[version: October 30, 2016 (final)] *Nonlinear Analysis* 153 (2017), 35-55
DOI: 10.1016/j.na.2016.10.012

On the structure of continua with finite length and Gołąb's semicontinuity theorem

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Dedicated to Nicola Fusco on the occasion of his 60th birthday

Abstract. The main results in this note concern the characterization of the length of continua[1] (Theorem 2.5) and the parametrization of continua with finite length (Theorem 4.4). Using these results we give two independent and relatively elementary proofs of Gołąb's semicontinuity theorem.

Keywords: continua with finite length, Hausdorff measure, Gołąb's semicontinuity theorem.

MSC (2010): 28A75, 54F50, 26A45, 49J45, 49Q20.

1. Introduction

Let $\mathscr{F}$ be the class of all continua K contained in $\mathbb{R}^d$, endowed with the Hausdorff distance. A classical result due to S. Gołąb (see [8], Section 3, or [6], Theorem 3.18) states that the length, that is, the function $K \mapsto \mathscr{H}^1(K)$, is lower semicontinuous on $\mathscr{F}$. Variants of this semicontinuity result, together with well-known compactness properties of $\mathscr{F}$, play a key role in the proofs of several existence results in the Calculus of Variations, from optimal networks [9] to image segmentation [2] and quasi-static evolution of fractures [3]. In particular, Gołąb's theorem has been extended to general metric spaces in [1], Theorem 4.4.17, and [9], Theorem 3.3.[2]

It should be noted that none of the proofs of Gołąb's theorem mentioned above is completely elementary. On the other hand, the counterpart of this result for paths, namely that the length of a path $\gamma : [0,1] \to X$ is lower semicontinuous with respect to the pointwise convergence of paths, is elementary and almost trivial. This sharp contrast is due to the fact that the definitions of length of a path and of one-dimensional Hausdorff measure of a set are utterly different, even though they aim to describe (essentially) the same geometric quantity. More precisely, the length of a path, being defined as a supremum of finite sums which are clearly continuous, is naturally lower semicontinuous, while the definition of Hausdorff measure is based on Caratheodory's construction, and is designed to achieve σ-subadditivity, not semicontinuity.

In this note we point out a couple of relations/similarities between the one-dimensional Hausdorff measure of continua and the length, which we then use to give two independent (and relatively elementary) proofs of Gołąb's theorem. We think, however, that these results are interesting in their own right.

[1] As usual, a *continuum* is a connected compact metric space (or subset of a metric space), and *length* stands for the one-dimensional Hausdorff measure $\mathscr{H}^1$.

[2] The proof in [1] is actually incomplete; the missing steps were given in [9].

Firstly, in Theorem 2.5 we show that for every continuum X there holds

$$\mathscr{H}^1(X) = \sup\Big\{\sum_i \operatorname{diam}(E_i)\Big\},$$

where the supremum is taken over all finite families $\{E_i\}$ of disjoint *connected* subsets of X. (Note the resemblance with the definition of length of a path.)

Secondly, in Theorem 4.4 we show that every continuum X with finite length admits some sort of canonical parametrization; more precisely, there exists a path $\gamma : I \to X$ with length equal $2\,\mathscr{H}^1(X)$ which "goes through almost every point of X twice, once moving in a direction, and once moving in the opposite direction", the precise statement requires some technical definitions and is postponed to Section 4.

This paper is organized as follows: Sections 2 and 4 contain the two results mentioned above (Theorems 2.5 and 4.4) and the corresponding proofs of Gołąb's theorem. Section 3 contains a review of some basic facts about paths with finite length in a metric space which are used in Section 4, and can be skipped by the expert reader. This review is self-contained and limited in scope; a more detailed presentation of the theory of paths with finite length in metric spaces can be found in [1], Chapter 4, while continua with finite length have been studied in detail in [4] (see also [7]).

Since the results described in this paper are rather elementary (in particular Theorem 2.5), we strove to keep the exposition self-contained, and avoid in particular the use of advanced results from Geometric Measure Theory. On the other hand, proofs are sometimes just sketched, with all steps clearly indicated but many details left to the reader.

2. A characterization of length

The main results in this section are the characterizations of the length of sets with countably many connected components (and in particular of continua) given in Theorem 2.5 and Proposition 2.8. Using the former result we give our first proof of Gołąb's theorem (Theorem 2.9).

2.1. Notation. Through this paper X is a metric space endowed with the distance d. Given $x \in X$ and E, E' subsets of X we set:

- $B(x,r)$ *closed* ball with center x and radius $r > 0$;
- $\operatorname{diam}(E)$ diameter of E, i.e., $\sup\{d(x,x') : x, x' \in E\}$;
- $\operatorname{dist}(x,E)$ distance between x and E, i.e., $\inf\{d(x,x') : x' \in E\}$;
- $\operatorname{dist}(E,E')$ distance between E and E', i.e., $\inf\{d(x,x') : x \in E,\ x' \in E'\}$;
- $d_H(E,E')$ Hausdorff distance between E and E', i.e., the minimum of all $r \geq 0$ such that $\operatorname{dist}(x,E') \leq r$ for every $x \in E$ and $\operatorname{dist}(x',E) \leq r$ for every $x' \in E'$;
- $\operatorname{Lip}(f)$ Lipschitz constant of a map f between metric spaces;
- $|E|$ $= \mathscr{L}^1(E)$ Lebesgue measure of a Borel set E contained in $\mathbb{R}$.

2.2. Hausdorff measure. For every set E contained in X, the one-dimensional Hausdorff measure of E is defined by

$$\mathscr{H}^1(E) := \sup_{\delta>0} \mathscr{H}^1_\delta(E) = \lim_{\delta\to 0} \mathscr{H}^1_\delta(E),$$

where, for every $\delta \in (0,+\infty]$,

$$\mathscr{H}^1_\delta(E) := \inf\Big\{\sum_i \operatorname{diam}(E_i)\Big\},$$

the infimum being taken over all countable families $\{E_i\}$ of subsets of X which cover E and satisfy $\operatorname{diam}(E_i) \leq \delta$.

2.3. Remark. Among the many properties of $\mathscr{H}^1$ we recall the following ones.

(i) $\mathscr{H}^1$ is a σ-subadditive set function (that is, an outer measure on X) and is σ-additive on Borel sets. Moreover $\mathscr{H}^1$ agrees with the (outer) Lebesgue measure $\mathscr{L}^1$ when $X = \mathbb{R}$.

(ii) Given a Lipschitz map $f : X \to Y$, for every set E contained in X there holds $\mathscr{H}^1(f(E)) \leq \operatorname{Lip}(f)\,\mathscr{H}^1(E)$.

(iii) If $\mathscr{H}^1_\delta(E) = 0$ for some $\delta \in (0,+\infty]$ then $\mathscr{H}^1(E) = 0$.

2.4. The set function L_δ. For every $\delta \in (0,+\infty]$ and every set E in X we define

$$L_\delta(E) := \sup\Big\{\sum_i \operatorname{diam}(E_i)\Big\},$$

where the supremum is taken over all finite, disjoint families $\{E_i\}$ of *continua* contained in E with $\operatorname{diam}(E_i) \leq \delta$.

2.5. Theorem. *Let E be a subset of X which is locally compact and has countably many connected components.*[3] *Then, for every $\delta \in (0,+\infty]$,*

$$\mathscr{H}^1(E) = L_\delta(E). \tag{2.1}$$

2.6. Remark. (i) The assumption that E has countably many connected components cannot be dropped. Indeed $L_\delta(E) = 0$ for every totally disconnected set E, but there are examples of such sets with $\mathscr{H}^1(E) > 0$, even compact and contained in $\mathbb{R}$.

(ii) Theorem 2.5, together with Lemma 2.11, implies the following weaker statement: for any set E as above, $\mathscr{H}^1(E)$ agrees with the supremum of $\sum_i \operatorname{diam}(E_i)$ over all finite disjoint families $\{E_i\}$ of connected subsets of E. Concerning this identity, it is not clear if the assumption that E is locally compact can be weakened or even removed. The role of compactness in our proof is briefly discussed in Remark 2.16.

Using Theorem 2.5 we show that $\mathscr{H}^1(E)$ can be approximated by $\sum_i \operatorname{diam}(E_i)$ using *any* partition of E made of connected subsets E_i with sufficiently small diameters. For a precise statement we need the following definition.

2.7. δ-Partitions. Let E be a subset of X and let $\delta \in (0,+\infty]$. We say that a countable family $\{E_i\}$ of subsets of E is a *δ-partition* of E if the sets E_i are Borel, connected, $\mathscr{H}^1$-essentially disjoint (i.e., $\mathscr{H}^1(E_i \cap E_j) = 0$ for every $i \neq j$), cover $\mathscr{H}^1$-almost all of E, and satisfy $\operatorname{diam}(E_i) \leq \delta$.

[3] A connected component of E is any element of the class of nonempty connected subsets of E which is maximal with respect to inclusion; the connected components are closed in E, disjoint, and cover E (for more details see [5], Chapter 6).

2.8. Proposition. *Let E be a subset of X. Then every δ-partition $\{E_i\}$ of E satisfies*

$$\mathscr{H}^1(E) \ge \sum_i \operatorname{diam}(E_i)\,. \tag{2.2}$$

If in addition E is locally compact and has countably many connected components, then for every $m < \mathscr{H}^1(E)$ there exists $\delta_0 > 0$ such that every δ-partition $\{E_i\}$ of E with $\delta \le \delta_0$ satisfies

$$\sum_i \operatorname{diam}(E_i) \ge m\,. \tag{2.3}$$

Using Theorem 2.5 we can also prove the following version of Gołąb's theorem.

2.9. Theorem. *For every $m = 1, 2, \dots,$ let $\mathscr{F}_m$ be the class of all compact subsets of X with at most m connected components. Then the function $K \mapsto \mathscr{H}^1(K)$ is lower-semicontinuous on $\mathscr{F}_m$ endowed with the Hausdorff distance.*

2.10. Remark. The statement of Gołąb's theorem is often restricted to the case $m = 1$, and in the metric setting reads as follows (cf. [1], Theorem 4.4.17): let be given a sequence of continua $\{K_n\}$, contained in a *complete* metric space X, which converge in the Hausdorff distance to some closed set K; then K is a continuum, and $\liminf \mathscr{H}^1(K_n) \ge \mathscr{H}^1(K)$. The assumption that X is complete is needed here to ensure that the limit K is compact and connected, but not to prove the semicontinuity of length.

The rest of this section is devoted to the proofs of Theorems 2.5 and 2.9, and Proposition 2.8. We begin with the proof of Theorem 2.5; the key estimate is contained in Lemma 2.17.

2.11. Lemma. *Let E be a connected set in X. Then $\mathscr{H}^1(E) \ge \operatorname{diam}(E)$.*

Proof. It suffices to prove that $\mathscr{H}^1(E) \ge d(x_0, x_1)$ for every $x_0, x_1 \in E$. Let indeed $f : X \to \mathbb{R}$ be the function defined by $f(x) := d(x, x_0)$. Then

$$\mathscr{H}^1(E) \ge |f(E)| = \operatorname{diam}(f(E)) \ge |f(x_1) - f(x_0)| = d(x_1, x_0)\,,$$

where the first inequality follows from Remark 2.3(ii) (and $\operatorname{Lip}(f) = 1$), while the first equality follows from the fact that $f(E)$ is an interval (because E is connected). □

2.12. Lemma. *For every set E in X and $\delta > 0$ there holds $\mathscr{H}^1(E) \ge L_\delta(E)$.*

Proof. Consider any family $\{E_i\}$ as in the definition of $L_\delta(E)$: Lemma 2.11 yields

$$\mathscr{H}^1(E) \ge \sum_i \mathscr{H}^1(E_i) \ge \sum_i \operatorname{diam}(E_i)\,,$$

and we obtain $\mathscr{H}^1(E) \ge L_\delta(E)$ by taking the supremum over all $\{E_i\}$. □

2.13. Lemma. *Let E be a subset of X and let $\{E_i\}$ be the family of all connected components of E. Then $L_\delta(E) = \sum_i L_\delta(E_i)$ for every $\delta > 0$.*[4]

The proof of this lemma is straightforward, and we omit it.

[4] The sum at the right-hand side of this equality is defined as the supremum of all finite subsums, and is well defined even if the family $\{E_i\}$ is uncountable.

2.14. Lemma. *Let U be a nonempty compact set in X, and let F be a connected component of U. If $F \cap \partial U = \varnothing$ then F is also a connected component of X. Accordingly, if X is connected and $F \ne X$ then $F \cap \partial U \ne \varnothing$.*

Proof. Let $\mathscr{F}$ be the family of all sets A such that $F \subset A \subset U$ and A is open and closed in U. Then $\mathscr{F}$ is closed by finite intersection, and F agrees with the intersection of all $A \in \mathscr{F}$ (see [5], Theorem 6.1.23).

If $F \cap \partial U = \varnothing$ then the intersection of the compact sets $A \cap \partial U$ with $A \in \mathscr{F}$ is empty, which implies that $A \cap \partial U$ is empty for at least one $A \in \mathscr{F}$.[5] This means that F is the intersection of all $A \in \mathscr{F}$ such that $A \cap \partial U = \varnothing$. Note that these sets A are open and closed in X, and then F is connected and agrees with the intersection of a family of open and closed sets. This implies that F is a connected component of X. □

2.15. Corollary. *Let E be a connected set in X, let $B = B(x, r)$ be a ball with center $x \in E$ such that $E \cap B$ is compact and $E \setminus B \ne \varnothing$, and let F be the connected component of $E \cap B$ that contains x. Then $\mathscr{H}^1(B \cap E) \ge \mathscr{H}^1(F) \ge r$.*

Proof. By applying Lemma 2.14 with E and $E \cap B$ in place of X and U, we obtain that F intersects ∂B. Then $\operatorname{diam}(F) \ge r$, and Lemma 2.11 yields $\mathscr{H}^1(F) \ge r$. □

2.16. Remark. The compactness assumptions in Lemma 2.14 and Corollary 2.15 are both necessary. Indeed it is possible to construct a bounded connected set E in $\mathbb{R}^2$ and a ball B with center $x \in E$ such that $E \setminus B \ne \varnothing$, but the connected component F of $E \cap B$ that contains x consists just of the point x; in particular $F \cap \partial B = \varnothing$, and $\mathscr{H}^1(F) = 0$.

2.17. Lemma. *Let E be a set in X which is connected and locally compact. Then $\mathscr{H}^1_\delta(E) \le L_\delta(E)$ for every $\delta > 0$.*

Proof. We can clearly assume that $L_\delta(E)$ is finite. We fix for the time being $\varepsilon > 0$, and choose a finite disjoint family $\{E_i\}$ of continua contained in E with $\operatorname{diam}(E_i) \le \delta$ such that

$$\sum_i \operatorname{diam}(E_i) \ge L_\delta(E) - \varepsilon\,. \tag{2.4}$$

Next we set $E' := E \setminus \big(\cup_i E_i\big)$. Since the union of all E_i is closed and E is locally compact, for every $x \in E'$ we can find a ball $B(x, r)$ with radius $r \le \delta/10$ such that $E \cap B(x, r)$ is compact and contained in E'. Using Vitali's covering lemma (cf. [1], Theorem 2.2.3), we can extract from this family of balls a subfamily of disjoint balls $B_j = B(x_j, r_j)$ such that the balls $B'_j := B(x_j, 5r_j)$ cover E'.

Then the balls B'_j together with the sets E_i cover the set E, and since their diameters do not exceed δ, the definition of $\mathscr{H}^1_\delta(E)$ yields

$$\mathscr{H}^1_\delta(E) \le \sum_i \operatorname{diam}(E_i) + \sum_j \operatorname{diam}(B'_j) \le L_\delta(E) + 10 \sum_j r_j\,. \tag{2.5}$$

On the other hand, by Corollary 2.15, for every j we can find a closed, connected set F_j contained in $B_j \cap E$ with diameter at least r_j. Since the balls B_j are disjoint and contained in E', we have that the sets F_j together with the sets E_i form a disjoint

[5] The basic fact behind this assertion is that every family of compact sets with empty intersection admits a finite subfamily with empty intersection.

family of continua contained in E with diameters at most δ, and therefore, using the definition of $L_\delta(E)$ and (2.4),

$$L_\delta(E) \geq \sum_i \operatorname{diam}(E_i) + \sum_j \operatorname{diam}(F_j) \geq L_\delta(E) - \varepsilon + \sum_j r_j \,,$$

which implies $\varepsilon \geq \sum_j r_j$. Hence (2.5) yields $\mathscr{H}^1_\delta(E) \leq L_\delta(E) + 10\,\varepsilon$, and the proof is complete because ε is arbitrary. □

Proof of Theorem 2.5. By Lemma 2.12, it suffices to prove that

$$L_\delta(E) \geq \mathscr{H}^1(E) \,. \tag{2.6}$$

We assume first that E is connected. In this case Lemma 2.17 and the definition of L_δ in Subsection 2.4 yield

$$L_\delta(E) \geq L_{\delta'}(E) \geq \mathscr{H}^1_{\delta'}(E) \quad \text{for every } 0 < \delta' \leq \delta,$$

and we obtain (2.6) by taking the limit as $\delta' \to 0$.

If E is not connected, then (2.6) holds for every connected component of E, and we obtain that it holds for E as well using Lemma 2.13, the subadditivity of $\mathscr{H}^1$, and the fact that E has countably many connected components. □

The next lemma is used in the proof of Proposition 2.8.

2.18. Lemma. *Let F be a continuum in X, let E be a Borel set in X which contains $\mathscr{H}^1$-almost all of F, and let $\{E_i\}$ be a δ-partition of E (see Subsection 2.7). Then $\sum_i \operatorname{diam}(E_i) \geq \operatorname{diam}(F)$.*

Proof. Take $x_0, x_1 \in F$ such that $d(x_0,x_1) = \operatorname{diam}(F)$, and let $f : X \to \mathbb{R}$ be the Lipschitz function given by $f(x) := d(x,x_0)$. Then

$$\begin{aligned}\sum_i \operatorname{diam}(E_i) &\geq \sum_i \operatorname{diam}(f(E_i)) \\ &= \sum_i |f(E_i)| \geq |f(F)| \geq \operatorname{diam}(F)\,,\end{aligned}$$

where the first inequality follows from the fact that $\operatorname{Lip}(f) = 1$, for the equality we use that each $f(E_i)$ is an interval, for the second inequality we use that the sets E_i cover $\mathscr{H}^1$-almost all of F and therefore the sets $f(E_i)$ cover $\mathscr{L}^1$-almost all of $f(F)$, and the last inequality follows from the fact that $f(F)$ is an interval that contains $f(x_0) = 0$ and $f(x_1) = d(x_0,x_1) = \operatorname{diam}(F)$. □

Proof of Proposition 2.8. To prove (2.2), it suffices to use the definition of δ-partition and estimate $\mathscr{H}^1(E_i) \geq \operatorname{diam}(E_i)$ (see Lemma 2.11):

$$\mathscr{H}^1(E) = \sum_i \mathscr{H}^1(E_i) \geq \sum_i \operatorname{diam}(E_i)\,.$$

To prove (2.3), we observe that by Theorem 2.5 we can find finitely many disjoint continua F_j contained in E such that

$$\sum_j \operatorname{diam}(F_j) \geq m\,. \tag{2.7}$$

Take now $\delta_0 > 0$ such that $\operatorname{dist}(F_j, F_k) > 2\delta_0$ for every $j \neq k$, and let $\{E_i\}$ be any δ-partition of E with $\delta \leq \delta_0$. For every j, let I_j the the collection of all indices i such that E_i intersects F_j. By the choice of δ_0 the collections I_j are pairwise disjoint, and therefore

$$\sum_i \operatorname{diam}(E_i) \geq \sum_j \sum_{i \in I_j} \operatorname{diam}(E_i) \geq \sum_j \operatorname{diam}(F_j)\,, \tag{2.8}$$

where the last inequality is obtained by applying Lemma 2.18 with F replaced by F_j, and E replaced by the union of all E_i with $i \in I_j$. Finally, equations (2.7) and (2.8) imply (2.3). □

We now pass to the proof of Theorem 2.9.

2.19. δ-chains and δ-connected sets. Given $\delta > 0$, a *δ-chain* in X is any finite sequence of points $\{x_i : i = 0, \dots, n\}$ contained in X such that $d(x_{i-1}, x_i) \leq \delta$ for every $i > 0$. We call x_0 and x_n *endpoints* of the δ-chain, and we say that the δ-chain connects x_0 and x_n. The *length* of the δ-chain is

$$\operatorname{length}(\{x_i\}) := \sum_{i=1}^n d(x_{i-1}, x_i)\,.$$

Finally, we say that a set E in X is δ-connected if every couple of points $x, x' \in E$ is connected by a δ-chain contained in E.

2.20. Lemma. *If E is a connected set in X then it is δ-connected for every $\delta > 0$.*

Proof. Fix $x \in E$ and let A_x be the set of all points $x' \in E$ which are connected to x by a δ-chain contained in E. We must show that $A_x = E$.

One easily checks that:

- A_x is closed in E and contains x;
- if $x' \in A_x$ then $B(x', \delta) \cap E$ is contained in A_x; thus A_x is open in E.

Since A_x is nonempty, open and closed in E, and E is connected, we conclude that $A_x = E$, as desired. □

2.21. Lemma. *Let K be a compact set in X with at most m connected components, which contains a δ-connected subset K'. Then*

$$\mathscr{H}^1(K) \geq \operatorname{diam}(K') - m\delta\,. \tag{2.9}$$

Proof. We can assume that m is finite. We then take $x_0, x_1 \in K'$ such that

$$\ell := d(x_0, x_1) \geq \operatorname{diam}(K') - \delta\,, \tag{2.10}$$

and let $H := f(K)$ where $f : X \to \mathbb{R}$ is defined by $f(x) := d(x, x_0)$. It is easy to check that:

(a) $\operatorname{Lip}(f) = 1$, and therefore $\mathscr{H}^1(K) \geq |H| \geq \ell - \big|(0,\ell) \setminus H\big|$;
(b) the sets $f(K')$ and H contain $0 = f(x_0)$ and $\ell = f(x_1)$;
(c) H has at most m connected components because so does K;
(d) the set $f(K')$ is δ-connected because K' is δ-connected and $\operatorname{Lip}(f) = 1$.

Statements (b) and (c) imply that the open set $(0,\ell) \setminus H$ has at most $m-1$ connected components, while statements (b) and (d) imply that each of these connected components has length at most δ; in particular

$$\big|(0,\ell) \setminus H\big| \leq (m-1)\delta\,. \tag{2.11}$$

Using the estimate in (a), (2.10), and (2.11) we finally obtain (2.9). □

2.22. Lemma. *Let K be a compact set in X with at most m connected components, let K' be a δ-connected subset of K, and let U be a closed set in X which contains K' and satisfies*

$$\operatorname{dist}(K', \partial U) \geq r \quad \text{for some } r > 0. \tag{2.12}$$

Then

$$\mathscr{H}^1(K \cap U) \geq \Big(1 - \frac{\delta}{r}\Big) \operatorname{diam}(K') - m\delta\,. \tag{2.13}$$

Proof. We can assume that $\mathscr{H}^1(K \cap U)$ and m are finite. By applying Lemma 2.21 to the set $K \cap U$ we obtain

$$\mathscr{H}^1(K \cap U) \geq \operatorname{diam}(K') - N\delta\,, \tag{2.14}$$

where N is the number of connected components of $K \cap U$. We prove next that

$$\mathscr{H}^1(K \cap U) \geq (N - m)r\,, \tag{2.15}$$

Note indeed that the connected components of $K \cap U$ which do not intersect ∂U are also connected components of K (use Lemma 2.14) and therefore their number is at most m. Thus the remaining connected components K_i intersect ∂U and therefore satisfy

$$\mathscr{H}^1(K_i) \geq \operatorname{diam}(K_i) \geq r$$

(use Lemma 2.11 and assumption (2.12)), and since their number is at least $N - m$ we obtain (2.15).

There are now two possibilities: either $(N - m)r \geq \operatorname{diam}(K')$, and then (2.13) follows from (2.15), or the opposite inequality holds, which means that

$$N < \frac{1}{r}\operatorname{diam}(K') + m\,,$$

and then (2.13) is obtained by plugging this inequality in (2.14). □

Proof of Theorem 2.9. We must show that for every sequence of compact sets $K_n \in \mathscr{F}_m$ which converge in the Hausdorff distance to some $K \in \mathscr{F}_m$ there holds $\liminf \mathscr{H}^1(K_n) \geq \mathscr{H}^1(K)$. Taking into account Theorem 2.5 it suffices to prove that

$$\liminf_{n\to+\infty} \mathscr{H}^1(K_n) \geq \sum_i \operatorname{diam}(E_i) \tag{2.16}$$

for every finite family $\{E_i\}$ of disjoint continua contained in K.

Since the sets E_i are compact and disjoint, we can find $r > 0$ and a family of disjoint closed sets U_i such that each U_i contains E_i and satisfies

$$\operatorname{dist}(E_i, \partial U_i) \geq r\,. \tag{2.17}$$

Then (2.16) follows by showing that, for every i,

$$\liminf_{n\to+\infty} \mathscr{H}^1(K_n \cap U_i) \geq \operatorname{diam}(E_i)\,. \tag{2.18}$$

Let us fix i and choose δ such that $0 < \delta < r$. Since E_i is connected, it is also δ-connected (Lemma 2.20) and therefore it contains a δ-chain $\{x_0, \dots, x_m\}$ with

$$d(x_0, x_m) \geq \operatorname{diam}(E_i) - \delta\,.$$

Consider now any n such that $d_H(K_n, K) \leq \delta$ (that is, any n sufficiently large). By the definition of Hausdorff distance, for every point x_j in the δ-chain we can choose a point $y_j \in K_n$ with $d(x_j, y_j) \leq \delta$, and we set $K'_n := \{y_0, \dots, y_m\}$. One readily checks that

(a) K'_n is a 3δ-chain, and therefore is 3δ-connected;

(b) $\operatorname{diam}(K'_n) \geq d(y_0, y_m) \geq d(x_0, x_m) - 2\delta \geq \operatorname{diam}(E_i) - 3\delta$;

(c) $\operatorname{dist}(K'_n, \partial U_i) \geq \operatorname{dist}(E_i, \partial U_i) - \delta \geq r - \delta$.

We can then apply Lemma 2.22 (with $K_n, K'_n, U_i, 3\delta, r - \delta$ in place of K, K', U, δ, r) and obtain

$$\begin{aligned}\mathscr{H}^1(K_n \cap U_i) &\geq \Big(1 - \frac{3\delta}{r - \delta}\Big) \operatorname{diam}(K'_n) - 3m\delta \\ &\geq \Big(1 - \frac{3\delta}{r - \delta}\Big) (\operatorname{diam}(E_i) - 3\delta) - 3m\delta\,.\end{aligned}$$

To obtain (2.18) we take the liminf as $n \to +\infty$ and then the limit as $\delta \to 0$. □

3. Basic properties of paths in metric spaces

In this section we recall some basic facts concerning paths with finite length, focusing in particular on two results that will be used in the following section, namely Propositions 3.4 and 3.5. Both statements are well-known at least in the Euclidean case.

3.1. Paths. A *path* in X is a continuous map $\gamma : I \to X$ where $I = [a_0, a_1]$ is a closed interval. Then $x_0 := \gamma(a_0)$ and $x_1 := \gamma(a_1)$ are called *endpoints* of γ, and we say that γ connects x_0 to x_1. If $x_0 = x_1$ we say that γ is *closed*.

The *multiplicity* of γ at a point $x \in X$ is the number (possibly equal to $+\infty$)

$$m(\gamma, x) := \#(\gamma^{-1}(x))\,.$$

The *length* of γ is

$$\operatorname{length}(\gamma) = \operatorname{length}(\gamma, I) := \sup\Big\{\sum_{i=1}^n d\big(\gamma(t_{i-1}), \gamma(t_i)\big)\Big\}$$

where the supremum is taken over all finite increasing sequences $\{t_0, \dots, t_n\}$ contained in I.[6]

The length of γ relative to a closed interval J contained in I, denoted by $\operatorname{length}(\gamma, J)$, is the length of the restriction of γ to J. If γ has finite length it is sometimes useful to consider the *length measure* associated to γ, namely the (unique) positive measure μ_γ on I which satisfies

$$\mu_\gamma([t_0, t_1]) = \operatorname{length}(\gamma, [t_0, t_1]) \quad \text{for every } [t_0, t_1] \subset I.$$

We say that γ is a *geodesic* if it has finite length and minimizes the length among all paths with the same endpoints.

We say that γ has *constant speed* if there exists a finite constant c such that

$$\operatorname{length}(\gamma, [t_0, t_1]) = c(t_1 - t_0) \quad \text{for every } [t_0, t_1] \subset I.$$

An (orientation preserving) *reparametrization* of γ is any path $\gamma' : I' \to X$ of the form

$$\gamma' = \gamma \circ \tau$$

where $\tau : I' \to I$ is an increasing homeomorphism.

[6] The length of γ is sometimes called *variation* and denoted by $\operatorname{Var}(\gamma, I)$; paths with finite length are called *rectifiable*.

3.2. Remark. Here are some elementary (and mostly well-known) facts.

(i) The length is lower semicontinuous with respect to the pointwise convergence of paths. More precisely, given a sequence of paths $\gamma_n : I \to X$ which converge pointwise to $\gamma : I \to X$, it is easy to check that

$$\operatorname{length}(\gamma) \leq \liminf_{n\to+\infty} \operatorname{length}(\gamma_n) .$$

(ii) Every path $\gamma : I \to X$ with finite length ℓ, which is not constant on any subinterval of I, admits a Lipschitz reparametrization $\gamma' : [0,1] \to X$ with constant speed ℓ, namely $\gamma' := \gamma \circ \sigma^{-1}$ where $\sigma : I \to [0,1]$ is the homeomorphism given by

$$\sigma(t) := \frac{1}{\ell}\operatorname{length}(\gamma, [a_0, t]) \quad \text{for every } t \in I = [a_0, a_1].$$

(iii) If γ is constant on some subinterval of I then the function σ defined above is continuous, surjective, but not injective. However, we can still consider the left-inverse τ defined by

$$\tau(s) := \min\{t : \sigma(t) = s\} \quad \text{for every } s \in [0,1],$$

and even though τ is not continuous, one readily checks that $\gamma' := \gamma\circ\tau$ is a continuous path with constant speed ℓ, and $m(\gamma', x) = m(\gamma, x)$ for all points $x \in X$ except countably many.

(iv) If γ is Lipschitz then $\operatorname{length}(\gamma, J) \leq \operatorname{Lip}(\gamma)\,|J|$ for every interval J contained in I, and more generally $\mu_\gamma(E) \leq \operatorname{Lip}(\gamma)\,|E|$ for every Borel set E contained in I. Thus the length measure μ_γ is absolutely continuous with respect to the Lebesgue measure on I, and more precisely it can be written as $\mu_\gamma = \rho\,\mathscr{L}^1$ with a density $\rho : I \to \mathbb{R}$ such that $0 \leq \rho \leq \operatorname{Lip}(\gamma)$ a.e.

(v) If γ has constant speed c then $\operatorname{Lip}(\gamma) = c$, $\operatorname{length}(\gamma) = c\,|I|$ and $\mu_\gamma = c\,\mathscr{L}^1$. Conversely, it is easy to check that if $\operatorname{Lip}(\gamma)\,|I| \leq \operatorname{length}(\gamma) < +\infty$ then γ has constant speed $c = \operatorname{Lip}(\gamma) = \operatorname{length}(\gamma)/|I|$.

3.3. Remark. The following result is worth mentioning, even though it will not be used in the following: if γ is Lipschitz and ρ is taken as in Remark 3.2(iv), then for a.e. $t \in I$ there holds

$$\lim_{h\to 0} \frac{d\big(\gamma(t+h), \gamma(t)\big)}{|h|} = \lim_{h\to 0^+} \frac{\operatorname{length}\big(\gamma, [t-h, t+h]\big)}{2h} = \rho(t) . \tag{3.1}$$

The second identity in (3.1) is a straightforward consequence of Lebesgue's differentiation theorem, while the first one is not immediate and will not be proved here. The first limit in (3.1) is called *metric derivative* of γ (see [1], Definition 4.1.2 and Theorem 4.1.6).

We can now state the main results of this section.

3.4. Proposition. *Let X be a continuum with $\mathscr{H}^1(X) < +\infty$, and let $x \neq x'$ be points in X. Then x and x' are connected by an injective geodesic $\gamma : [0,1] \to X$ with constant speed and length $\ell \leq \mathscr{H}^1(X)$.*

If X is a subset of $\mathbb{R}^n$, this statement can be found for example in [6], Lemma 3.12. A slightly more general version of this statement (in the metric setting) can be found in [1], Theorem 4.4.7. For the sake of completeness we give a proof below, which follows essentially the one in [6].

3.5. Proposition. *Let $\gamma : I \to X$ be a path with finite length. Then the multiplicity $m(\gamma, \cdot) : X \to [0, +\infty]$ is a Borel function and*

$$\operatorname{length}(\gamma) = \int_X m(\gamma, x)\, d\mathscr{H}^1(x) . \tag{3.2}$$

In particular $m(\gamma, x)$ is finite for $\mathscr{H}^1$-a.e. $x \in X$.

3.6. Remark. (i) Formula (3.2) can be viewed as the one-dimensional area formula in the metric setting, in particular if coupled with the existence of the metric derivative, see Remark 3.3.

(ii) Formula (3.2) can easily re-written in local form: for every Borel function $f : X \to [0, +\infty]$ there holds

$$\int_I f(\gamma(t))\, d\mu_\gamma(t) = \int_X f(x)\, m(\gamma, x)\, d\mathscr{H}^1(x) . \tag{3.3}$$

This means that the push-forward of the length measure μ_γ according to the map γ agrees with the measure $\mathscr{H}^1$ on X multiplied by the density $m(\gamma, \cdot)$; in short $\gamma_\#(\mu_\gamma) = m(\gamma, \cdot)\, \mathscr{H}^1$.

The rest of this section is devoted to the proofs of Propositions 3.4 and 3.5. We begin with some preliminary lemmas.

3.7. Lemma. *Take X, x, x' as in Proposition 3.4. Then, for every $\delta > 0$, x and x' are connected by a δ-chain $\{x_i : i = 0, \dots, n\}$ (see Subsection 2.19) such that*

$$\operatorname{length}(\{x_i\}) \leq 4\, \mathscr{H}^1(X) . \tag{3.4}$$

Proof. We can assume $\delta < d(x, x')$, otherwise it suffices to take the δ-chain consisting just of the points x, x' and use Lemma 2.11 to obtain (3.4).

By Lemma 2.20, x and x' are connected a δ-chain $\{x_i\}$, and possibly removing some points from the chain, we can further assume that

$$d(x_i, x_j) > \delta \quad \text{if } |j - i| \geq 2. \tag{3.5}$$

Consider now the balls $B_i := B(x_i, \delta/2)$ with $i = 0, \dots, n$, and note that $\mathscr{H}^1(B_i) \geq \delta/2$ by Corollary 2.15, while (3.5) implies that B_i and B_j do not intersect if $|j-i| \geq 2$, which means that every point in X belongs to at most two balls in the family $\{B_i\}$. Using these facts and the estimate $\delta \geq d(x_{i-1}, x_i)$ we obtain

$$2\, \mathscr{H}^1(X) \geq \sum_{i=0}^n \mathscr{H}^1(B_i) \geq \sum_{i=0}^n \frac{\delta}{2} \geq \frac{1}{2} \operatorname{length}(\{x_i\}) . \qquad \square$$

3.8. Lemma. *For every path $\gamma : I \to X$ there holds $\mathscr{H}^1(\gamma(I)) \leq \operatorname{length}(\gamma)$.*

Proof. It suffices to show that $\mathscr{H}^1_\delta(\gamma(I)) \leq \operatorname{length}(\gamma)$ for every $\delta > 0$ (cf. Subsection 2.2). Using the continuity of γ, we partition I into finitely many closed intervals I_i with disjoint interiors so that

$$\operatorname{diam}(\gamma(I_i)) \leq \delta \quad \text{for every } i.$$

Using the definition of $\mathscr{H}^1_\delta$ and the fact that $\operatorname{diam}(\gamma(I_i)) \leq \operatorname{length}(\gamma, I_i)$, we obtain

$$\mathscr{H}^1_\delta(\gamma(I)) \leq \sum_i \operatorname{diam}(\gamma(I_i)) \leq \sum_i \operatorname{length}(\gamma, I_i) = \operatorname{length}(\gamma) . \qquad \square$$

3.9. Lemma. *If $\gamma : I \to X$ is an injective path then* $\operatorname{length}(\gamma) = \mathscr{H}^1(\gamma(I))$.

Proof. By Lemma 3.8 and the definition of length it suffices to show that for every increasing sequence $\{t_0, \dots, t_n\}$ in I there holds

$$\sum_{i=1}^n d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \le \mathscr{H}^1(\gamma(I)) \,.$$

Since the sets $E_i := \gamma([t_{i-1}, t_i])$ are connected, then $\operatorname{diam}(E_i) \le \mathscr{H}^1(E_i)$ (Lemma 2.11), and since γ is injective, the intersection $E_i \cap E_j$ contains at most one point for every $i \neq j$, and in particular is $\mathscr{H}^1$-null. Hence

$$\sum_{i=1}^n d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \le \sum_{i=1}^n \operatorname{diam}(E_i) \le \sum_{i=1}^n \mathscr{H}^1(E_i) \le \mathscr{H}^1(\gamma(I)) \,. \qquad \square$$

Proof of Proposition 3.4. The idea is simple: for every $\delta > 0$ we take the (almost) shortest δ-chain $\{x_i^\delta\}$ that connects x and x', and consider the set Γ_δ of all couples $(t_i^\delta, x_i^\delta) \in [0,1] \times X$ with suitably chosen times $t_i^\delta \in [0,1]$. Passing to a subsequence, we can assume that the compact sets Γ_δ converge in the Hausdorff distance to some limit set Γ as $\delta \to 0$; we then show that Γ is the graph of a path $\gamma : [0,1] \to X$ with the desired properties.

We set $I := [0,1]$ and $L := \mathscr{H}^1(X)$. The proof is divided in several steps.

Step 1: construction of the δ-chain $\{x_i^\delta\}$. Fix $\delta > 0$, and let $\mathscr{F}_\delta$ be the class of all δ-chains with initial point x and final point x', and let L_δ be the infimum of the length over all δ-chains in $\mathscr{F}_\delta$. By Lemma 3.7 we know that $\mathscr{F}_\delta$ is not empty and $L_\delta \le 4L$.

We then choose a δ-chain $\{x_i^\delta : i = 0, \dots, n_\delta\}$ in $\mathscr{F}_\delta$ whose length ℓ_δ satisfies

$$L_\delta \le \ell_\delta \le L_\delta + \delta \le 4L + \delta \,. \tag{3.6}$$

Step 2: construction of the set Γ_δ. Fix $\delta > 0$ and let $\{x_i^\delta : i = 0, \dots, n_\delta\}$ be the δ-chain chosen in Step 1. We can clearly assume that the points x_i^δ are all different, and find an increasing sequence of numbers t_i^δ with $i = 0, \dots, n_\delta$ such that the first one is 0, the last one is 1, and the differences $t_i^\delta - t_{i-1}^\delta$ are proportional to the distances $d(x_{i-1}^\delta, x_i^\delta)$. This means that

$$\frac{d(x_{i-1}^\delta, x_i^\delta)}{t_i^\delta - t_{i-1}^\delta} = \ell_\delta \tag{3.7}$$

for every $i = 1, \dots, n_\delta$, and in particular we have that

$$t_i^\delta - t_{i-1}^\delta = \frac{d(x_{i-1}^\delta, x_i^\delta)}{\ell_\delta} \le \frac{\delta}{\ell_\delta} \le \frac{\delta}{d(x,x')} \,. \tag{3.8}$$

Finally, we set

$$\Gamma_\delta := \big\{(t_i^\delta, x_i^\delta) : i = 0, \dots, n_\delta\big\} \,.$$

Step 3: construction of the set Γ. The sets Γ_δ defined in Step 2 are contained in the compact metric space $[0,1] \times X$, and by Blaschke's selection theorem (see for example [1], Theorem 4.4.15) we have that, possibly passing to a subsequence, they converge in the Hausdorff distance as $\delta \to 0$ to some compact set Γ contained in $[0,1] \times X$.

Step 4: Γ is the graph of a Lipschitz path $\gamma : I \to X$. Formula (3.7) implies that each Γ_δ is the graph of a map γ_δ from a subset of I to X with Lipschitz constant ℓ_δ.

This immediately implies that Γ is the graph of a Lipschitz map γ from a subset of I to X with $\operatorname{Lip}(\gamma) \le \ell$ where (recall (3.6))

$$\ell := \liminf_{\delta \to 0} \ell_\delta = \liminf_{\delta \to 0} L_\delta \le 4L \,. \tag{3.9}$$

Moreover the projection of Γ_δ on I is the set $I_\delta := \{t_i^\delta\}$, and taking into account estimate (3.8) and the fact that I_δ contains 0 and 1, we get that I_δ converges to I in the Hausdorff distance as $\delta \to 0$. This implies that the projection of Γ on I is I itself, which means that the domain of γ is I.

Step 5: γ connects x and x'. Since $x_0^\delta = x$ and $x_{n_\delta}^\delta = x'$ for every $\delta > 0$, each Γ_δ contains the points $(0, x)$ and $(1, x')$, and therefore so does Γ, which means that $\gamma(0) = x$ and $\gamma(1) = x'$.

Step 6: $\ell \le \operatorname{length}(\gamma)$. For every $\delta > 0$ we can extract from the image of γ a δ-chain that connects x and x' and has length at most $\operatorname{length}(\gamma)$. This implies that $\ell_\delta \le \operatorname{length}(\gamma)$ (cf. Step 1), and using (3.9) we obtain the claim.

Step 7: γ has constant speed, and $\operatorname{length}(\gamma) = \ell$. By Step 4 and Step 6 we have that $\operatorname{Lip}(\gamma) \le \ell \le \operatorname{length}(\gamma)$. Then the claim follows from Remark 3.2(v).

Step 8: γ is a geodesic. Let γ' be any path connecting x and x'. Arguing as in Step 6 we obtain $\ell \le \operatorname{length}(\gamma')$, which implies $\operatorname{length}(\gamma) \le \operatorname{length}(\gamma')$ by Step 7.

Step 9: γ is a injective. Assume by contradiction that there exists $t_0 \in I$ and $s > 0$ such that $\gamma(t_0) = \gamma(t_0 + s)$. Then the path $\gamma' : [0, 1-s] \to X$ defined by

$$\gamma'(t) := \begin{cases} \gamma(t) & \text{if } 0 \le t \le t_0, \\ \gamma(t+s) & \text{if } t_0 < t \le 1-s. \end{cases}$$

is well-defined, connects x and x', and has length $\ell(1-s)$, which is strictly smaller than the length of γ, contrary to the fact that γ is a geodesic.

Step 10: $\operatorname{length}(\gamma) \le \mathscr{H}^1(X)$. Apply Lemma 3.9. $\square$

We pass now to the proof of Proposition 3.5.

3.10. Piecewise regular paths. Let I be a closed interval. We say that a finite family $\{I_i\}$ of closed intervals is a *partition* of I if the intervals I_i have pairwise disjoint interiors and cover I, and we say that a path $\gamma : I \to X$ is *piecewise regular* on the partition $\{I_i\}$ if it is either constant or injective on each I_i.

3.11. Lemma. *Let $\gamma : I \to X$ be a path with finite length, and let $\{I_i\}$ be a partition of I. Then there exists a path $\gamma' : I \to X$ such that:*

(i) *γ' is piecewise regular on the partition $\{I_i\}$;*
(ii) *γ' agrees with γ at the endpoints of each I_i and $\gamma'(I_i) \subset \gamma(I_i)$;*
(iii) $\operatorname{length}(\gamma', I_i) = \mathscr{H}^1(\gamma'(I_i)) \le \mathscr{H}^1(\gamma(I_i)) \le \operatorname{length}(\gamma, I_i)$ *for every i.*

Proof. We define γ' on each interval $I_i = [a_i, a_i']$ as follows:

- if $\gamma(a_i) = \gamma(a_i')$ we let γ' be the constant path $\gamma(a_i)$;
- if $\gamma(a_i) \neq \gamma(a_i')$, we let γ' be any injective path from I_i to $X' := \gamma(I_i)$ which connects $\gamma(a_i)$ to $\gamma(a_i')$ (such path exists because X' has finite length, cf. Proposition 3.4).

The path γ' satisfies statements (i) and (ii) by construction, while (iii) follows from Lemmas 3.8 and 3.9. $\square$

Proof of Proposition 3.5. The proof is divided in three cases.

Step 1: γ is injective. In this case the multiplicity $m(\gamma,\cdot)$ is the characteristic function of the compact set $\gamma(I)$, and therefore is Borel, while identity (3.2) follows from Lemma 3.9.

Step 2: γ is piecewise regular. We easily reduce to the previous step.

Step 3: the general case. We choose a sequence of piecewise regular paths $\gamma_n : I \to X$ that approximate γ in the following sense:

(a) γ_n converge to γ uniformly;
(b) $\mathrm{length}(\gamma_n) \le \mathrm{length}(\gamma)$ for every n;
(c) $m(\gamma_n, x) \le m(\gamma, x)$ for every $x \in X$ and every n;
(d) $m(\gamma_n, x) \to m(\gamma, x)$ as $n \to +\infty$ for every $x \in X$.

More precisely, we construct γ_n as follows: for every n we choose a partition $\{I_i^n\}$ of I so that

$$\max_i \mathrm{diam}(I_i^n) \to 0 \quad \text{as } n \to +\infty. \tag{3.10}$$

and then take γ_n according to Lemma 3.11. Then statements (a), (b), (c) and (d) can be readily derived from (3.10) and statements (ii) and (iii) in Lemma 3.11.

We can now prove that the multiplicity $m(\gamma,\cdot)$ is Borel and (3.2) holds. The first part of this claim follows by the fact that $m(\gamma,\cdot)$ agrees with the pointwise limit of the multiplicities $m(\gamma_n,\cdot)$ (statement (d)), which are Borel measurable because the paths γ_n fall into Step 2. To prove (3.2), note that statements (a) and (b) and the semicontinuity of length imply that

(e) $\mathrm{length}(\gamma_n) \to \mathrm{length}(\gamma)$ as $n \to +\infty$,

while statements (c) and (d) and Fatou's lemma yield

(f) $\int_X m(\gamma_n,\cdot)\, d\mathscr{H}^1 \to \int_X m(\gamma,\cdot)\, d\mathscr{H}^1$ as $n \to +\infty$.

We already know that (3.2) holds for each γ_n, and using statements (e) and (f) we can pass to the limit (as $n \to +\infty$) and obtain that (3.2) holds for γ as well. □

4. Parametrizations of continua with finite length

In this section we address the following question: can we parametrize a continuum X by a single path $\gamma : I \to X$, and if yes, what can we require of γ?

First of all, note that in general a continuum cannot be parametrized by a one-to-one path, and not even by a path with multiplicity equal to 1 at almost every point (take for example any network with a triple junction).[7] On the other hand, it is easy to see that every network can be parametrized by a closed path that goes through every arc in the network twice, once in a direction and once in the opposite direction.

In Theorem 4.4 we show that something similar holds for every continuum X with finite length, and more precisely there exists a closed path that *goes through (almost) every point of X twice, once in a direction and once in the opposite direction.*

Before stating the result, we must give a precise formulation of the requirement in italic. If X is a network made of regular arcs of class C^1 in $\mathbb{R}^n$, we simply require that γ has multiplicity equal to 2 and *degree* equal to 0 at every point of X except junctions. The problem in extending this condition to general continua is that the usual definition of degree cannot be easily adapted to the metric setting. To get around this issue, in Subsection 4.1 we introduce a suitable weaker notion of path with degree zero.

Unless further specification is made, in the following X is a metric space.

4.1. Paths with degree zero. Given a Lipschitz path $\gamma : I \to X$, a locally bounded Borel function $f : X \to \mathbb{R}$, and a Lipschitz function $g : X \to \mathbb{R}$, we introduce the notation

$$\int_\gamma f\, dg := \int_I (f\circ\gamma)\, \frac{d}{dt}(g\circ\gamma)\, dt\,. \tag{4.1}$$

Note that $g\circ\gamma$ is Lipschitz, and therefore the derivative in the integral at the right-hand side is well-defined at almost every $t \in I$ and bounded in t, and the integral itself is well-defined.

We say that γ has *degree zero* (at almost every point of its image) if

$$\int_\gamma f\, dg = 0 \quad \text{for every } f, g : X \to \mathbb{R} \text{ Lipschitz.} \tag{4.2}$$

4.2. Remark. (i) A simple approximation argument shows that if γ has degree zero then $\int_\gamma f\, dg = 0$ for every Lipschitz function $g : X \to \mathbb{R}$ and every bounded Borel function $f : X \to \mathbb{R}$.

(ii) If X is a finite union of oriented regular arcs in $\mathbb{R}^n$, or more generally an oriented 1-rectifiable set, and $\gamma : I \to X$ is a Lipschitz path, then for $\mathscr{H}^1$-almost every $x \in X$ one can define the degree of γ at x, denoted by $\deg(\gamma, x)$. Moreover for every $f, g : X \to \mathbb{R}$ there holds

$$\int_\gamma f\, dg := \int_X f(x)\, \frac{\partial g}{\partial\tau}(x)\, \deg(\gamma, x)\, d\mathscr{H}^1(x)\,, \tag{4.3}$$

where $\partial g/\partial\tau$ is the tangential derivative of g. Using this formula it is easy to check that (4.2) holds if and only if $\deg(\gamma, x) = 0$ for $\mathscr{H}^1$-a.e. $x \in X$. This justifies the use of the expression "path with degree zero" in Subsection 4.1.

(iii) Formula (4.2) can be reinterpreted in the framework of metric currents by saying that the push-forward according to γ of the canonical 1-current associated to the (oriented) interval I is trivial.

4.3. Proposition. *Let $\gamma : I \to X$ be a Lipschitz path, and let $f, g : X \to \mathbb{R}$ be Lipschitz functions. Then the following statements hold.*

(i) [Invariance under reparametrization] *Let $\tau : I' \to I$ be an increasing homeomorphism such that $\gamma\circ\tau$ is Lipschitz. Then*

$$\int_\gamma f\, dg = \int_{\gamma\circ\tau} f\, dg\,. \tag{4.4}$$

In particular, γ has degree zero if and only if $\gamma\circ\tau$ has degree zero.

(ii) [Stability] *Given a sequence of paths $\gamma_n : I \to X$ which are uniformly Lipschitz and converge uniformly to γ, then*

$$\int_{\gamma_n} f\, dg \to \int_\gamma f\, dg \quad \text{as } n \to +\infty.$$

In particular, if each γ_n has degree zero, then γ has degree zero.

[7] By network we mean here a *connected* union of finitely many arcs (that is, images of injective Lipschitz paths) which intersect at most at the endpoints; a point which agrees with n endpoints, $n \ge 3$, is called an n-junction.

(iii) [Parity] *If γ has degree zero then the multiplicity $m(\gamma, x)$ is finite and even for $\mathscr{H}^1$-a.e. $x \in X$.*

We can now state the main result of this section.

4.4. Theorem. *Let X be a continuum with finite length. Then there exists a path $\gamma : [0,1] \to X$ with the following properties:*

(i) *γ is closed, Lipschitz, surjective, and has degree zero;*
(ii) *$m(\gamma, x) = 2$ for $\mathscr{H}^1$-a.e. $x \in X$, and $\mathrm{length}(\gamma) = 2\,\mathscr{H}^1(X)$;*
(iii) *γ has constant speed, equal to $2\,\mathscr{H}^1(X)$.*

4.5. Remark. (i) The existence of a Lipschitz surjective path $\gamma : [0,1] \to X$ with $\mathrm{Lip}(\gamma) \le 2\,\mathscr{H}^1(X)$ was first proved in [10]. The key improvement here is pointing out that γ can be taken of degree zero.

(ii) An immediate corollary of this result is that every continuum X with finite length is a rectifiable set of dimension 1.

(iii) If X is contained in $\mathbb{R}^n$, then one can apply Rademacher's differentiability theorem to the parametrization γ and prove with little effort that X admits a tangent line in the classical sense at $\mathscr{H}^1$-a.e. point.

The rest of this section is devoted to the proofs of Proposition 4.3 and Theorem 4.4. At the end of the section we give another proof of Gołąb's theorem based on the latter.

4.6. Additional notation. Let be given a Lipschitz path $\gamma : I \to X$ and a Lipschitz function $g : X \to \mathbb{R}$. We write $h := g \circ \gamma$, and denote by N the set of all $s \in \mathbb{R}$ such that one of the following properties fails:

(a) the set $h^{-1}(s)$ is finite;
(b) the derivative of h exists and is not 0 at every $t \in h^{-1}(s)$.

Thus for every $s \in \mathbb{R} \setminus N$ and every $x \in g^{-1}(s)$, the following sum is well-defined and finite:

$$p(x) := \sum_{t \in \gamma^{-1}(x)} \mathrm{sgn}\big(\dot h(t)\big)\,, \tag{4.5}$$

where, as usual, $\mathrm{sgn}(x) := 1$ if $x > 0$, $\mathrm{sgn}(x) := -1$ if $x < 0$, and $\mathrm{sgn}(0) = 0$.

4.7. Lemma. *Take γ, g, h, N and p as in Subsection 4.6. Then $|N| = 0$ and for every Lipschitz function $f : X \to \mathbb{R}$ there holds*

$$\int_\gamma f\,dg = \int_{\mathbb{R}\setminus N} \bigg[\sum_{x \in g^{-1}(s)} f(x)\,p(x) \bigg] ds\,. \tag{4.6}$$

Proof. To prove that $|N| = 0$ we write

$$N = N_0 \cup h(E_0) \cup h(E_1)\,,$$

where N_0 is the set of all $s \in \mathbb{R}$ such that $h^{-1}(s)$ is infinite, E_0 is the set of all $t \in I$ where the derivative of h exists and is 0, E_1 is the set of all $t \in I$ where the derivative of h does not exists.

We observe now that $|N_0| = 0$ and $|h(E_0)| = 0$ by the one-dimensional area formula applied to the Lipschitz function $h : I \to \mathbb{R}$,[8] while $|E_1| = 0$ by Rademacher's theorem and then $|h(E_1)| = 0$ because h is Lipschitz. We conclude that $|N| = 0$.

Let us prove (4.6). Using (4.1), the area formula, and $|N| = 0$, we get

$$\begin{aligned}\int_\gamma f\,dg = \int_I (f\circ\gamma)\,\dot h\,dt &= \int_I (f\circ\gamma)\,\mathrm{sgn}(\dot h)\,|\dot h|\,dt \\ &= \int_{\mathbb{R}\setminus N} \bigg[\sum_{t\in h^{-1}(s)} f(\gamma(t))\,\mathrm{sgn}(\dot h(t)) \bigg] ds\,,\end{aligned}$$

and we obtain (4.6) by suitably rewriting the sum within square brackets. □

Proof of Proposition 4.3(i). Given g and γ, we take h, N and p as in Subsection 4.6, and let h' N' and p' be the analogous quantities where γ is replaced by $\gamma\circ\tau$. Thanks to Lemma 4.7, identity (4.4) can be proved by showing that $p(x) = p'(x)$ for every x such that $g(x) \notin N \cup N'$.

Taking into account (4.5) and the fact that $h' = h\circ\tau$, the identity $p(x) = p'(x)$ reduces to the following elementary statement: given $t \in I$ such that the derivative of h at t exists and is nonzero, and the derivative of $h' = h\circ\tau$ at $t' := \tau^{-1}(t)$ exists and is nonzero, then these derivatives have the same sign (recall that τ is increasing). □

Proof of Proposition 4.3(ii). In view of (4.1) it suffices to show that

$$\int_I (f\circ\gamma_n)\,\frac{d}{dt}(g\circ\gamma_n)\,dt \to \int_I (f\circ\gamma)\,\frac{d}{dt}(g\circ\gamma)\,dt \quad \text{as } n \to +\infty.$$

This is an immediate consequence of the fact that the functions $f\circ\gamma_n$ converge to $f\circ\gamma$ uniformly, and therefore strongly in $L^1(I)$, while the derivatives of the functions $g\circ\gamma_n$ converge to the derivative of $g\circ\gamma$ in the weak* topology of $L^\infty(I)$. □

The next lemmas are used in the proof of Proposition 4.3(iii).

4.8. Lemma. *Let $\gamma : I \to X$ be a path with finite length, and let μ_γ be the corresponding length measure (Subsection 3.1). Then for μ_γ-a.e. $t \in I$ there holds*

$$\rho(t) := \liminf_{r\to 0} \frac{\mathrm{diam}\big(\gamma(B(t,r))\big)}{r} > 0\,. \tag{4.7}$$

This lemma would be an immediate consequence of formula (3.1), which however we did not prove. The proof below is self-contained.

Proof. Let $E := \{t \in I : \rho(t) = 0\}$. We must prove that $\mu_\gamma(E) = 0$.

Let $\varepsilon > 0$ be fixed for the time being. For every $t \in E$ we can find $r_t > 0$ such that the ball (i.e., centered interval) $B(t, r_t)$ is contained in I and

$$\mathrm{diam}\big(\gamma(B(t,r_t))\big) \le \varepsilon r_t\,.$$

[8] The one-dimensional area formula we need reads as follows: if $h : I \to \mathbb{R}$ is Lipschitz and $f : I \to \mathbb{R}$ is either positive or in $L^1(I)$ then

$$\int_I f\,|\dot h|\,dt = \int_{\mathbb{R}} \bigg[\sum_{t\in h^{-1}(s)} f(t) \bigg] ds\,.$$

In particular $\int_I |\dot h|\,dt = \int_{\mathbb{R}} m(h,s)\,ds$ where $m(h,s)$ is the multiplicity of h at s, which implies that $m(h,s)$ is finite for a.e. $s \in \mathbb{R}$.

Consider now the family $\mathscr{F}$ of all balls $B(t, r_t/5)$ with $t \in E$. Using Vitali's covering lemma (see for example [1], Theorem 2.2.3), we can extract from $\mathscr{F}$ a subfamily of disjoint balls $B_j = B(t_j, r_j/5)$ such that the balls $B'_j := B(t_j, r_j)$ cover E. Thus the sets $\gamma(B'_j)$ cover $\gamma(E)$ and can be used to estimate $\mathscr{H}^1_\infty(E)$ (see Subsection 2.2):

$$\begin{aligned}\mathscr{H}^1_\infty(\gamma(E)) &\le \sum_j \operatorname{diam}(\gamma(B'_j))\\ &\le \varepsilon \sum_j r_j = \frac{5\varepsilon}{2} \sum_j |B_j| \le \frac{5\varepsilon}{2}\,|I|\end{aligned}$$

(for the last inequality we used that the balls B_j are disjoint and contained in I). Since ε is arbitrary, we obtain that $\mathscr{H}^1_\infty(\gamma(E)) = 0$ and therefore $\mathscr{H}^1(\gamma(E)) = 0$ (cf. Remark 2.3(iii)). Using formula (3.3) we finally get

$$\mu_\gamma(E) = \int_{\gamma(E)} m(\gamma, x)\, d\mathscr{H}^1(x) = 0\,. \qquad \square$$

4.9. Lemma. *Let $\gamma : I \to X$ be a path with finite length, and let E be a Borel subset of $\gamma(I)$ with $\mathscr{H}^1(E) > 0$. Then there exists a Lipschitz function $g : X \to \mathbb{R}$ such that $|g(E)| > 0$.*

Proof. We can assume that γ has constant speed 1 (Remark 3.2(iii)), which implies that γ has Lipschitz constant 1 and the length measure μ_γ agrees with the Lebesgue measure on I.

We set $F := \gamma^{-1}(E)$ and $F' := I \setminus F$. Since $E = \gamma(F)$ is not $\mathscr{H}^1$-negligible, F must have positive Lebesgue measure, and using Lemma 4.8 and Lebesgue's density theorem we can find a point $t \in F$ where (4.7) holds and F has density 1, and accordingly F' has density 0.

We define $g : X \to \mathbb{R}$ by $g(x) := d(x, \gamma(t))$, and set $h := g \circ \gamma$. By (4.7) there exists $\delta > 0$ such that $\operatorname{diam}(\gamma(B(t,r))) \ge 2\delta r$ for every ball $B(t,r)$ contained in I. This implies that

$$\sup_{t' \in B(t,r)} h(t') = \sup_{t' \in B(t,r)} d(\gamma(t'), \gamma(t)) \ge \delta r\,.$$

Thus the interval $h(B(t,r))$ contains $[0, \delta r]$ and then

$$\big|h(B(t,r))\big| \ge \delta r\,. \tag{4.8}$$

On the other hand, the fact that h is Lipschitz and F' has density 0 at t implies

$$\big|h(F' \cap B(t,r))\big| = o(r)\,. \tag{4.9}$$

Finally, the inclusion

$$g(E) = h(F) \supset h(B(t,r)) \setminus h(F' \cap B(t,r))\,,$$

together with estimates (4.8) and (4.9), yields

$$|g(E)| \ge \delta r - o(r)\,,$$

and we conclude by observing that $\delta r - o(r) > 0$ for r small enough. $\square$

Proof of Proposition 4.3(iii). We already know that $m(\gamma, x)$ is finite for $\mathscr{H}^1$-a.e. $x \in X$ (Proposition 3.5). Let then E be the set of all $x \in X$ such that $m(\gamma, x)$ is finite and *odd*, and assume by contradiction that $\mathscr{H}^1(E) > 0$.

By Lemma 4.9 there exists a Lipschitz function $g : X \to \mathbb{R}$ such that $|g(E)| > 0$. Then we take N and p as in Subsection 4.6, and let $f : X \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} \operatorname{sgn}(p(x)) & \text{if } x \in E \setminus g^{-1}(N),\\ 0 & \text{otherwise.}\end{cases}$$

For this choice of g and f, the sum between square brackets in formula (4.6) is a positive *odd* integer for every $s \in g(E) \setminus N$ and is 0 otherwise, and therefore (4.6) yields

$$\int_\gamma f\, dg \ge |g(E) \setminus N| = |g(E)| > 0\,.$$

This contradicts the assumption that γ has degree zero (cf. Remark 4.2(i)). $\square$

The following construction is used in the proof of Theorem 4.4.

4.10. Joining paths. Let $I := [0,1]$, let $\gamma : I \to X$ be a *closed* path, and let $\gamma' : I \to X$ be a path whose endpoint $\gamma'(0)$ belongs to the image of γ. We join these paths to form a *closed* path $\gamma \ltimes \gamma' : I \to X$ as follows: we choose t_0 such that $\gamma'(0) = \gamma(t_0)$ and set [9]

$$(\gamma \ltimes \gamma')(t) := \begin{cases} \gamma(3t) & \text{if } 0 \le t \le t_0/3,\\ \gamma'(3t - t_0) & \text{if } t_0/3 < t \le (t_0+1)/3,\\ \gamma'(t_0 + 2 - 3t) & \text{if } (t_0+1)/3 < t \le (t_0+2)/3,\\ \gamma(3t-2) & \text{if } (t_0+2)/3 < t \le 1.\end{cases}$$

The next lemma collects some straightforward properties of $\gamma \ltimes \gamma'$ that will be used later. We omit the proof.

4.11. Lemma. *Take γ, γ' and $\gamma \ltimes \gamma'$ as in Subsection 4.10. The following statements hold:*

(i) *if γ and γ' are Lipschitz, then $\gamma \ltimes \gamma'$ is Lipschitz;*
(ii) $\operatorname{length}(\gamma \ltimes \gamma') = \operatorname{length}(\gamma) + 2\operatorname{length}(\gamma')$*;*
(iii) *if γ and γ' have bounded multiplicities, so does $\gamma \ltimes \gamma'$;*
(iv) *if the path γ has multiplicity 2 at all points in its image except finitely many, γ' has multiplicity 1 at all points in its image except finitely many, and the set $\gamma(I) \cap \gamma'(I)$ is finite, then $\gamma \ltimes \gamma'$ has multiplicity 2 at all points in its image except finitely many;*
(v) *for every $f : X \to \mathbb{R}$ bounded and Borel, and every $g : X \to \mathbb{R}$ Lipschitz there holds*
$$\int_{\gamma \ltimes \gamma'} f\, dg = \int_\gamma f\, dg\,;$$
(vi) *if γ has degree zero (cf. Subsection 4.1) then $\gamma \ltimes \gamma'$ has degree zero.*

Proof of Theorem 4.4. Let $I := [0,1]$. We obtain the path $\gamma : I \to X$ with the required properties as limit of the closed paths $\gamma_n : I \to X$ constructed by the inductive procedure described in the next two steps.

[9] The notation $\gamma \ltimes \gamma'$ is not quite appropriate, because this path does not depend only on γ and γ', but also on the choice of t_0.

Step 1: construction of γ_1. We choose $x'_0 \in X$ and take $x_0 \in X$ which maximizes the distance from x'_0. By Proposition 3.4, there exists an *injective* Lipschitz path $\gamma'_0 : I \to X$ that connects x'_0 to x_0. We then set

$$\gamma_1(t) := \begin{cases} \gamma'_0(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma'_0(2-2t) & \text{if } 1/2 < t \le 1. \end{cases}$$

Note that that γ_1 is closed, has degree 0, and its multiplicity is 2 at all points of $\gamma_1(I)$ except $\gamma_1(1/2)$, where it is 1. Clearly $\mathrm{length}(\gamma_1) = 2\,\mathrm{length}(\gamma'_0)$.

Step 2: construction of γ_{n+1}, given γ_n. We assume that $\gamma_n(I)$ is a proper subset of X.[10] Then we take a point $x_n \in X$ which maximizes the distance from $\gamma_n(I)$, and an *injective* Lipschitz path $\gamma'_n : I \to X$ that connects x_n to some point $x'_n \in \gamma_n(I)$.

By "cutting off a piece of γ'_n" we can assume that this path intersects $\gamma_n(I)$ *only* at the endpoint x'_n. We can also assume that $x'_n = \gamma'_n(0)$. Then we set

$$\gamma_{n+1} := \gamma_n \ltimes \gamma'_n .$$

Step 3: properties of γ_n. Using Lemma 4.11, one easily proves that each γ_n is closed and Lipschitz, has degree zero, and satisfies

$$\ell_n := \mathrm{length}(\gamma_n) = 2\sum_{m=0}^{n-1} \mathrm{length}(\gamma'_m) . \tag{4.10}$$

Moreover the multiplicity of γ_n is bounded and equal to 2 for all points in $\gamma_n(I)$ except finitely many. This last property, together with formula (3.2), yields

$$\ell_n := \mathrm{length}(\gamma_n) \le 2\,\mathscr{H}^1(X) . \tag{4.11}$$

Step 4: reparametrization of γ_n. Since the multiplicity of γ_n is bounded, γ_n is not constant on any subinterval of I, and therefore it admits a reparametrization with constant speed equal to ℓ_n (Remark 3.2(ii)). In the rest of the proof we replace γ_n by this reparametrization, which still satisfies all the properties stated in Step 3.

Step 5: construction of γ. The paths $\gamma_n : I \to X$ are closed and uniformly Lipschitz, and more precisely $\mathrm{Lip}(\gamma_n) = \ell_n \le 2\,\mathscr{H}^1(X)$. Therefore, possibly passing to a subsequence, the paths γ_n converge uniformly to a path $\gamma : I \to X$ which is closed and Lipschitz, and satisfies $\mathrm{Lip}(\gamma) \le 2\,\mathscr{H}^1(X)$.

Step 6: γ is surjective. Equations (4.10) and (4.11) imply that the sum of the lengths of all paths γ'_n is finite, and then

$$\mathrm{length}(\gamma'_n) \to 0 \quad \text{as } n \to +\infty.$$

Now, recalling the choice of x_n and the fact that γ'_n connects x_n to $\gamma_n(I)$ (cf. Step 2) we obtain that

$$d_n := \sup_{x\in X} \mathrm{dist}(x, \gamma_n(I)) = \mathrm{dist}(x_n, \gamma_n(I)) \le \mathrm{length}(\gamma'_n) ,$$

and therefore d_n tends to 0 as $n \to +\infty$, which means that the union of all $\gamma_n(I)$ is dense in X.

Now, $\gamma_m(I)$ contains $\gamma_n(I)$ for every $m \ge n$, and then $\gamma(I)$ contains $\gamma_n(I)$ for every n. Hence $\gamma(I)$ contains a dense subset of X, and since it is closed, it must agree with X.

[10] This inductive procedure stops if γ_n is surjective; when this happens, we simply reparametrize γ_n so that it has constant speed, and set $\gamma := \gamma_n$. In this case it is quite easy to verify that γ has the required properties (we omit the details).

Step 7: completion of the proof. Since the paths γ_n have degree zero, so does γ (Proposition 4.3(ii)), and the proof of statement (i) is complete. This fact, the surjectivity of γ, and Proposition 4.3(iii) imply that

$$m(\gamma, x) \ge 2 \quad \text{for } \mathscr{H}^1\text{-a.e. } x \in X. \tag{4.12}$$

On the other hand, estimate (4.11) and the semicontinuity of the length (Remark 3.2(i)) imply

$$\mathrm{length}(\gamma) \le 2\,\mathscr{H}^1(X) . \tag{4.13}$$

Now, equations (4.12) and (4.13), together with (3.2), imply that equality must hold both in (4.12) and in (4.13), and statement (ii) is proved.

To prove statement (iii), note that $\mathrm{Lip}(\gamma) \le 2\,\mathscr{H}^1(X) = \mathrm{length}(\gamma)$ (cf. Step 5), and then γ must have constant speed (cf. Remark 3.2(v)). □

We conclude this section by another proof of Gołąb's theorem.

Second proof of Theorem 2.9 for $m = 1$. We must show that for every sequence of continua K_n contained in X which converge in the Hausdorff distance to some continuum K, there holds $\liminf \mathscr{H}^1(K_n) \ge \mathscr{H}^1(K)$.

We can clearly assume that the measures $\mathscr{H}^1(K_n)$ are uniformly bounded. For every n, we apply Theorem 4.4 to the continuum K_n and find a path $\gamma_n : I \to X$ with $I := [0,1]$ such that $\gamma_n(I) = K_n$, γ_n has constant speed and degree zero, and $\mathrm{length}(\gamma_n) = \mathrm{Lip}(\gamma_n) = 2\,\mathscr{H}^1(K_n)$.

Note that the paths γ_n are uniformly Lipschitz, and therefore, possibly passing to a subsequence, they converge uniformly to some path $\gamma : I \to X$, and clearly $\gamma(I) = K$.

Moreover Proposition 4.3(ii) implies that γ has degree zero, and Proposition 4.3(iii) implies that $m(\gamma, x) \ge 2$ for $\mathscr{H}^1$-a.e. $x \in K$. Then formula (3.2) implies that $\mathrm{length}(\gamma) \ge 2\,\mathscr{H}^1(K)$.

We can now conclude, using the semicontinuity of length (cf. Remark 3.2(i)):

$$\liminf_{n\to+\infty} \mathscr{H}^1(K_n) = \frac{1}{2} \liminf_{n\to+\infty} \mathrm{length}(\gamma_n) \ge \frac{1}{2}\mathrm{length}(\gamma) \ge \mathscr{H}^1(K) . \qquad \square$$

ACKNOWLEDGEMENTS

The research of the first author has been partially supported by the University of Pisa through the 2015 PRA Grant "Variational methods for geometric problems", and by the European Research Council (ERC) through the Advanced Grant "Local structure of sets, measures and currents".

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