A NOTE ON THE GLOBAL INTEGRABILITY, FOR ANY FINITE POWER, OF THE FULL GRADIENT FOR A CLASS OF GENERALIZED POWER LAW MODELS, $p < 2$.

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Abstract

In the following we consider a class of non-linear systems that covers some well known generalized Navier-Stokes systems with shear dependent viscosity of power law type, $p < 2$. We show that weak solutions to our class of systems have integrable gradient up to the boundary, with any finite exponent. This result extends, up the the boundary, some of the interior regularity results known in the literature for systems of the above power law type. Boundedness of the gradient, up to the boundary, remains an open problem.

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1 Main result

In the following we prove $W^{1,q}(\Omega)$-regularity up to the boundary, for any finite power q, for solutions of the system (1.5) under very weak assumptions on the non-linear term $G(x, \nabla u)$. Our proof, based on a bootstrap argument and Stokes-elliptic regularization (see [4]), is elementary. Nevertheless, the Theorem 1.1 below extend to a larger class of operators, and up to the boundary, some of the well known results in the literature.

In the sequel Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class C^2.
Below we consider solutions to the following class of stationary Navier-Stokes equations for flows with shear (more generally, gradient) dependent viscosity

$$(1.1) \qquad \begin{cases} -\nabla \cdot T(u,\pi) + (u \cdot \nabla)\, u = f, \\ \nabla \cdot u = 0\,, \end{cases}$$

under suitable boundary conditions. T denotes the Cauchy stress tensor

$$(1.2) \qquad T = -\pi\, I + \nu_0\, \mathcal{D}\, u + G(x, \nabla u)$$

and $\mathcal{D}\, u$ is the symmetric gradient, i.e.,

$$\mathcal{D}\, u = \frac{1}{2}\,(\nabla\, u + \nabla\, u^T)\,.$$

Here ν_0 is a strictly positive constant and G is a tensor with components G_{ij}, $i, j = 1, 2, 3$. Note that

$$G = G(x, \mathcal{D}u)$$

is a particular case of $G(x, \nabla u)$. We set

$$|S|^2 = \sum S_{k\,l}^2 \,,$$

where $S = S_{kl}$ is a tensor. In the following we assume that $G(x, S)$ satisfies the classical Caratheodory conditions (measurability in x for each S, continuity in S for a.a. x), together with

$$|G(x, S)| \leq c\,(1 + |S|)^{p-1} \tag{1.3}$$

for some $p \in (1, 2)$. As a particular case of (1.3), one may have

$$|G(x, S)| \leq c\,(1 + |S|)^{\mu(x)-1} \tag{1.4}$$

provided that $\mu(x) \leq p < 2$, almost everywhere in Ω. This particular case is related to the theory of electro-rheological fluids. See [10].

It is worth noting that $G(x, \nabla u)$ may depend on each of the first order derivatives $\partial_i\, u_j$ in a totaly independent way. In particular, there are not convexity-related assumptions here.

Without loss of generality we assume that $\nu_0 = 1$. From (1.1) we get

$$\begin{cases} -\Delta\, u + \nabla \cdot G(x, \nabla\, u) + (u \cdot \nabla)\, u + \nabla\, \pi = f\,, \\ \nabla \cdot u = 0\,. \end{cases} \tag{1.5}$$

In order to fix ideas we assume here the homogeneous Dirichlet boundary condition

$$u_{|\Gamma} = 0\,. \tag{1.6}$$

However, many other boundary conditions fall within the above scheme. Actually, it is sufficient that an estimate like (2.11) holds for the usual Stokes linear system (2.10) under the desired boundary condition. We may also assume a non-homogeneous Dirichlet boundary condition $u_{|\Gamma} = a(x)$, if $a \in W^{1,+\infty}(\Gamma)$ satisfy the necessary compatibility condition

$$\int_\Gamma a \cdot n\, d\Gamma = 0\,.$$

We take into account any possible weak solution u,

$$u \in W_0^{1,p}\,. \tag{1.7}$$

Note that (1.7) may be replaced by $u \in W_0^{1,2}$ (see (2.1) and (2.2)). Spaces $W_0^{1,q}$ are endowed here with the norm $\|\nabla u\|_q$.

Remark 1.1. *Our assumptions do not necessarily imply the existence of a solution. However, and this is a crucial point here, many well known existence theorems fall within the assumptions made here.*

Our main result is the following.

Theorem 1.1. *Assume that* (1.3) *holds and that*

$$f \in L^3 \,. \tag{1.8}$$

Let u *be a solution of problem* (1.5), (1.6) *in the class* (1.7). *Then*

$$u \in W^{1,q}(\Omega) \quad \forall\, q < +\infty \,. \tag{1.9}$$

In particular

$$u \in C^{0,\alpha}(\overline{\Omega}) \,, \quad \forall\, \alpha < 1 \,. \tag{1.10}$$

Theorem 1.1 allows the extension up to the boundary of some of the interior regularity results known in the literature, obtained under much restrictive assumptions on G. In [7], Theorem 3.2.3, it is proved that $\mathcal{D}u \in L^\infty_{loc}(\Omega)$. In Remark 3.2.5 it is pointed out that this last result implies (1.9) locally in Ω. In Theorem 3.2.1 (1.10) is proved locally in Ω. See also [6]. Finally, in reference [5], the author proves (1.9) under more classical assumptions on G.
It remains open, in particular, the conjecture proposed in [7] Remark 3.2.9, namely, to prove that $\mathcal{D}u \in L^\infty(\Omega)$. Actually, by taking into account (1.9), we may even expect that, in our very general context, $\nabla u \in L^\infty(\Omega)$.
For interior partial regularity results we also refer the reader to [2] (see, in particular, Lemma 3.14) and to [1] and references therein.
Finally we refer to [8] where the authors prove the existence of globally smooth solutions in the two dimensional case under suitable conditions.

2 Proof of Theorem 1.1

From (1.5), (1.6) we get the "energy estimate"

$$\|\nabla u\|_2^2 \leq \|f\|_{-1,2}\, \|\nabla u\|_2 + c\, \|\nabla u\|_1 + c\, \|\nabla u\|_p^p \,. \tag{2.1}$$

In particular, it readily follows that

$$\|\nabla u\|_2^2 \leq c(\, 1 + \|f\|_{-1,2}\,) \,. \tag{2.2}$$

The symbol c denotes, here and in the sequel, positive constants that may depend, at most, on Ω. The same symbol may denote distinct constants.
Note that under any of the typical assumptions that lead to an existence theorem, the term $\|\nabla u\|_p^p$ appears in the left hand side instead of in the right hand side of equation (2.1).

We start by the following result.

Lemma 2.1. *Let* $u \in W^{1,p}_0$ *be a weak solution of problem* (1.5), (1.6). *Then*

$$\nabla u \in L^3 \,. \tag{2.3}$$

Proof. We start by noting that

$$f \in L^3 \subset W^{-1,q} \,, \quad \forall q < \infty \,, \tag{2.4}$$

and that (since $\nabla u \in L^2$)

$$(u \cdot \nabla)\, u \in L^{\frac{3}{2}} \subset W^{-1,\,3}\,. \tag{2.5}$$

Set, for each non-negative integer m ,

$$r_m = \min\Big\{\,3,\ \frac{2}{(p-1)^m}\Big\}\,. \tag{2.6}$$

Let us show that

$$\nabla u \in L^{r_m}\,, \quad \forall\, m\,. \tag{2.7}$$

Clearly (2.7) holds for $m = 0$. Next we assume that (2.7) holds for some m and show it for $m+1$. If $r_m = 3$ then $r_{m+1} = 3$, and the thesis follows. If $r_m < 3$, by appealing to (1.3) one shows that

$$G(x,\, \nabla\, u) \in L^{\frac{2}{(p-1)^{m+1}}} \subset L^{r_{m+1}}\,. \tag{2.8}$$

In particular, from (2.4), (2.5) and (2.8) it follows that

$$f - \nabla \cdot G(x, \nabla\, u) - (u\, \cdot\, \nabla)\, u \in W^{-1,\,r_{m+1}}\,. \tag{2.9}$$

Well know regularity results, see [4], for the linear Stokes system

$$\begin{cases} -\,\Delta\, w + \nabla\, \widetilde{\pi} = F, \\ \nabla \cdot w = 0\,, \\ w_{|\Gamma} = 0\,, \end{cases} \tag{2.10}$$

state that

$$\|\nabla w\|_q \leq C_q\, \|F\|_{-1,q}\,, \tag{2.11}$$

where C_q denotes a suitable positive constant. So, from (2.9), it follows that (2.7) holds with m replaced by $m+1$. Hence it holds for each index m . This shows (2.3). □

Lemma 2.2. *Let $u \in W_0^{1,p}$ be a weak solution of problem* (1.5), (1.6). *Then*

$$\nabla u \in L^6\,. \tag{2.12}$$

Proof. The proof follows that of the previous lemma. Now, since $\nabla\, u \in L^3$, it follows that $(u \cdot \nabla)\, u \in L^2 \subset W^{-1,\,6}$. This is used here in place of (2.5). Furthermore, the exponents r_m , defined by (2.13), are now replaced by

$$s_m = \min\Big\{\,6,\ \frac{3}{(p-1)^m}\Big\}\,. \tag{2.13}$$

Details are left to the reader. □

The proof of Theorem 1.1 follows the same lines. Now we appeal to the fact that $\nabla u \in L^6$ yields

$$f - (u \cdot \nabla) u \in L^3 \subset W^{-1,q}, \quad \forall q < \infty .$$

Set $t_m = \frac{2}{(p-1)^m}$. Clearly $\nabla u \in L^{t_0}$. If $\nabla u \in L^{t_m}$ then

$$f - \nabla \cdot G(x, \nabla u) - (u \cdot \nabla) u \in W^{-1, t_{m+1}} .$$

This leads to $\nabla u \in L^{t_{m+1}}$. So $\nabla u \in L^{t_m}$ for all m. This proofs (1.9).

Remark 2.1. *It is worth noting that if the constants C_q were uniformly bounded for large values of q then we could easily prove the uniform boundedness of the norms $\|\nabla u\|_q$, hence the Lipschitz continuity of u up to the boundary. However, it may be proved, see [11], that for the scalar equation $-\Delta w = F$ under the homogeneous Dirichlet boundary condition, one has $\|w\|_{2,q} \leq C_q \|F\|_q$, where $C_q \approx q$, as $q \to \infty$. At best, we expect this same behavior for the constants C_q in the case of the Stokes problem* (2.10)*. By merely assuming this behavior, specific estimates obtained by following our proof do not lead to the above uniform boundedness.*

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