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On the Sharp Vanishing Viscosity Limit of Viscous Incompressible Fluid Flows

H. Beirão da Veiga

Abstract. We consider the classical problem of the convergence of local-in-time regular solutions of the Navier-Stokes equations to a solution of the Euler equations, as the viscosity ν goes to zero. Initial data are given in an $H^k(\Omega)$ space, where $k > 1 + \frac{n}{2}$. Solutions are continuous in time, with values in the initial-data's space. We look for convergence of the solutions v of the Navier-Stokes equations to the solution w of the Euler equations in the space $C([0, T]; H^k)$. We are interested in proofs that apply to the case $n = 3$. This convergence result, in the strong topology, is due to T. Kato, see [8]. We show here a very elementary proof. We assume, together with the convergence of ν to zero, the convergence of the initial data in the H^k norm.

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1. Introduction

Our main concern is showing that (1.4) holds, where v_ν and w are the solutions to the systems (1.1) and (1.2) respectively. See Theorem 1.2 and Corollary 1.2 below. This result was essentially proved, many years ago, by Kato, see [8], by appealing to a completely different method, based on rather general theorems on abstract equations. The proof followed here is borrowed from reference [3], where a substantially more difficult problem is considered (we take this occasion to quote our recent review [4], where an introduction to our methods to prove sharp singular limit results is given). We also refer to Ebin and Marsden, cf. [6], where the limit of zero viscosity is considered in H^s, for $s > 5 + \frac{n}{2}$. See [6], Section 15.4, p. 152.

In considering problems like vanishing viscosity limits, incompressible limits, dependence on initial data, etc., the results are called here *sharp* if convergence is shown in $C([0, T]; X)$, where X is the initial data's space. As remarked by T. Kato in reference [9] this is the more difficult part in a theory dealing with nonlinear equations of evolution. Note that sufficiently strong a priori estimates, independent of ν, for the solutions to the Navier-Stokes equations immediately lead to non-sharp convergence results, by appealing to suitable compactness theorems and to the uniqueness of the strong solution to the Euler equations. For instance, by assuming that the initial data, a_ν, are bounded in H^s, for some $s > k$, (1.4) follows easily. Many non-sharp vanishing viscosity limit results are known in the literature. Classical, specific references, are [7] and [12]. A simpler approach is given in [5].

In the sequel k_0 denotes the smallest integer such that $k_0 \geq n/2$ and k is a fixed integer satisfying $k \geq k_0 + 1$. The canonical norm in H^k is denoted by $\|\cdot\|_k$. The norm in L^2 is simply denoted by $\|\cdot\|$.

We set

$$H^k_\sigma(\Omega) =: \left\{ u \in H^k(\Omega) : \nabla \cdot u = 0 \right\}.$$

We denote by $\|\cdot\|_{l,T}$ the standard norm in $C([0,T]; H^l)$ and by $[\cdot]_{l,T}$ that in $L^2(0,T; H^l)$.

In the sequel $\Omega = [0,1]^n$ is the n-dimensional torus, $n \geq 2$. Obvious modifications in the proofs allow one to assume that $\Omega = \mathbb{R}^n$. The motion of a viscous, incompressible, fluid is described by the system

$$\begin{cases} \partial_t v_\nu + (v_\nu \cdot \nabla) v_\nu + \nabla p_\nu = \nu \Delta v_\nu \quad \text{in} \quad Q_T, \\ \nabla \cdot v_\nu = 0 \quad \text{in} \quad Q_T, \\ v_\nu(0) = a_\nu(x), \end{cases} \tag{1.1}$$

where $\nabla \cdot a_\nu = 0$ in Ω, and $\nu \in \mathbb{R}^+_0$, the set of nonnegative reals. We also consider the "limit problem"

$$\begin{cases} \partial_t w + (w \cdot \nabla) w + \nabla \pi = \overline{\nu} \Delta w \quad \text{in} \quad Q_T, \\ \nabla \cdot w = 0 \quad \text{in} \quad Q_T, \\ w(0) = b, \end{cases} \tag{1.2}$$

where $\nabla \cdot b = 0$. Note that in the more interesting case, namely $\overline{\nu} = 0$, we are dealing with the Euler equation for non-viscous fluids

$$\begin{cases} \partial_t w + (w \cdot \nabla) w + \nabla \pi = 0 \quad \text{in} \quad Q_T, \\ \nabla \cdot w = 0 \quad \text{in} \quad Q_T, \\ w(0) = b(x). \end{cases} \tag{1.3}$$

We are interested in showing that

$$\lim \|v_\nu - w\|_{C([0,T]; H^k)} = 0, \tag{1.4}$$

as $(a_\nu, \nu) \to (b, \overline{\nu})$ in $H^k \times \mathbb{R}^+_0$.

We recall the following well-known existence and regularity theorem for local-in-time smooth solutions of (1.1). For the reader's convenience, in the next section we give a sketch of the proof.

Theorem 1.1. *Assume that*

$$\|a_\nu\|_{k_0+1} \leq c_1 \tag{1.5}$$

and

$$\|a_\nu\|_k \leq c_2\,. \tag{1.6}$$

Then there is a positive constant T *depending only on* c_1 *such that the problem* (1.1) *has a unique solution in* $[0,\,T]$. *Moreover,*

$$\|v_\nu\|^2_{k,\,T} + \nu\,[\nabla\,v_\nu]^2_{k,\,T} \leq C\,, \tag{1.7}$$

and

$$\|\partial_t\,v_\nu\|^2_{k-2,\,T} + \nu\,[\nabla\,\partial_t\,v_\nu\,]^2_{k-2,\,T} \leq C\,. \tag{1.8}$$

Constants C may depend on k and n, on an arbitrarily fixed upper bound for the values ν, and on c_1 and c_2. For convenience we do not show the explicit dependence of the various constants C on c_1 and c_2.

Due to (1.11) below, the reader may assume that the initial data a_ν satisfy the constraint $\|a_\nu\|_k \leq \|b\|_k + 1$, so that T and the constants C that appear in equations (1.7) and (1.8) are fixed once and for all.

Corollary 1.1. *Under the assumption* (1.11) *one has*

$$v_\nu \rightharpoonup w \quad in \;\; L^\infty(0,\,T;\,H^k)\text{-}weak^* \quad and\ in \;\; C(0,\,T;H^{k-\epsilon})\,, \tag{1.9}$$

for $\epsilon > 0$ *small enough. Moreover,*

$$\partial_t\,v_\nu \rightharpoonup \partial_t\,w \quad in \;\; L^\infty(0,\,T;\,H^{k-2})\text{-}weak^* \quad and\ in \;\; C(0,\,T;H^{k-2-\epsilon})\,. \tag{1.10}$$

Corollary 1.1 follows immediately from the uniform estimates (1.7), (1.8), by appealing to well-known compact embedding theorems. These theorems guarantee that we may pass to the limit in equation (1.1), as $\nu \to 0$. The uniqueness of the strong solution w of equation (1.2) is used in order to show that all the sequences v_ν converge to the same limit w.

The following is the main result here, especially when $\overline{\nu} = 0$.

Theorem 1.2. *Let* $\overline{\nu} \geq 0$ *and* $a_\nu,\, b \in H^k_\sigma(\Omega)$. *Assume that*

$$\lim_{\nu \to \overline{\nu}} \|a_\nu - b\|_k = 0\,. \tag{1.11}$$

Then

$$\lim_{\nu \to \overline{\nu}} \big(\|v_\nu - w\|^2_{k,\,T} + \overline{\nu}\,[v_\nu - w]^2_{k+1,\,T}\big) = 0\,. \tag{1.12}$$

In particular, (1.4) *holds.*

Corollary 1.2. *Under the assumptions of the above theorem one has*

$$\lim_{\nu \to \overline{\nu}} \big(\|\partial_t\,v_\nu - \partial_t\,w\|^2_{k-2,\,T} + \|\nabla\,p_\nu - \nabla\,\pi\|^2_{k-1,\,T} + \overline{\nu}\,[\partial_t\,v_\nu - \partial_t\,w]^2_{k-1,\,T}\big) = 0\,. \tag{1.13}$$

Remark 1.1. *Under the sole assumptions of Theorem* 1.2 *the equation*

$$\lim_{\nu \to \overline{\nu}} \left(\|\partial_t v_\nu - \partial_t w\|^2_{k-1,T} + \overline{\nu}\, [\partial_t v_\nu - \partial_t w]^2_{k,T} \right) = 0 \tag{1.14}$$

is false in general. Obviously it holds under stronger regularity assumptions on the initial data, and for $t > 0$.

2. Preliminaries

For the reader's convenience, we give in this section a sketch of the proof of equations (1.7) and (1.8). Here the parameter ν is fixed. Hence we denote v_ν simply by v and $\partial_t v_\nu$ by v_t.

We start by some useful results.

For convenience, we denote integrals $\int_\Omega f(x)\, dx$ simply by $\int f(x)$, or even by $\int f$. If D^α denotes partial differentiation, $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$\widetilde{D}^\alpha \{f\, g\} = D^\alpha (f\, g) - f\, D^\alpha g$$

and $|D^m f|^2 = \sum_{|\alpha|=m} |D^\alpha f|^2$. In the sequel we appeal to the following three results.

Lemma 2.1. *Let* $|\alpha| \leq l$. *Then*

$$\|\widetilde{D}^\alpha \{f\, g\}\| \leq c\, (|D\, f|_\infty \|g\|_{l-1} + |g|_\infty \|D\, f\|_{l-1}\,. \tag{2.1}$$

For a proof see [10], Lemma A.1.

Lemma 2.2. *For* $0 \leq |\alpha| \leq m \leq k$,

$$\|D^\alpha (f\, g)\| \leq c\, \|f\|_m \|g\|_{k-1} + c\, \delta^m_k\, |f|_\infty \|g\|_k\,. \tag{2.2}$$

See [3], equation (3.4).

Lemma 2.3. *Let* $k > 1 + n/2$ *and* $1 \leq l \leq k$. *If* $|\alpha| \leq l$ *then*

$$\|\widetilde{D}^\alpha \{f\, g\}\| \leq c\, \|D\, f\|_{k-1} \|g\|_{l-1}\,. \tag{2.3}$$

For a proof see [1] Appendix A, Corollary A.4.

By applying the operator D^α to both sides of (1.1), by multiplying by $D^\alpha v$, and by integrating in Ω, we show that

$$\frac{1}{2} \frac{d}{dt} \|D^\alpha v\|^2 + \int \widetilde{D}^\alpha \{(v \cdot \nabla)\, v\} \cdot D^\alpha v + \nu\, \|\nabla D^\alpha v\|^2 = 0\,. \tag{2.4}$$

Then we add the above equations, side by side, for $0 \leq |\alpha| \leq m$. By taking into account (2.1), and also $|\cdot|_\infty \leq c\, \|\cdot\|_{k_0}$, it readily follows that

$$\frac{1}{2} \frac{d}{dt} \|v\|^2_m + \nu\, \|\nabla v\|^2_m \leq c\, \|v\|_{k_0+1} \|v\|^2_m\,. \tag{2.5}$$

By setting $m = k_0 + 1$, well-known methods lead to (1.7) for $k = k_0$, (with dependence of T only on c_1). The estimate (1.7) for $k = k_0$, together with (2.5) written for $m = k$, shows (1.7) for k.

Lemma 2.4. *Assume that* (1.5) *and* (1.6) *hold. Let l be an integer satisfying $0 \leq l \leq k-2$. Then there is a constant C such that*

$$\|v_t\|^2_{l,T} + \nu\,[\nabla v_t]^2_{l,T} \leq C\,. \tag{2.6}$$

In particular (1.8) *holds.*

Proof. From (1.1) it follows that

$$\partial_{tt}\, v + (v\cdot\nabla)\, v_t + (v_t\cdot\nabla)\, v + \nabla p_t = \nu\,\Delta\, v_t\,. \tag{2.7}$$

Next apply D^α, $|\alpha| \leq l$, to both sides of the above equation, multiply by $D^\alpha v_t$ and integrate over Ω. This gives

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha v_t\|^2 + \int \widetilde{D}^\alpha\{(v\cdot\nabla)v_t\}\cdot D^\alpha v_t + \int D^\alpha[(v_t\cdot\nabla)v]\cdot D^\alpha v_t + \nu\|\nabla D^\alpha v_t\|^2 = 0. \tag{2.8}$$

By using (2.3) and (2.2) we show that

$$\frac{1}{2}\,\frac{d}{dt}\,\|D^\alpha\, v_t\|^2 + \nu\,\|\nabla\, D^\alpha\, v_t\,\|^2 \leq c\,\|D\,v\|_{k-1}\,\|\,v_t\|_l\,\|D^\alpha\, v_t\|\,.$$

Hence, for $|\,\alpha\,| \leq l$,

$$\frac{1}{2}\,\frac{d}{dt}\,\|D^\alpha\, v_t\|^2 + \nu\,\|\nabla\, D^\alpha\, v_t\,\|^2 \leq C\,\|v_t\|^2_l, \tag{2.9}$$

and a well-known argument leads to (2.6). Note that, by applying the divergence operator to both sides of the first equation (1.1), we show that $\|\nabla\, p\,\|_{k-2,T} \leq C$. In particular, it readily follows that $\|\,v_t(0)\|_{k-2} \leq C$. □

3. Proof of Theorem 1.2

In the sequel we appeal to Fourier series

$$\phi(x) = \sum_{\xi} \widehat{\phi}(\xi)\, e^{2\pi i\,\xi\cdot x}\,,$$

$$\widehat{\phi}(\xi) = \int_\Omega e^{-2\pi i\,\xi\cdot x}\,\phi(x)\,dx\,.$$

The ξ_i's are nonnegative integers, and $\xi = (\xi_1, \ldots, \xi_n)$. The Euclidian norm of ξ is denoted by $|\xi|$. For each nonnegative real s one has

$$\|\phi\|^2_s = \sum_{\xi} (1 + |\xi|^2)^s\, |\widehat{\phi}(\xi)|^2\,.$$

Given $\delta \in]0,\,1]$, we define linear operators

$$T^\delta\,\phi = \sum_{|\xi|\leq 1/\delta} \widehat{\phi}(\xi)\, e^{2\pi i\,\xi\cdot x}\,, \tag{3.1}$$

where ϕ is a scalar or a vector field, and set

$$a^\delta_\nu = T^\delta\, a_\nu\,, \quad b^\delta = T^\delta\, b\,. \tag{3.2}$$

Since T^δ commutes with the divergence operator, a^δ_ν and b^δ are divergence free. Clearly, for each nonnegative real s, T^δ is a bounded linear operator. In particular, $|||\, T^\delta \,|||_{s,\,s} \leq 1$ where, in general, we denote by $|||\cdot|||_{s,\,r}$ the canonical norm in the space of bounded linear operators from H^s to H^r. So, a^δ_ν satisfies the assumptions (1.5), (1.6) with the same constants c_1 and c_2.

Also note that

$$|||T^\delta\,|||_{s,\,m} \leq (2/\,\delta)^{m-\,s}\,, \quad |||T^\delta - I\,|||_{m,\,s} \leq \delta^{m-\,s}\,, \tag{3.3}$$

if $0 \leq s \leq m$, where s and m are nonnegative integers. In particular

$$\|a^\delta_\nu\,\|_{k_0+1} \leq c_1\,, \quad \|a^\delta_\nu\|_{k+1} \leq \frac{2\,c_2}{\delta}\,. \tag{3.4}$$

and

$$\|a^\delta_\nu - b^\delta\|_{k+1} \leq \frac{2}{\delta}\,\|a_\nu - b\|_k\,. \tag{3.5}$$

Note that

$$a^\delta_\nu \to b^\delta \quad \text{in}\, H^{k+1} \qquad \text{if} \quad a_\nu \to b \quad \text{in}\, H^k\,.$$

The following system plays here a very central role:

$$\begin{cases} \partial_t\, v^\delta_\nu + (v^\delta_\nu \cdot \nabla)\, v^\delta_\nu + \nabla\, p^\delta_\nu = \nu\, \Delta\, v^\delta_\nu \quad \text{in} \quad Q_T\,, \\ \nabla \cdot v^\delta_\nu = 0 \quad \text{in} \quad Q_T\,, \\ v^\delta_\nu(0) = a^\delta_\nu\,. \end{cases} \tag{3.6}$$

We also consider the (inviscid, if $\overline{\nu} = 0$) counterpart of the system (3.6), namely

$$\begin{cases} \partial_t\, w^\delta + (w^\delta \cdot \nabla)\, w^\delta + \nabla\, \pi^\delta = \overline{\nu}\, \Delta\, w^\delta \quad \text{in} \quad Q_T\,, \\ \nabla \cdot w^\delta = 0 \quad \text{in} \quad Q_T\,, \\ w^\delta(0) = b^\delta\,. \end{cases} \tag{3.7}$$

From Corollary 1.1, with k replaced by $k+1$, applied to the solutions v^δ and w^δ of the above problems, and also by taking into account (3.4) and (3.5), one shows the following result.

Proposition 3.1. *Under the assumptions of Theorem* 1.2 *one has*

$$\lim_{\nu \to \overline{\nu}} \left(\|v^\delta_\nu - w^\delta\|^2_{k,\,T} + \overline{\nu}\, [v^\delta_\nu - w^\delta]^2_{k+1,\,T} \right) = 0\,, \tag{3.8}$$

for each fixed $\delta > 0$.

The following estimate will be useful in the sequel:

$$\|a^\delta_\nu - a_\nu\|^2_k \leq 2\,\|b - a_\nu\|^2_k + 2 \sum_{|\xi| > 1/\,\delta} (1 + |\xi|^2)^k\, |\widehat{b}(\xi)|^2\,. \tag{3.9}$$

The proof is left to the reader.

In the sequel we denote by δ^m_k the Kronecker symbol and set

$$\overline{v} = v^\delta_\nu - v_\nu\,, \quad \overline{p} = p^\delta_\nu - p_\nu\,.$$

Clearly, $\overline{v}$ and $\overline{p}$ depend on δ and ν.

Our next step is to prove the following result.

Theorem 3.1. *Let* $0 \leq m \leq k$. *Then, for each* $\delta > 0$,

$$\frac{1}{2}\frac{d}{dt}\|\overline{v}\|_m^2 + \nu\,\|\nabla\,\overline{v}\|_m^2 \leq C\,\|\overline{v}\|_m^2 + c\,\delta_k^m\,\|v_\nu^\delta\|_{k+1}\,|\overline{v}|_\infty\,\|\overline{v}\|_m\,. \tag{3.10}$$

Proof. In the calculations that follow the reader should take into account that the quantities $\|v_\nu\|_{k,T}$, $\|v_\nu^\delta\|_{k,T}$, $\nu\,[\,v_\nu\,]_{k+1,T}$ and $\nu\,[\,v_\nu^\delta\,]_{k+1,T}$ are uniformly bounded by constants C.

By taking the termwise difference between the equations (3.6) and (1.1) we find that

$$\overline{v}_t + (v_\nu\cdot\nabla)\,\overline{v} + \nabla\,\overline{p} = -(\overline{v}\cdot\nabla)\,v_\nu^\delta + \nu\,\Delta\,\overline{v}\,. \tag{3.11}$$

Apply D^α to (3.11), multiply by $D^\alpha\,\overline{v}$ and integrate on Ω. Using previous estimates and formulae (in particular (2.3) and (2.2)), straightforward manipulations show that

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\,\|D^\alpha\,\overline{v}\|^2 + \nu\,\|\nabla\,D^\alpha\,\overline{v}\,\|^2\\ &\leq C\,\|\overline{v}\|_m^2 + c\,\delta_k^m\,\|v_\nu^\delta\|_{k+1}\,|\overline{v}|_\infty\,\|\overline{v}\|_m\,.\end{aligned} \tag{3.12}$$

Equation (3.10) follows. □

Next, fix a real β_0 such that $0 < \beta_0 < k_0 - (n/2)$. Clearly, $0 < \beta_0 < 1$. Since $k_0 - \beta_0 > n/2$, one has $|\cdot|_\infty \leq c\,\|\cdot\|_{k_0-\beta_0}$. Well-known interpolation results for L^2-Sobolev spaces show that

$$|\cdot|_\infty \leq c\,\|\cdot\|_{k_0-1}^{\beta_0}\,\|\cdot\|_{k_0}^{1-\beta_0}\,. \tag{3.13}$$

Theorem 3.2. *For each* $\delta > 0$,

$$|\overline{v}|_{\infty,T} \leq C\,\delta^{2\,(k-k_0+\beta_0)}\,. \tag{3.14}$$

Proof. Let $0 \leq m \leq k-1$. From (1.7) one has $\nu\,[\nabla\,v_\nu^\delta]_{k,T}^2 \leq C$. Hence, by appealing to (3.10), it follows that

$$\|\overline{v}(t)\|_m^2 \leq C\,\|\overline{v}(0)\|_m^2\,,\quad \forall\, t \in [0,\,T]\,.$$

So,

$$\|\overline{v}\|_{m,T}^2 \leq C\,\|a_\nu^\delta - a_\nu\|_m^2\,.$$

By appealing to this inequality for $m = k_0$ and $m = k_0 - 1$, and by taking into account (3.13), we show that

$$|\overline{v}|_{\infty,T}^2 \leq C\,\|a_\nu^\delta - a_\nu\|_{k_0-1}^{2\,\beta_0}\,\|a_\nu^\delta - a_\nu\|_{k_0}^{2\,(1-\beta_0)}\,. \tag{3.15}$$

By using $(3.3)_2$ for $m = k$ and $m = k_0 - 1$, we obtain

$$\|a_\nu^\delta - a_\nu\|_{k_0-1}^2 \leq \delta^{2\,(k-k_0+1)}\,\|a_\nu\|_k^2\,. \tag{3.16}$$

Again by $(3.3)_2$, one has

$$\|a_\nu^\delta - a_\nu\|_{k_0}^2 \leq \delta^{2\,(k-k_0)}\,\|a_\nu\|_k^2\,. \tag{3.17}$$

The estimates (3.15), (3.16) and (3.17) lead to (3.14). □

Corollary 3.1. *One has, for each* $\delta \in]\,0,\,1\,]$,

$$|\overline{v}|_{\infty,\,T}\,\|\,v_\nu^\delta\|_{k+1,\,T} \leq \,C\,\delta^{k-\,k_0-\,1+\,\beta_0}\,. \tag{3.18}$$

Proof. By applying the estimate (1.7) to the solution v_ν^δ, with k replaced by $k+1$, and by appealing to $(3.3)_1$ for $m = k+1$ and $s = k$, it follows that

$$\|\,v_\nu^\delta\|_{k+1,\,T}^2 \leq \,C/\,\delta^2\,. \tag{3.19}$$

This estimate together with (3.14) shows (3.18). □

Theorem 3.3. *For each* $\delta \in]\,0,\,1\,]$,

$$\|\overline{v}\|_{k,\,T}^2 + \nu \int_0^T \|\nabla\,\overline{v}(t)\|_k^2 dt \leq \,C\,(\,\|a_\nu^\delta -\,a_\nu\|_k^2 +\,\delta^{2\,\beta_0})\,. \tag{3.20}$$

Proof. From equation (3.10) for $m = k$, together with (3.18), we get

$$\frac{1}{2}\,\frac{d}{d\,t}\,\|\overline{v}(t)\|_k^2 +\,\nu\,\|\nabla\,\overline{v}\|_k^2 \leq C\,\|\overline{v}\|_k^2 +\,C\,\|\overline{v}\|_k\,\delta^{\beta_0}\,. \tag{3.21}$$

Standard techniques yield

$$\|\overline{v}\|_{k,\,T} \leq \,e^{C\,T}\,(\|\overline{v}(0)\|_k + \delta^{\beta_0}\,)\,. \tag{3.22}$$

Equation (3.20) follows easily. Note that $e^{C\,T}$ is a constant of type C. □

Proof of Theorem 1.2

Define

$$|||\,u|||^2 =: \,\|u\|_{k,\,T}^2 +\,\overline{\nu}\,[\,\nabla\,u\,]_{k,\,T}^2\,.$$

Let $\epsilon > 0$ be fixed. From (3.20) and (3.9) it follows that

$$|||\overline{v}|||_{k,\,T}^2 \leq \,C\,\Big(\,\|b -\,a_\nu\|_k^2 + \sum_{|\xi|>\,1/\delta}\,(\,1 +\,|\xi|^2\,)^k\,|\widehat{b}(\xi)|^2 \,+\,\delta^{2\,\beta_0}\,+\,|\nu -\,\overline{\nu}|\,\Big)\,.$$

In particular,

$$\|\overline{v}\|_{k,\,T}^2 \leq \,C\,\big(\,\|b -\,a_\nu\|_k^2 +\,\widehat{h}(\delta) +\,|\nu -\,\overline{\nu}|\,\big)\,, \tag{3.23}$$

where $\widehat{h}(\delta)$ depends only on δ (b and k are fixed), and satisfies

$$\lim_{\delta\to\,0}\,\widehat{h}(\delta) =\,0\,.$$

We fix (once and for all) $\delta = \,\delta(\epsilon)$ such that $C\,\widehat{h}(\delta) \leq \,\epsilon/\,3$. It follows that

$$|||v_\nu^\delta -\,v_\nu\,|||_{k,\,T}^2 < \,\frac{\epsilon}{3} +\,C\,\big(\,\|b -\,a_\nu\|_k^2 +\,|\nu -\,\overline{\nu}|\,\big)\,. \tag{3.24}$$

The same argument applied to the particular case in which $(a_\nu,\,\nu) = \,(b,\,\overline{\nu})$ shows that

$$|||w^\delta -\,w\,|||_{k,\,T}^2 \leq \,\frac{\epsilon}{3}\,. \tag{3.25}$$

On the other hand, Proposition 3.1 shows that there is $\lambda = \,\lambda(\,\delta(\epsilon),\,\epsilon\,)$ for which

$$|||v_\nu^\delta -\,w^\delta|||_{k,\,T}^2 \leq \,\epsilon\,, \tag{3.26}$$

if $|\nu -\,\overline{\nu}| < \,\lambda\,$.

In short, from (3.24), (3.25) and (3.26) it follows that given $\epsilon > 0$ there is a $\lambda = \lambda(\epsilon)$ such that

$$|||v_\nu - w|||^2_{k,\,T} \leq \epsilon\,, \tag{3.27}$$

if $|\nu - \overline{\nu}| < \lambda$. This proves (1.12).

Proof of Corollary 1.2

Proof. One has

$$\begin{aligned}\partial_t\,(v_\nu - w) + (v_\nu \cdot \nabla)\,(v_\nu - w) + \big((v_\nu - w)\cdot\nabla\big)\,w + \nabla\,(p_\nu - \pi)\\ = \nu\,\Delta\,(v_\nu - w)\,, \quad \text{in} \quad Q_T\,.\end{aligned} \tag{3.28}$$

In particular, by applying the divergence operator to both sides of (3.28), one gets

$$-\,\Delta\,(p_\nu - \pi) = \nabla\cdot\big\{(v_\nu\cdot\nabla)\,(v_\nu - w) + \big((v_\nu - w)\cdot\nabla\big)\,w\big\}\,.$$

It readily follows, by appealing to previous estimates, that

$$\|\,(v_\nu\cdot\nabla)\,(v_\nu - w) + \big((v_\nu - w)\cdot\nabla\big)\,w\,\|_{k-1,\,T} \leq C\,\|v_\nu - w\|_{k,\,T}\,.$$

The pressure-estimate in equation (1.13) follows from classical regularity results for solutions to elliptic equations $-\,\Delta\,u = f$, together with (1.12).

Now, the time-derivative estimates in equation (1.13) follow from (3.28). Note that more elaborate manipulations lead to better results concerning the convergence of $\partial_t\,v_\nu$ to $\partial_t\,w$, but not to (1.14). □

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H. Beirão da Veiga
((Please insert address)) ⟸