BUILDING EXOTIC MANIFOLDS I

GOAL OF THE TALK :

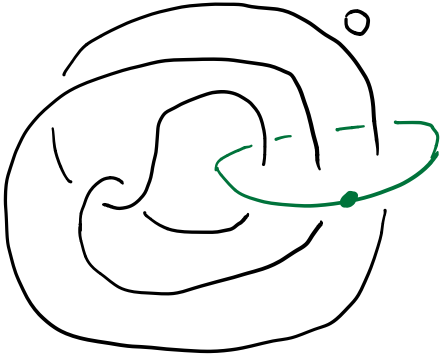
- SHOW THE EXISTENCE OF CORKS
- SHOW THE EXISTENCE OF EXOTIC 4-MFDS WITH BOUNDARY

Def A CORK is a pair (W^4, f) where:

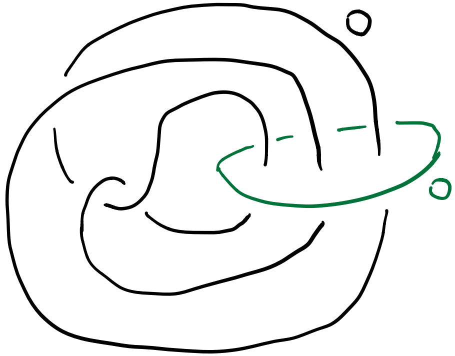
- W^4 is compact, contractible, with ∂
- $f: \partial W \to \partial W$ diffeo s.t. f extends to $F: W \to W$ homeo but f doesn't extend to a diffeo of W

Mazur manifold

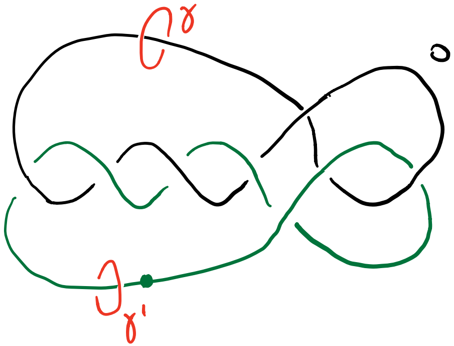
$W^4 =$

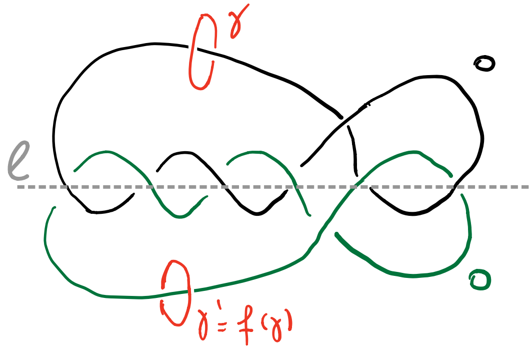


$\partial W =$

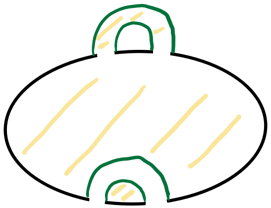


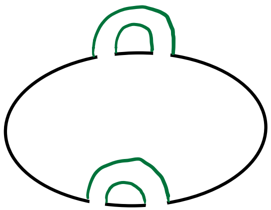
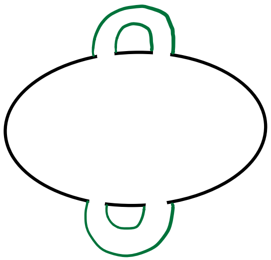
W admits a symmetrical diagram

$W =$ 

$\partial W =$ 

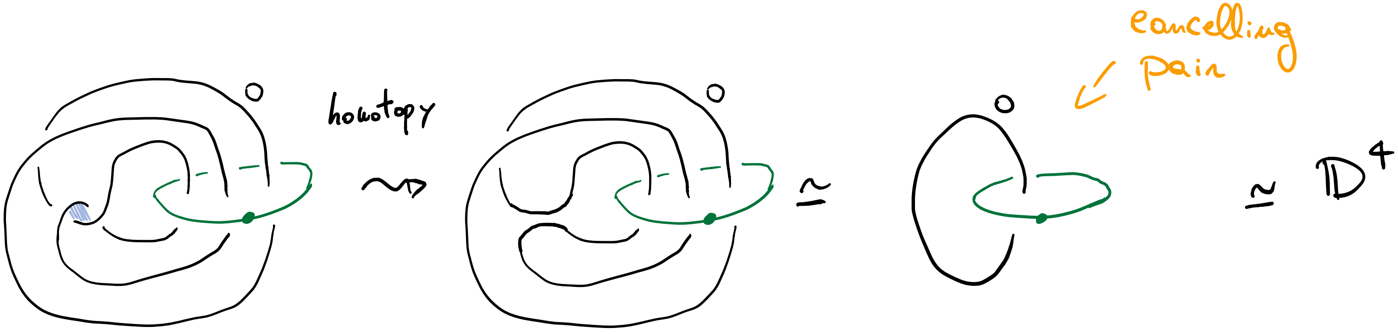
$f : \partial W \longrightarrow \partial W$ is given by the π-rotation along ℓ.

$W =$ 

$\partial W =$  $=$ 

Thm: The pair (W, f) is a cork.

Pf 1) W is contractible



2) f extends to $F: W \to W$ homeo

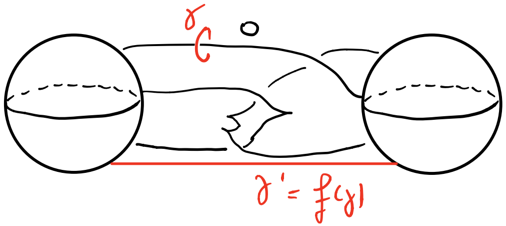
[Freedman] X^4 compact, contractible with ∂. Then any diffeo $f: \partial X \to \partial X$ can be extended to a homeo $F: X \to X$

3) f can not be extended to a diffeo.

Key Lemma
- X^4 compact Stein
- $F^2 \subset X^4$ smooth prop. emb., $\partial F = K$ Legendrian in ∂X^4 (wrt the induced contact structure)
- n the framing induced from a trivialization of the normal bundle of F.

Then we have: $$-\chi(F) \geq \left(TB(K) - n\right) + |\mathrm{rot}_F(K)|$$

W is Stein



If f extends to a diffeo $\Rightarrow$ γ' is smoothly slice.

Key Lemma $\Rightarrow$ $\underbrace{-\chi(D^2)}_{=-1} \geq \underbrace{TB(\gamma')}_{=0} - \underbrace{|\mathrm{rot}(\gamma')|}_{=0}$ ↯

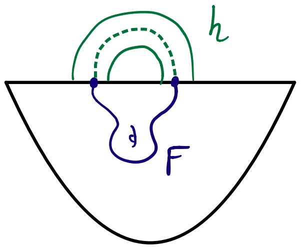
Key Lemma

- X^4 compact Stein
- $F^2 \subset X^4$ smooth prop. emb., $\partial F = K$ Legendrian in ∂X^4 (wrt the induced contact structure)
- n the framing induced from a trivialization of the normal bundle of F.

Then we have :
$$-\chi(F) \geq \big(TB(K) - n\big) + |\mathrm{rot}_F(K)|$$

Pf of Key Lemma.

Attach a 2-handle h to X^4 along K with framing $TB(K)-1$



$Z = X \cup h$ $\quad\quad$ $\Sigma = F \cup_K D^2$

↖Stein

It holds :

1) $[\Sigma] \neq 0$ in $H_2(Z, \mathbb{Z})$
2) $\Sigma \cdot \Sigma = \big(TB(K) - n\big) - 1$
3) $\mathrm{rot}_F(K) = \langle c_1(Z), \Sigma \rangle$ [Gompf. Handlebody construction of Stein surfaces]

Lisca-Matić: Z^4 embeds as a Stein domain in S closed minimal Kähler of general type with $b_2^+>1$

FACTS ABOUT KÄHLER SURFACES [Ozbagci-Stipsicz, Surgery on contact 3-manifolds and Stein surfaces]

1) [Adjunction inequality] X^4 closed minimal Kähler of general type with $b_2^+>1$.

If Σ is smooth surface, $[\Sigma]\neq 0$ in $H_2(X,\mathbb{Z})$, then

$$2g(\Sigma)-2 \geq \Sigma\cdot\Sigma + |\langle c_1(X),\Sigma\rangle|$$

2) Closed minimal Kähler surfaces with $b_2^+>1$ cannot contain nontrivial smooth spheres with self intersection ≥ -1 (NEEDED FOR LATER)

So we have

$$\begin{array}{ccccc} W & \xrightarrow{\cup\, 2\text{-}h} & Z & \xhookrightarrow[\text{Matic}]{\text{Lisca}} & S \text{ Kähler} \\ \cup & & & & \\ F & \xrightarrow{\cup\, \mathbb{D}^2} & \Sigma & \hookrightarrow & \Sigma \end{array}$$

If $[\Sigma] \neq 0$ in $H_2(S,\mathbb{Z}) \overset{\text{Adj. ineq.}}{\Longrightarrow} \underbrace{2g(\Sigma)-2}_{\substack{\| \\ -\chi(F)-1}} \geq \underbrace{\Sigma\cdot\Sigma}_{\substack{\| \\ (TB(K)-n)-1}} + \underbrace{|\langle \underbrace{c_1(S)}_{\substack{\| \\ c_1(Z)}}, \Sigma\rangle|}_{rot_F(K)}$

this can be achieved

Cor $K \subset (S^3, \xi_{st})$ legendrian. If K is smoothly slice $\Rightarrow TB(K) \leq -1$

Pf Key Lemma $\Rightarrow \underbrace{-\chi(\mathbb{D}^2)}_{-1} \geq TB(K) + |rot(K)| \geq TB(K)$

Def We say that X' is an exotic copy X relative to the boundary if

- $\partial X' = \partial X$
- There exists a homeomorphism $X' \to X$ which is the identity on ∂
- There exists no diffeo $X' \to X$ which is the identity on ∂

Cor The Mazur cork W admits an exotic structure rel. to the ∂.

Pf Consider $F: W \to W$ extending the involution f.

W' is W with the pulled back smooth structure.

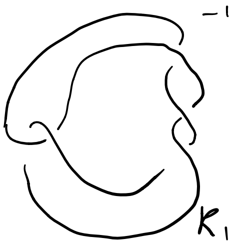
- $\mathrm{id}: W' \to W$ is homeo since F is homeo
- Suppose $\varphi: W' \to W$ diffeo $\varphi|_\partial = \mathrm{id}$. Then

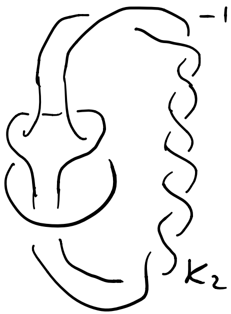
we have

$$\begin{tikzcd} W' \arrow[r, "\varphi", "\sim \mathrm{diff}"'] & W \\ W \arrow[u, "F", "\wr \mathrm{diff}"'] \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

$G: W \to W$ diffeo extending f ↯

Thm The manifolds Q_1 and Q_2 are homeomorphic but not diffeomorphic

$Q_1 =$ 

$Q_2 =$ 

Pf

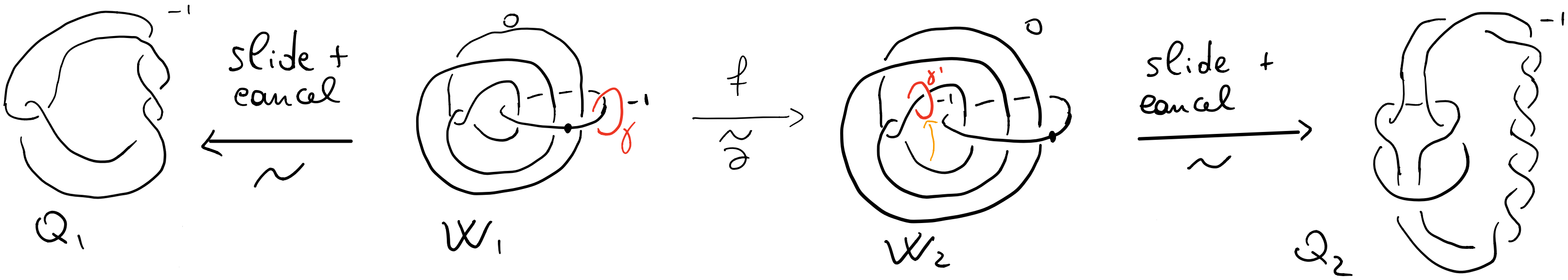
NOT DIFFEO:

K_2 is slice $\Rightarrow$ $H_2(Q_2, \mathbb{Z})$ is generated by a smooth sphere Σ with $\Sigma \cdot \Sigma = -1$

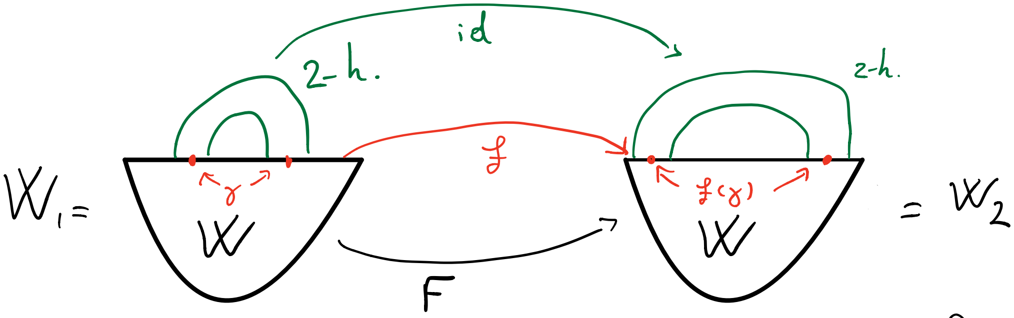
Q_1 is Stein $\Rightarrow$ Q_1 embeds in S minimal Kähler.

$Q_1 \simeq_{diff} Q_2$ $\Rightarrow$ S contains a smooth sphere with self intersection -1 in S

Homeo



$W_1 \simeq_{homeo} W_2$:



W = Mazur cork $\qquad F: W \to W$ homeo extending f.

In the previous theorem Q_1 admits a Stein structure while Q_2 does not.

By using the Key Lemma one can show that there exist also compact Stein manifolds homeomorphic but non diffeomorphic

- Theorem 9.7. Akbulut: 4-manifolds
- Yasui: Corks, exotic 4-manifolds and knot concordance.