A hyperbolic 4-manifold with a perfect circle-valued Morse function.

joint work w/ Bruno Martelli.

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1) INTRODUCTION:
- Why perfect circle-valued Morse function?
- Topological properties;

2) METHODS:
- Colourings;
- Diagonal maps;
- States;

3) CONCLUSION:
- The manifold that admits a perfect circle-valued Morse function;

INTRODUCTION.

3-dim.: TOPOLOGY $\longleftrightarrow$ GEOMETRY
↳ Hyperbolic geom.

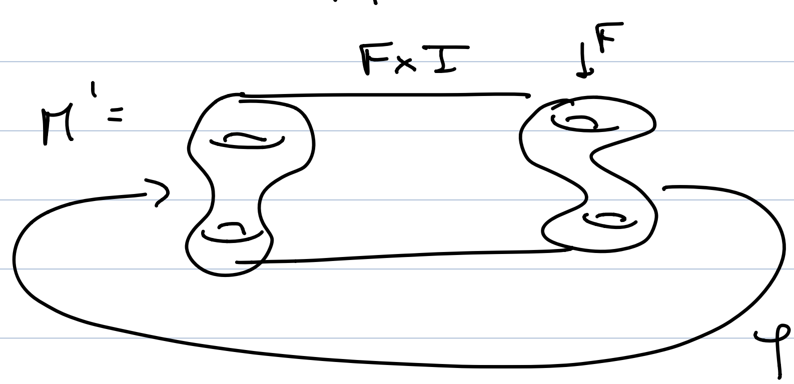
Thm (Agol-Wise) [Virtually fibered conj.]:

If M is a fin. vol. hyp. 3-manifolds,
M virtually fibers over the circle.

$$\begin{tikzcd} F \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\text{finite cover.}"] & S^1 \\ & M & \end{tikzcd}$$

f is a bundle.

$M' =$



Can we extend this in dim. 4?
[I will assume M^4 to be compact]
We cannot expect the same result

- Gauss-Bonnet gen. formula $\chi \sim \text{vol} \Rightarrow \chi > 0$
- Fibration $\Rightarrow \chi(M) = \chi(F) \cdot \underset{0}{\underset{\shortparallel}{\chi(S^1)}} = 0 \Rightarrow \underset{\lightning}{\chi = 0}$.

Bundle over S^1 $\Leftrightarrow$ $f: M \to S^1$ without critical points.

Def.: A circle valued Morse function is a map $f: M \to S^1$ with $<\infty$ of critical points, that are non-degenerate.

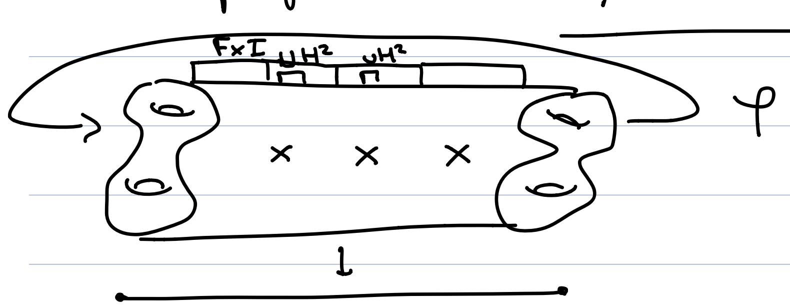
Prop.: If $f: M \to S^1$ is a c.v.M.f. with c_i critical points of index i $\Rightarrow$

$$\chi(M) = \sum_{i=0}^{n} c_i (-1)^i .$$

$$\Rightarrow \quad \sum c_i \geq |\chi(M)|$$

Def.: A "perfect" c.v.M.f. $f: M \to S^1$ is a c.v.M.f. with $|\chi(M)|$ critical points.

Q: Does there exist a hyp. manifold M^4 with a perfect c.v.M.f.?



- P. c.v. M. f. is a generalization of bundles over S^1
- The existance of such a function allow us to study the topology of M.

METHODS

1) Colourings. [build $\overset{\text{hyp./flat}}{\text{manifolds}}$]

Let P be a right-angled polytope in $X^n = \mathbb{H}^n$ or $\mathbb{R}^n$.
let $\{c_1, -, c_n\}$ be a set of colours.

<u>Def.</u> : A colouring of P is a map

$$\lambda : \{F \text{ facets of } P\} \longrightarrow \{c_1, -, c_s\}$$

s.t. :
- λ is surjective;
- If F_i is adjacent to $F_j \Rightarrow \lambda(F_i) \neq \lambda(F_j)$.

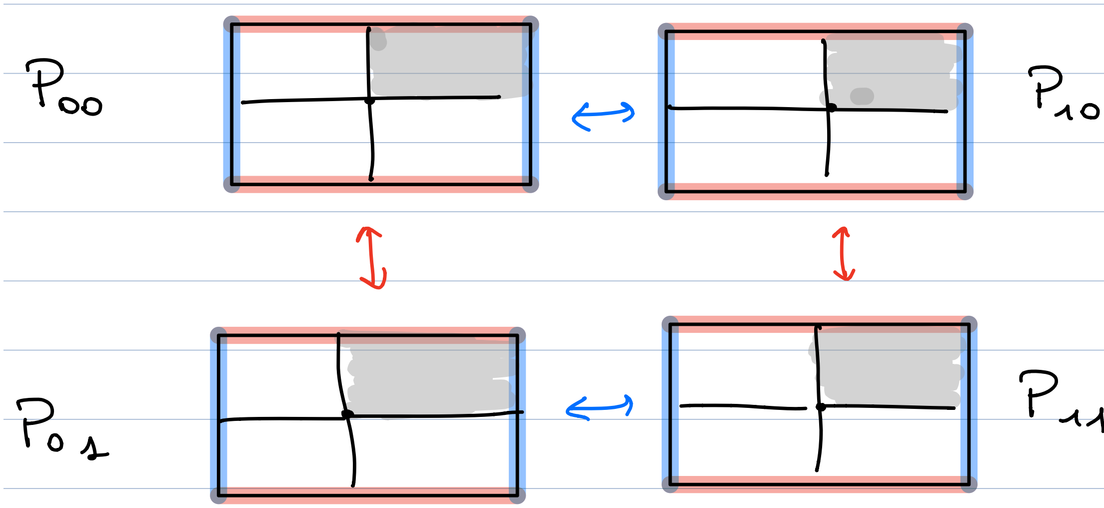
← This is a colouring.

Colouring $\to$ Manifold.

Let A be the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by $\{c_1, \ldots, c_s\}$.

Pick a copy of P for each $v \in A$.
Glue P_v, face F with $P_{v+\lambda(F)}$ face F.

2^s copies



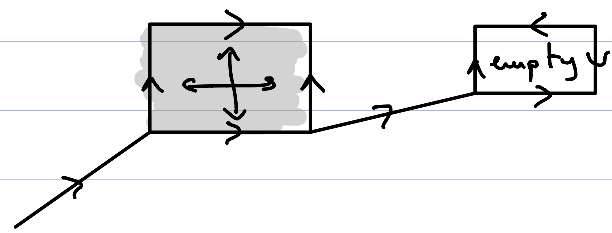
Manifold tessellated by P has a dual tessellation which is a Cube Complex.

Polytope with colouring $\Rightarrow$ Manifold containing a Cube complex homeomorphic to the manifold.

Diagonal maps [build f: Cube complex $\downarrow$ S^1]

Let C be a cube complex.

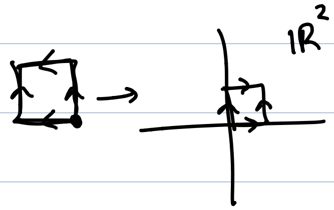
<u>Def</u>: A "coherent orientation" for C is a choice of an orientation for its edges s.t. opposite sides of squares are cooriented.



← A coherent orientation.

Let $Q \in C$ be a cube of C.

Coherent orient. $\Rightarrow Q \subseteq \mathbb{R}^n$



$$\mathbb{R}^n \supseteq Q \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$$
$$[x_1, \ldots, x_n] \mapsto \sum x_i \mapsto [\sum x_i].$$

<u>Fact</u>: Coherent or. $\Rightarrow$ these maps glue to a global map $C \to S^1$.

Assume that C is the dual of a tessellation of a Man. made by R.A. polytopes.

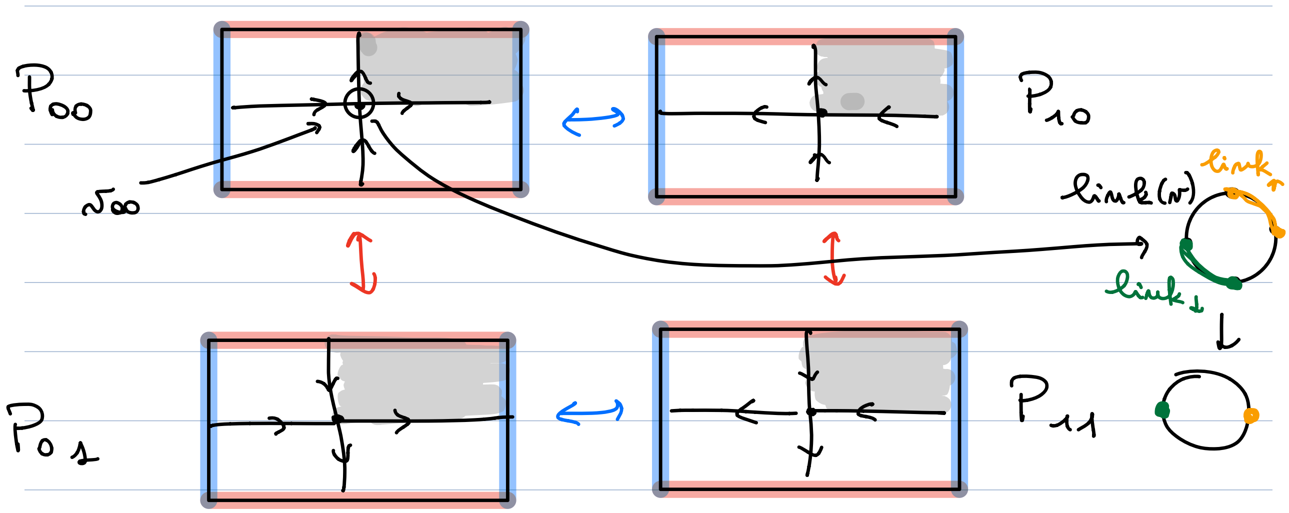
Def: The "Hopf link of index k" is the submanifold of S^{n-1} given by:

$$S^{n-1} = \partial D^n = \partial(D^k \times D^{n-k}) \supseteq \underbrace{\partial D^k \times \{0\} \sqcup \{0\} \times \partial D^{n-k}}_{\text{Hopf. link ind. } k}$$

Example: $n=4$, $k=2$ we have the usual Hopf link.

Given $v \in C$ and a coh. orient., consider $\operatorname{link}(v) \cong S^{n-1}$.

Def: $\operatorname{link}_\uparrow(v)$ [resp. $\operatorname{link}_\downarrow(v)$] is the subcomplex of $\operatorname{link}(v)$ generated by vertices corresponding to edges that point outward [resp. inward].



Theorem: Let C have a coherient orientation s.t. for every $v \in C$ vertex $\text{link}_{\uparrow}^{(v)} \sqcup \text{link}_{\downarrow}^{(v)}$

- collapses to a pair of points (we say v is "trivial")
- collapses to a Hopf link of index k; (we say v is of Hopf type of index k).

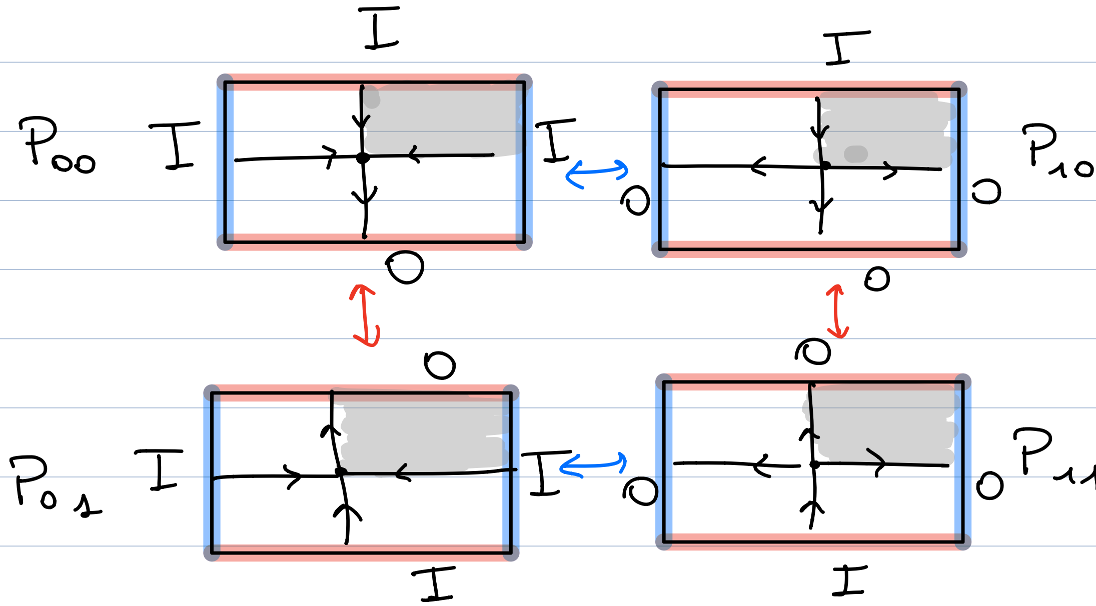
$\Rightarrow$ The Diagonal map can be smoothened to a c.v.M.f. with one cr. point of index k for each vertex of C of Hopf type of index k.

Coh. orient. $\Rightarrow$ • Map $M \to S^1$, with control over the indeces of critical points.

STATE

A state is a choice of a letter {"O" or "I"} for each facet of a R.A. polytope.

If I have a colouring of P, colours act on the state to produce a state for each $\{P_v\}_{v \in A}$, switching the status of facets with that colour.



Colouring + State $\Rightarrow$

Manifold + Coherient orientation $\Rightarrow$

$\Rightarrow$ Manifold with a map $M \to S^1$ with some control over the critical points.