p + c\,\nu\,\|\,\zeta\,\|_{3\,p}^p + c\,\nu\,\int_\Omega |\zeta|^{p-2}\,|\,\nabla\,\zeta|^2\,dx \\ &+c\,\nu\,\int_\Omega \left|\,\nabla\,|\,\zeta\,|^{\frac{p}{2}}\,\right|^2 dx \,\le\, \int_\Omega |\,\nabla\,u\,|\,|\,\nabla\,\omega\,|\,|\,\zeta\,|^{p-1}\,dx \\ &+c\,\nu\,\|\,\zeta\,\|_p^p + \,c\,\int_\Gamma |\,\zeta\,|^{p-1}\,|\,u\,|\,|\,\omega\,|\,d\Gamma\,, \end{aligned} \tag{3.18}$$

where $\zeta = -\Delta\,u$.

Unfortunately, this estimate seems still insufficient to prove the desired strong inviscid limit result, in the presence of convective terms. For the time being, we restrict ourselves to the strong vanishing limit result in the framework of Stokes problems.

Proof of Theorem 1.4. To prove the Theorem 1.4, it is sufficient to show how to control the boundary integral on the right hand side of (3.9), since the convective term in equation (1.1) is not present. In this case, by appealing to the equation

$$\partial_t\,\omega + \,\nu\,curl\,\zeta = \,0$$

it follows that

$$\underline{n} \times \,curl\,\zeta = \,0 \tag{3.19}$$

on Γ . So, for the Stokes problem, the estimate (3.9) holds without the two integrals occurring in the right hand side. This leads to the results stated in Theorem 1.4.

4 Proof of theorems 1.5 and 1.6.

In this section we show that the estimates proved in the previous section immediately lead to regularity results in $W^{k,\,p}(\Omega)$.

We start by proving the Theorem 1.6, since the proof is simpler than that of Theorem 1.5, in spite of a stronger conclusion. For the reader's convenience, we start by assuming that $f = 0$, so that we may appeal to the estimates in the form stated in the previous section. Then we show how to treat the supplementary terms which appear if an external force is present.

Proof. We prove the Theorem 1.6 by appealing to (3.18). It is sufficient to show that

$$\int_\Gamma |\,\zeta\,|^{p-1}\,|\,u\,|\,(\,|\,u\,|+\,|\nabla\,u\,|\,)\,d\Gamma \,\le \epsilon\,\|\,\nabla\,|\zeta\,|^{\frac{p}{2}}\,\|_2^2 + \,c\,\|\,\zeta\,\|_p^{2\,p} + C_\epsilon\,\|\,\zeta\,\|_p^{p+1}\,. \tag{4.1}$$

Since $p > \frac{3}{2}$ it follows that

$$\|\,u\,\|_{\infty,\,\Gamma} \le \,c\,\|\,u\,\|_2 \le \,c\,\|\,\zeta\,\|_p\,.$$

On the other hand,

$$\|\,\nabla\,u\,\|_{p;\,\Gamma} \le \,c\,\|\nabla\,u\,\|_{1,\,p} \le \,c\,\|\,\zeta\,\|_p\,.$$

Hence, by appealing to Hölder's inequality, we show that

$$\int_\Gamma |\,\zeta\,|^{p-1}\,|\,u\,|\,(\,|\,u\,| + \,|\nabla\,u\,|\,)\,d\Gamma \,\le\, c\,\|\,\zeta\,\|_{p;\,\Gamma}^{p-1}\,\|\,\zeta\,\|_p^2\,.$$

Finally, by taking into account (3.8), (4.1) follows. □

Next we prove Theorem 1.5. We start with a result based on (3.3).

Theorem 4.1. *Let ω be a divergence free vector field in Ω such that $\omega \times \underline{n} = 0$ on Γ. Then*

$$|\nabla(\omega \cdot \underline{n}) \cdot \omega| \leq c\,|\omega|^2\,, \tag{4.2}$$

on Γ.

In particular, the following estimate holds.

Lemma 4.2.

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p + \frac{\nu}{2}\int_\Omega |\omega|^{p-2}|\nabla\omega|^2\,dx + \nu\,\frac{4(p-2)}{p^2}\int_\Omega \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \\ &\leq 2\int_\Omega |\nabla u|\,|\omega|^p\,dx + c\,\nu\,\|\omega\|_p^p\,. \end{aligned} \tag{4.3}$$

Let us prove (4.3), by assuming (4.2). By appealing to (4.2), we may replace in equation (3.3) the boundary integral term by $c\,\nu\,\|\omega\|^p_{p;\,\Gamma}$. Next, by exploiting (3.7), we prove (3.8) where ζ is replaced by ω. This proves (4.3).

Next we prove (4.2).

Proof. We turn back to the system of coordinates used in the proof of Theorem 3.2. Here we denote by R terms which are, pointwise, bounded by $c\,|\omega(x)|^2$. Moreover, the constants c may depend on the positive scale functions, and its derivatives. Also note that $\nabla(\omega \cdot \underline{n}) \cdot \omega$ is an invariant.

By appealing to the expression of the gradient of a scalar field, and by taking into account that ω is normal to Γ, it readily follows that (see (3.13))

$$\nabla(\omega \cdot \underline{n}) \cdot \omega = \frac{1}{h_3}\,\frac{\partial(\omega \cdot \underline{n})}{\partial \xi_3}\,\omega_3 = \frac{\omega_3}{h_3}\,\frac{\partial \omega_3}{\partial \xi_3} + R\,.$$

On the other hand, by taking into account that $div\,\omega = 0$, and by appealing to the expression of the divergence of a vector field in orthogonal curvilinear coordinates, it follows that

$$h_2\,h_3\,\frac{\partial \omega_1}{\partial \xi_1} + h_3\,h_1\,\frac{\partial \omega_2}{\partial \xi_2} + h_1\,h_2\,\frac{\partial \omega_3}{\partial \xi_3}$$

is bounded in absolute value by $c\,|\omega|$. Since $\frac{\partial \omega_j}{\partial \xi_j} = 0$ for $\xi_3 = 0$ and $j = 1, 2$, our thesis follows. □

Lemma 4.3. *Let be $p \geq \frac{3}{2}$ and set $\alpha = \frac{3}{2\,p}$. Then*

$$\|h\|_{\frac{p^2}{p-1}} \leq c\left(\|\nabla|h|^{\frac{p}{2}}\|_2^{\frac{2}{p}}\right)^\alpha \|h\|_p^{1-\alpha} + c\,\|h\|_p\,. \tag{4.4}$$

Proof. By interpolation, and by appealing to the embedding of $W^{1,2}$ in L^6, one has

$$\|g\|_{\frac{2p}{p-1}} \leq \|g\|_6^\alpha\,\|g\|_2^{1-\alpha} \leq c\,\|\nabla g\|_2^\alpha\,\|g\|_2^{1-\alpha} + c\,\|g\|_2\,.$$

By setting $g = |h|^{\frac{p}{2}}$, the desired result follows. □

Corollary 4.1. *Let be* $p > \frac{3}{2}$. *Then, given* $\epsilon > 0$ *there is a constant* C_ϵ *such that*

$$\int_\Omega |\nabla u|\,|\omega|^p\,dx \leq c\,\epsilon\,\|\nabla|\omega|^{\frac{p}{2}}\|_2^2 + C_\epsilon\,\|\omega\|_p^{\frac{p(2p-1)}{2p-3}} + c\,\|\omega\|_p^{p+1}\,. \tag{4.5}$$

The result follows easily from the previous lemma, by appealing to the Hölder's inequality with exponents p and $\frac{p}{p-1}$ and then to the Young's inequality with exponents $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$, $\alpha = \frac{3}{2p}$.

From equations (4.3) and (4.5) it follows that

$$\begin{aligned} &\frac{1}{p}\,\frac{d}{dt}\|\omega\|_p^p + \frac{\nu}{2}\int_\Omega |\omega|^{p-2}\,|\nabla\omega|^2\,dx + \nu\,\frac{4(p-2)}{p^2}\int_\Omega \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \\ &\leq c\,\|\omega\|_p^p + c\,\|\omega\|_p^{\frac{p(2p-1)}{2p-3}}\,, \end{aligned} \tag{4.6}$$

where we have used $\frac{p(2p-1)}{2p-3} > p+1$. Note that the constant c may blow up as ν goes to zero. By appealing to (4.6), well known manipulations lead to Theorem 1.5.

Assume now that an external force f is present. We start by considering the Theorem 1.5. In this case it comes out an additional term $curl f$ on the right hand side of (1.5). This leads to the addition of the term

$$\|curl f\|_p\,\|\omega\|_p^{p-1}$$

to the right hand sides of equations (3.2), (3.3), (4.3) and (4.6). Since

$$\|curl f\|_p\,\|\omega\|_p^{p-1} \leq c\,\|curl f\|_p^p + c\,\|\omega\|_p^p\,,$$

the theorem follows if f satisfies (1.18), with $k = 1$. Note that stronger results may be easily obtained by appealing to the third term in the left hand side of (4.6) in order to estimate the "$f-$terms". But this looks unessential.

Concerning the Theorem 1.6, if the external force f is present, it comes out an additional term $curl^2 f$ on the right hand side of (1.5). This leads to the addition of the term

$$\|curl^2 f\|_p\,\|\zeta\|_p^{p-1}$$

to the right hand side of equation (3.5). This term must be systematically added to the right hand sides of our "ζ-estimates". In parallel to the previous case, if f satisfies (1.18) with $k = 2$, this term is innocuous. However, in the "ζ-case", a new boundary integral appears. In fact, we have to add the term $curl f$ to the right hand side of (3.14). This leads to the addition of

$$\int_\Gamma |\zeta|^{p-1}\,|curl f|\,d\Gamma$$

to the right hand side of (3.18). This integral is bounded by $c\,\|\zeta\|_{p;\Gamma}^p + c\,\|curl f\|_{p;\Gamma}^p$. By appealing to (3.1) and to

$$\|curl f\|_{p;\Gamma}^p \leq c\,\|f\|_{2,p}\,,$$

the thesis follows.

5 Other boundary conditions.

On flat portions of the boundary, the boundary condition (1.2) coincides with the slip boundary condition

$$\left\{ \begin{array}{l} u \cdot \underline{n} = 0, \\ \underline{t} \cdot \underline{\tau} = 0\,, \end{array} \right. \tag{5.1}$$

where $\underline{\tau}$ stands for any arbitrary unit tangential vector (or, equivalently, two linearly independent representatives). The stress vector $\underline{t}$ is defined by $\underline{t} = \mathcal{T} \cdot \underline{n}$ and the stress tensor $\mathcal{T}$ by

$$\mathcal{T} = -\pi\, I + \frac{\nu}{2}\, (\nabla u + \nabla u^T)\,.$$

If the boundary is not flat,

$$\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}\, (\omega \times \underline{n}) \cdot \underline{\tau} \,-\, \nu\, u \cdot \frac{\partial\, \underline{n}}{\partial\, \underline{\tau}}\,. \tag{5.2}$$

The last term on the right hand side is a lower order term. Note that $\omega \times \underline{n}$ and $\frac{\partial\, \underline{n}}{\partial\, \underline{\tau}}$ are tangential to Γ, while $|\, \frac{\partial\, \underline{n}}{\partial\, \underline{\tau}}\, |$ is the normal curvature in the $\underline{\tau}$ direction. Actually, $\frac{\partial\, \underline{n}}{\partial\, \underline{\tau}} = \mathcal{K}_\tau\, \underline{\tau}$, where $\mathcal{K}_\tau$ is the principal curvature in the $\underline{\tau}$ direction, positive if the corresponding center of curvature lies inside Ω.

Since the boundary conditions (1.2) and (5.1) differ only by lower order terms, the corresponding regularity results for solutions to the Navier-Stokes equations should be similar. In this direction, we recall that in reference [7] we added to the left hand side of the second boundary condition (5.1) a lower order term (which behaves like the second term on the right hand side of (5.2)), and showed the classical $L^2(0,\, T;\, W^{2,\,2}) \cap W^{1,\,2}(0,\, T;\, L^2)$ regularity result for this problem.

6 Appendix. Estimating the convective integrals.

As announced in section 2, we show here how to estimate the convective integrals and how to obtain estimate (2.25). See [9] for more details in the case $k_0 = 3$.

From (2.24), it follows that the proof of Proposition 2.5 is accomplished once the following L^p-norm is estimated in a suitable way

$$\|\, \Delta^{k-1} \left((\omega^1 \cdot\, \nabla)\, u - \,(u \cdot\, \nabla)\, \omega^1 \right) \|_p\,.$$

Recall that, by identity (2.21), the highest order term $(\, u \cdot \nabla)\, \omega^k$ (corresponding to $|\gamma| = 0,\, l = 0\,, m = k-1$ in (2.15)) does not appear. By recalling the expression of Δ^{k-1} given in Lemma 2.6, by increasing the L^p- norm of the sum with the sum of the L^p- norms, it suffices to estimate terms of the following kind:

$$I_1 = \int_\Omega |\, D^\gamma\, \omega_j^{k-|\gamma|-l}\, |^{\,p}\, |\, \nabla D^\gamma\, \zeta_i^l\, |^{\,p}\, dx\,,$$

for each $|\gamma| \in \{\, 0, \cdots,\, k-1\, \}$ and each $l \in \{\, 0, \cdots,\, k-1-|\gamma|\, \}$, and

$$I_2 = \int_\Omega |\, D^\gamma\, \zeta_j^l\, |^{\,p}\, |\, \nabla D^\gamma\, \omega_i^{k-|\gamma|-l}\, |^{\,p}\, dx\,,$$

for each $|\gamma| \in \{0, \cdots, k-1\}$ and each $l \in \{0, \cdots, k-1-|\gamma|\}$, with $(|\gamma|, l) \neq (0, 0)$. Recall that $|\gamma|$ and l are non-negative integers, and $p > \frac{3}{2}$. As far as I_1 is concerned, we get

$$I_1 \leq \| D^\gamma \omega_j^{k-|\gamma|-l} \|_p^p \, \| \nabla D^\gamma \zeta_i^l \|_\infty^p \leq c \, \|\omega^k\|_p^{2p} \tag{6.1}$$

for each $|\gamma| \in \{0, \cdots, k-1\}$ and each non-negative integer l such that $k - |\gamma| - l > l+1$ (that is $l < \frac{k-|\gamma|-1}{2}$). In the last step we have used Sobolev's embedding and the norm-equivalences given in Lemma 2.10. On the other hand, if $|\gamma| \in \{0, \cdots, k-1\}$ and $l \geq \frac{k-|\gamma|-1}{2}$, by using Sobolev's embedding and Lemma 2.10, we estimate I_1 as follows

$$I_1 \leq \| D^\gamma \omega_j^{k-|\gamma|-l} \|_\infty^p \, \| \nabla D^\gamma \zeta_i^l \|_p^p \leq c \, \|\omega^k\|_p^{2p} . \tag{6.2}$$

Note that the previous distinction in estimating I_1, depending on the value of l, is needed to appeal to the embedding of $W^{2,p}$ in L^∞.

Similarly,

$$I_2 \leq \| D^\gamma \zeta_j^l \|_\infty^p \, \| \nabla D^\gamma \omega_i^{k-|\gamma|-l} \|_p^p \leq c \, \|\omega^k\|_p^{2p} \tag{6.3}$$

for each $|\gamma| \in \{0, \cdots, k-1\}$ and each l such that $k - |\gamma| - l > l$ (that is $l < \frac{k-|\gamma|}{2}$), with $(|\gamma|, l) \neq (0, 0)$. Moreover

$$I_2 \leq \| D^\gamma \zeta_j^l \|_p^p \, \| \nabla D^\gamma \omega_i^{k-|\gamma|-l} \|_\infty^p \leq c \, \|\omega^k\|_p^{2p} \tag{6.4}$$

for each $|\gamma| \in \{0, \cdots, k-1\}$ and each $l \geq \frac{k-|\gamma|}{2}$.

Finally, by appealing to the estimates from (6.1) to (6.4), we get

$$\| \Delta^{k-1} \left((\omega^1 \cdot \nabla) \, u - (u \cdot \nabla) \, \omega^1 \right) \|_p \leq c \, \|\omega^k\|_p^2 ,$$

which, together with (2.24), proves Proposition 2.5.

References

[1] H.-O. Bae and B.J. Jin, Regularity for the Navier-Stokes equations with slip boundary condition, *Proc. Amer. Math. Soc.*, **136** (2008), 2439-2443.

[2] C. Bardos, Existence et unicité de la solution de l'équation de Euler en dimension deux, *J. Math. Anal. Appl.*, **40** (1972), 769-790.

[3] G. S. Beavers and D. D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid mech.*, **30** (1967), 197-207.

[4] H. Beirão da Veiga, Existence and asymptotic behaviour for strong solutions of the Navier–Stokes equations in the whole space, *Indiana University Math. Journal*, **36** (1987), 149–166.

[5] H. Beirão da Veiga, Regularity of solutions to a non homogeneous boundary value problem for general Stokes systems in $\mathbb{R}^n_+$, *Math. Annalen*, **331** (2005), 203-217.

[6] H. Beirão da Veiga, Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip type boundary conditions, *Advances in Diff. Equations*, **9** (2004), no.9-10, 1079-1114.

[7] H. Beirão da Veiga, Remarks on the Navier-Stokes equations under slip type boundary conditions with linear friction, *Portugaliae Math.*, **64** (2007), 377-387.

[8] H. Beirão da Veiga, On the sharp vanishing viscosity limit of viscous incompressible fluid flows, to appear in a volume to the memory of A. Kazhikhov in the Birkhaueser book series “Advances in Mathematical Fluid Mechanics”.

[9] H. Beirão da Veiga and F. Crispo, Sharp inviscid limit results under Navier type boundary conditions. An L^p theory, *J. Math. Fluid Mech.*, (2009), DOI 10.1007/s00021-009-0295-4.

[10] A. Bendali, J.M. Dominguez and S. Galic, A variational approach for the vector potential formulation of the Stokes and Navier-Stokes problems in three dimensional domains, *J. Math. Anal. Appl.* **107** (1985), 537-560.

[11] L.C. Berselli and M. Romito, On the existence and uniqueness of weak solutions for a vorticity seeding model, *SIAM J. Math. Anal.*, **37** (2006), 1780-1799.

[12] D. Bresch, F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido, Asymptotic derivation of a Navier condition for the primitive equations, *Asymptot. Anal.*, **33** (2003), 237-259.

[13] D. Cioranescu, P. Donato and H.I. Ene, Homogenization of the Stokes problem with non-homogeneous slip boundary conditions, *Math. Methods Appl. Sci.*, **19** (1996), 857-881.

[14] T. Clopeau, A. Mikelic and R. Robert, On the vanishing viscosity limit for the 2-D incompressible Navier-Stokes equations with the friction type boundary conditions, *Nonlinearity*, **11** (1998), 1625-1636.

[15] P. Constantin and C. Foias, Navier-Stokes Equations, The University of Chicago Press, Chicago, IL, 1988.

[16] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations:Linearized Steady Problems*, Springer Tracts in Natural Philosophy,**38**, Second corrected printing, Springer-Verlag,1998.

[17] G.P. Galdi and W. Layton, *Approximation of the larger eddies in fluid motion: A model for space filtered flow*, Math. Models and Meth. in Appl. Sciences, **3** (2000), 343-350.

[18] V. Girault, *Curl-conforming finite element methods for Navier-Stokes equations with non-standard boundary conditions in* $\mathbb{R}^3$, Lecture Notes in Mathematics, **1431**, Springer Verlag, Berlin-Heidelberg, 1988.

[19] G. Grubb, Nonhomogeneous time-dependent Navier-Stokes problems in L_p Sobolev spaces, *Diff. Int. Eq.* **8** (1995), 1013-1046.

[20] G. Grubb and V.A. Solonnikov, boundary value problems for the non-stationary Navier-Stokes equations treated by pseudo-differential methods, *Math. Scand.* **69** (1991), 217-290.

[21] D. Iftimie and G. Planas, Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions, *Nonlinearity*, **19** (2006), 899-918.

[22] D. Iftimie, G. Raugel and G. R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, *Indiana Univ. Math. J.*, **56** (2007), 1083-1156.

[23] W. Jäger and A. Mikelić, Couette flows over a rough boundary and drag reduction, *Comm. Math. Phys.*, **232** (2003), 429-455.

[24] V.I. Judovich, Non-stationary flow of an ideal incompressible liquid, *Zh. Vychisl. Mat. i Mat. Fiz.*, **3** (1963), 1032-1066.

[25] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *J. Functional Anal.*, **9** (1972), 296-305.

[26] T. Kato, Quasi-linear equations of evolution with applications to partial differential equations, in "Spectral Theory and Differential Equations", *Lecture Notes in Mathematics*, **448**, Springer-Verlag (1975).

[27] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41** (1988), 891-907.

[28] Y. Li and K. Li, Penalty finite element method for Stokes problem with nonlinear slip boundary conditions, *Appl. Math. and Computation*, **204** (2008), 216-226.

[29] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[30] P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid, *J. Math. Pures Appl.*, **77** (1998), 585-627

[31] M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas, On the inviscid limit for two-dimensional incompressible flow with Navier friction condition, *SIAM J. Math. Anal.*, **36** (2005), 1130-1141.

[32] N. Masmoudi, Remarks about the Inviscid Limit of the NavierStokes System, *Commun. Math. Phys.*, **270** (2007), 777-788.

[33] N. Masmoudi and L. Saint-Raymond, From the Boltzmann equation to the Stokes-Fourier system in a bounded domain, *Comm. Pure Appl. Math.*, **56** (2003), 1263-1293.

[34] C. L. M. Navier, Mémoire sur les lois du mouvement des fluides, *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, 1 (1816).

[35] S. Rionero and G.P. Galdi, The weight function approach to uniqueness of viscous flows in unbounded domains, *Arch. Rat. Mech. Anal.*, **69** (1979), 37-52.

[36] P. Secchi, Flussi non stazionari di fluidi incompressibili viscosi e ideali in un semipiano, *Ricerche di Matematica*, **34** (1984), 27-44.

[37] V.A. Solonnikov and V.E. Ščadilov, On a boundary value problem for a stationary system of Navier-Stokes equations, *Proc. Steklov Inst. Math.*,**125** (1973), 186-199.

[38] H. S. G. Swann, The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, **157** (1971), 373-397.

[39] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Func. Anal.*, **20** (1975), 32-43.

[40] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam, 1984.

[41] J. Watanabe, On incompressible viscous fluid flows with slip boundary conditions, *J. Comp. and Appl. Math.*, **159** (2003), 161-172.

[42] Y. Xiao and Z. Xin, On the vanishing viscosity limit for the 3-D Navier-Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.*, **60** (2007), 1027-1055.