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Generalized N-property and Sard Theorem for Sobolev maps

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Abstract. I report on some recent extensions of the Lusin N-property and the Sard theorem for Sobolev maps, which have been obtained in a joint work with M. Csörnyei, E. D'Aniello, and B. Kirchheim. Our research was originally motivated by questions related to the uniqueness of weak solutions for the continuity equation associated to a vector field with Sobolev regularity.[1]

Keywords: Sobolev maps, Lusin N-property, Sard theorem, flow associated to a vector field, continuity equation.

MSC (2010): 35F10 (35L03, 46E35, 26B35, 28A75)

1. Introduction

In this paper I describe some extensions of the Lusin N-property and the Sard theorem for Sobolev maps which have been recently obtained in collaboration with M. Csörnyei, E. D'Aniello, and B. Kirchheim [3], [4]; since this work is still in progress, some of the results I will mention here are not yet in definitive form.

The N-property (see Section 4 for the definition) has been widely studied, mostly in connection with the area formula for Sobolev maps and other classes of weakly differentiable maps. However, the variant of this property that we are interested in arises as a key ingredient of our proof of the optimal form of Sard theorem for Sobolev maps. We were led to consider this version of Sard theorem in the attempt—which eventually failed—to produce a counterexample to a certain uniqueness statement for the flow associated to a vector field with Sobolev regularity; this statement is in turn related to the uniqueness of weak solutions of the continuity equation (or the transport equation) associated to the same vector field.

In the following I plan to explain the connections between these problems (N-property, Sard Theorem, uniqueness for the flow and for the continuity equation associated to a divergence-free vector field), and then illustrate some of our results at least in simple cases, giving when possible an outline of the proof. In writing this note I tried to improve readability at the expenses of precision by omitting most technical details. I hope that nevertheless these pages will convey some meaning.

[1] This paper orignates from a lecture that the I gave at the Conference "Geometric Function Theory", which took place at the Accademia dei Lincei on November 3rd, 2011.

Let me finally add that similar results on the N-property and the Sard theorem for Sobolev maps have been obtained by J. Bourgain, M.V. Korobkov, and J. Kristensen [8] at about the same time as us (but with different motivations in the background).

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2. Uniqueness for the continuity equation

Let us consider the continuity equation

$$u_t + \operatorname{div}(bu) = 0 \tag{pde}$$

where b is a vector field on $\mathbb{R}^n$ and the unknown u is a scalar function on $[0,T)\times\mathbb{R}^n$ subject to the initial condition $u(0,\cdot) = u_0$, with u_0 a given initial datum.

To understand what follows it is convenient to keep in mind the standard mechanical interpretation of (pde): consider a continuous distribution of point particles in $\mathbb{R}^n$ such that the trajectory $x = x(t)$ of each particle satisfies the ordinary differential equation

$$\dot{x} = b(x)\,, \tag{ode}$$

and let $u = u(t,x)$ be the corresponding density—that is, mass per unit volume at time t and position x. Then u satisfies (pde).

This interpretation suggests that existence and uniqueness of solutions of the Cauchy problem for (pde) are strictly related to existence and uniqueness for the Cauchy problem for (ode).

2.1. Existence. Assume for the time being that b is bounded and smooth. Under these assumptions we can construct the flow associated to (ode), namely the one-parameter family of diffeomorphisms of $\mathbb{R}^n$

$$\{\Phi_t\}_{t\geq 0}$$

defined by the fact that for every $x \in \mathbb{R}^n$ the map $t \mapsto \Phi_t(x)$ solves the equation (ode) with initial value $\Phi_0(x) = x$.

If b is divergence-free then the flow is volume-preserving (that is, each diffeomorphism Φ_t is volume-preserving), and therefore a solution of (pde) with initial datum u_0 is [2]

$$u(t,x) := u_0\big(\Phi_t^{-1}(x)\big)\,. \tag{2.1}$$

[2] The heuristic idea behind formula (2.1) is clear: if $B = B(x,r)$ is a ball centered at x with small radius r, the density $u(t,x)$ is (up to a small error) the mass $m(B,t)$ of the particles contained in B at time t divided by the volume of B. But the particles contained in B at time t are those contained in $B' := \Phi_t^{-1}(B)$ at time 0, and therefore $m(B,t) = m(B',0)$, while the volume of B is the same as that of B' because Φ_t is volume-preserving. Hence $u(t,x)$ agrees with $m(B',0)$ divided by the volume of B', which is the density at time 0 and position $\Phi_t^{-1}(x)$.

It follows immediately that if u_0 is bounded then

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty \quad \text{for all } t. \tag{2.2}$$

Assume now that the vector field b is bounded, divergence-free (in the sense of distribution) but no longer smooth. We construct a solution of (pde) with initial datum u_0 as follows: let b_ε be a regularization of b by convolution (so b_ε is bounded, divergence-free, and *smooth*), and let u_ε be the solution of (pde) with b_ε in place of b given by formula (2.1); then we can use the bound (2.2) to pass to the limit in u_ε as $\varepsilon \to 0$, and obtain bounded function u that solves (pde) for all positive times (in the sense of distribution).

To make this argument work it is not needed that $\operatorname{div} b = 0$, but it suffices that $\operatorname{div} b \ge -m$ for some finite m; in this case (2.1) should be replaced by

$$u(t,x) := u_0\big(\Phi_t^{-1}(x)\big) \cdot \det\big(\nabla\Phi_t^{-1}(x)\big)\,,$$

and since the derivative of $\det(\nabla\Phi_t(x))$ with respect to the variable t agrees with $\operatorname{div} b(x)$, which is larger than $-m$, then the bound (2.2) becomes

$$\|u(t,\cdot)\|_\infty \le e^{mt}\|u_0\|_\infty \quad \text{for all } t.$$

Note that without assumptions on the divergence of b the existence of bounded solutions *for all times* may no longer hold, because it can happen that all particles end up in the same point and remain there; therefore after some time the particle density becomes a measure with an atom, and is no longer represented by a function (let alone a bounded function). For example, this is the case when

$$b(x) := \begin{cases} -x/\sqrt{|x|} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2.2. Uniqueness. Under the only assumption that b is bounded and has bounded (or even vanishing) divergence there is in general no uniqueness for the Cauchy problem for the continuity equation (pde).[3] However, in the fundamental paper [12], R.J. DiPerna and P.-L. Lions proved that uniqueness holds under the additional assumption that b is (locally) of Sobolev class $W^{1,1}$, and later on L. Ambrosio [5] improved this result by showing that it suffices that b is (locally) of class BV.[4]

Note that in both papers uniqueness is proved within the class of distributional solutions of (pde) that are functions for all times (actually some additional bound on the solution u is also needed, for example $\|u(t,\cdot)\|_\infty$ uniformly bounded in t for all finite time-intervals). In other words, the possibility that particles concentrate in a negligible set [5] is excluded a priori, and not proved impossible.

[3] Among the existing examples we mention the one in [11]: a time-dependent, bounded, divergence-free vector field b on $\mathbb{R}^2$ such that (pde) admits a nontrivial (distributional) solution with vanishing initial datum. It is easy to modify this construction and obtain an example of non-uniqueness for a time-independent, divergence-free vector field on $\mathbb{R}^3$. An example of time-independent vector field on $\mathbb{R}^2$, quite more complicated, is given in [1].

[4] The relevance of these uniqueness results lies in the applications to other hyperbolic problems, which I am not going to discuss here.

[5] Here and in the following the terms “negligible” and “measure” refer, unless stated otherwise, to the Lebesgue measure on the ambient space.

It should also be noted that both results give conditions which are sufficient for uniqueness, but not necessary (cf. §2.4).

In view of the mechanical interpretation described above, one would expect that uniqueness for (pde) is related to uniqueness for (ode), and the heuristic argument should be the following: let N be the *set of non-uniqueness* associated to b, that is, the set of all points $z \in \mathbb{R}^n$ such that the differential equation (ode) has at least two solutions $x_z(t)$ and $\tilde{x}_z(t)$ with initial datum z. Consider now an initial distribution of particles contained in N: there are at least two possible evolutions of this distribution, one obtained by moving each particle initially located at the point z according to the trajectory $x_z(t)$, and the other one obtained by moving it according to $\tilde{x}_z(t)$. We thus expect that the densities u and $\tilde{u}$ associated to these two evolutions give different solutions of (pde) with the same initial datum.

Now, this would certainly be the case if our notion of solution included measure-valued solutions, that is, if we allowed the particle density at time t to be represented by a measure instead of a function. But since by solutions we mean functions, and sometimes even bounded functions, we quickly realize that to make the previous constructions work we need some additional assumptions.

First of all we need an initial distribution of particles with positive total mass whose density is a function and not a measure, and therefore we must assume that the non-uniqueness set N has positive measure.

Secondly, we need that at every time $t > 0$ the densities of the two distributions considered above are functions and not measures, which is obtained by assuming that the families of trajectories $\{x_z\}$ and $\{\tilde{x}_z\}$ do not "concentrate", where non-concentration (for $\{x_z\}$) means that for every set E with positive measure contained in N and every $t > 0$, the set $E_t := \{x_z(t) : z \in E\}$ has positive measure. (This is the weakest notion of non-concentration: to makes sure that the solutions u and $\tilde{u}$ constructed above are bounded functions, and not just functions, one has to impose some explicit lower bound for the measure of E_t, such as $\operatorname{meas}(E_t) \geq m \operatorname{meas}(E)$ for some positive constant m.)

The argument I have just presented has been made rigorous by Ambrosio in [5] using a suitable weak notion of flow (compare it with the classical one in §2.1): a *regular Lagrangian flow* associated to a vector field b on $\mathbb{R}^n$ is a family of maps $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ parametrized by time t such that

(i) $t \mapsto \Phi_t(x)$ solves (ode) for *almost every* $x \in \mathbb{R}^n$,
(ii) there exists a positive constant m such that $\operatorname{meas}(\Phi_t(E)) \geq m \operatorname{meas}(E)$ for every set E and every time t (non-concentration).

Two Lagrangian flows are said to be equivalent if they agree for almost every x and every t, and, as shown in [5], the existence of two non-equivalent regular Lagrangian flows implies non-uniqueness of bounded solutions for (pde). In particular, the uniqueness result for (pde) in [12] and [5] imply the uniqueness of regular Lagrangian flows up to equivalence.

For more details on the connection between (pde) and flows for (ode), and for a review of related uniqueness results I refer the reader to [9], [6].

2.3. An open question. The uniqueness of regular Lagrangian flows (up to equivalence) can be loosely interpreted as uniqueness for (ode) for *almost every* initial position. However, these two conditions are not equivalent: while the latter clearly implies the former (because of assumption (i) in the definition of regular Lagrangian flow), the converse is not true (essentially because for certain vector fields b there exist flows that satisfy condition (i) but not (ii)).

In particular, it is not know whether the uniqueness results for (pde) in [12] and [5] imply uniqueness for (ode) for almost every initial position.

We are thus led to the following question, which is still open: *Is there a continuous vector field b on $\mathbb{R}^n$ with bounded divergence and of class $W^{1,p}$ for some $p \geq 1$ (that is, a vector field to which the uniqueness result in* [12] *applies) such that the non-uniqueness set N has positive measure?*

2.4. Relation with Sard theorem. In this paragraph we restrict our attention to vector fields b on $\mathbb{R}^2$ that are bounded and divergence-free. Under these assumptions there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$, called *potential* of b, such that

$$b = (\nabla f)^{\perp} \tag{2.3}$$

where $v^{\perp}$ stands for the rotation of the vector v by ninety degrees counterclockwise (f exists because the rotation of b by ninety degrees clockwise is curl-free).

In [1, Theorem 4.7] it is proved that that the vector fields b such that there is uniqueness for the corresponding continuity equation (pde) can be characterized in terms of the critical set of the potential f.

Let me give an idea of the proof. In view of the mechanical interpretation of (pde) given at the beginning of this section, we can rephrase the first step of this proof as follows: a particle that belongs to some level set $f^{-1}(y)$ at time 0, remains for all subsequent times in the same level set, and more precisely in the same connected component of the same level sets. This is not surprising because b is orthogonal to ∇f and therefore tangent to the level sets of f at almost every point.[6]

It follows that solving (pde) is equivalent to solve a partial differential equation similar to (pde) on every nontrivial connected component E of a generic level set $f^{-1}(y)$ (here "nontrivial" means "containing more than one point"; "generic" means "for almost every y").

Moreover a nontrivial connected component E of a generic level set is a simple rectifiable curve (see [2, Theorem 2.5]) and therefore uniqueness for (pde) reduces to uniqueness for a family of variants of the continuity equation in one space dimension. It turns out that uniqueness for these one-dimensional continuity equations is strictly related to the intersection of the connected component E and the set of critical points

$$S := \{x : \nabla f(x) = 0\} .$$

[6] In other words, the level sets of f play the role of characteristic curves for (pde).

In particular, if a generic level set of f does not contains critical points (that is, if f has the Sard property—see Section 3) then there is uniqueness for all these one-dimensional equations, and therefore also for the original two-dimensional equation (pde).[7]

In the rest of this paragraph I follow this line of thought and claim that a negative answer to the question raised at the end of §2.3 could be given by a suitable example of Sobolev function without the Sard property.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function of class $W^{2,p}$ and with compact support, and let V be the set of all values $y \in \mathbb{R}$ such that there exists a nontrivial connected component E_y of the level set $f^{-1}(y)$ which contains one and only one critical point of f, denoted by x_y. Finally let b be the vector field with potential f, that is, the one defined by (2.3), and let N be the non-uniqueness set associated to b (see §2.2).

I claim that *if the set V has positive measure then the set N has positive measure, and therefore the answer to the question in §2.3 is negative.*

Let me argue in favour of this claim. I first recall that for almost every $y \in \mathbb{R}$ the set E_y is a rectifiable, simple, closed curve, and I observe that

(i) a particle that moves along E_y reaches x_y in finite time;
(ii) after the particle has reached the critical point x_y it can stay there for any given amount of time and then start moving again.

Statement (ii) is essentially a consequence of statement (i) (applied with reversed time) and of the fact that b vanishes in x_y. To prove statement (i), note that the time T_y taken by the particle to go all the way through the curve E_y is

$$T_y = \int_{E_y} \frac{1}{|b|} = \int_{E_y} \frac{1}{|\nabla f|} \le \int_{f^{-1}(y)} \frac{1}{|\nabla f|} \,,$$

and therefore

$$\int_V T_y \, dy \le \int_{-\infty}^{+\infty} \left[\int_{f^{-1}(y)} \frac{1}{|\nabla f|} \right] dy \le \operatorname{meas}(\operatorname{supp}(f)) < +\infty$$

(the second inequality follows by the coarea formula and the fact that $f^{-1}(y)$ is contained in the support of f for all $y \neq 0$; the last inequality is due to the fact that the support of f is assumed to be compact, and therefore it has finite measure).

Hence T_y is finite for almost every $y \in V$, which implies statement (i).

Now notice that statements (i) and (ii) together imply that for every point z contained in E_y with $y \in V$ there are infinitely many solutions of (ode) with initial datum z, and therefore E_y is contained in the non-uniqueness set N of the vector field b. Finally, the coarea formula and the fact that V has positive measure imply that the union of all E_y with $y \in V$, and therefore also N, are sets of positive measure in the plane.

2.5. Conclusions. The fact that the set V in the previous construction has positive measure implies that the function f does not have the Sard property. When

[7] The uniqueness result in [1] actually requires that f satisfies a weaker version of the Sard property; the precise definition is a bit technical, and therefore it has been omitted.

we started working on these problems it was only known that Sard theorem holds for functions $f : \mathbb{R}^2 \to \mathbb{R}$ of class $W^{2,p}$ with $p > 2$ but nothing was known for $p \le 2$ (see the next section). So we looked for a counterexample, with the hope that it would eventually lead to a negative answer to the question raised in §2.3. Unfortunately (or fortunately) we found out in the end that there are no counterexamples, and that Sard theorem holds for all $p \ge 1$.

3. Sard theorem

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \le n$, the critical set of f is

$$S := \bigl\{x : \ \mathrm{rank}(\nabla f(x)) < m\bigr\}$$

We say that f has the Sard property if $f(S)$ is negligible, that is, if a generic level set of f contains no critical points.

In the classical form (see [18]), Sard theorem states that if f is of class C^{n-m+1} then it has the Sard property. Note that the regularity exponent $n-m+1$ is sharp: there exist maps of class C^{n-m} without the Sard property (see [19], [13, §3.4.4]).

A more precise version of Sard theorem was given in [13, Theorem 3.4.3]: given a map $f : \mathbb{R}^n \to \mathbb{R}^m$ of class C^k (without restrictions on n and m) and $h = 0, 1, \dots,$ then the set

$$S_h := \bigl\{x : \ \mathrm{rank}(\nabla f(x)) \le h\bigr\} . \tag{3.1}$$

is $\mathscr{H}^{h+(n-h)/k}$-negligible, where $\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure. This result was later extended in [7] to maps of class $C^{k,\alpha}$.

Concerning Sobolev maps, L. De Pascale proved in [10] that continuous maps of class $W^{n-m+1,p}$ with $p > n > m$ have the Sard property. A simpler proof of this statement was later given in [14]. Note that the counterexamples mentioned before show that the differentiability exponent $n - m + 1$ is sharp. On the other hand, there are no examples showing that the bound $p > n$ on the summability exponent is optimal (and indeed it is not, as I am going to explain).

In the rest of this section I restrict for simplicity to the case $n = 2$ and $m = 1$, that is, to functions f on $\mathbb{R}^2$ to $\mathbb{R}$. (For $n = m$ Sard theorem is just a consequence of the area formula, and therefore the "interesting" cases are those with $n > m$; among these the case $n = 2$ and $m = 1$ is the simplest, and is also the one which is relevant to the construction explained in §2.4.)

In this case the critical set S agrees with the set S_0 of all points where the gradient ∇f vanishes, and the result by De Pascale states that a continuous function in $W^{2,p}$ with $p > 2$ has the Sard property. Next I will give a detailed outline of the proof of this result, and then indicate how it can be extended to $W^{2,1}$.

3.1. Proof of Sard theorem for $p > 2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuos function of class $W^{2,p}$ for some $p > 2$; we assume for simplicity that the singular set S_0 ha finite measure.

The starting point is the following estimate: for every ball $B = B(x,r)$ with center x and radius r there holds

$$\operatorname{osc}(f,B) \lesssim r\,|\nabla f(x)| + r^2 \left(\fint_B |\nabla^2 f|^p \right)^{1/p}, \tag{3.2}$$

where $\operatorname{osc}(f,B)$ stands for the oscillation of f over the set B (that is, the difference between the supremum and the infimum), the symbol $\lesssim$ means that the inequality holds up to some (universal) multiplicative factor, and the dashed integral stands for the average.

Since estimate (3.2) is scaling and translation invariant, it suffices to prove it when $B = B(0,1)$. Since $W^{2,p}$ embeds in L^∞, we can bound the oscillation of f by its $W^{2,p}$-norm (on B). Now recall that an equivalent norm on $W^{2,p}$ is given by the sum of the L^p-norm of $\nabla^2 f$ and any continuous seminorm ϕ on $W^{2,p}$ which does not vanishes on nontrivial affine functions, for example $\phi(f) := |f(0)| + |\nabla f(0)|$ (the equivalence with the usual norm of $W^{2,p}$ follows by a standard argument, see [20, Chapter 4]). Thus

$$\operatorname{osc}(f,B) \lesssim |f(0)| + |\nabla f(0)| + \|\nabla^2 f\|_{L^p(B)}\,. \tag{3.3}$$

Moreover, since $\operatorname{osc}(f,B)$ is invariant under the addition of a constant to f, we can assume $f(0) = 0$ and drop the first addendum on the right-hand side of this inequality, and so we finally obtain (3.2).

Note that if x belongs to S_0 then $\nabla f(x) = 0$ and (3.2) becomes

$$\operatorname{osc}(f,B) \lesssim r^{2-2/p} \left(\int_B |\nabla^2 f|^p \right)^{1/p}. \tag{3.4}$$

We now choose an open set A that contains S_0, and cover S_0 with a collections of balls $B_i = B(x_i, r_i)$ such that $x_i \in S_0$ and $B_i \subset A$. Thus the sets $f(B_i)$ cover the set $f(S_0)$, and we can use this cover to estimate the measure of $f(S_0)$:

$$\operatorname{meas}(f(S_0)) \le \sum_i \operatorname{meas}(f(B_i))\,.$$

Since the measure of the set $f(B_i)$ is less than its diameter, which is $\operatorname{osc}(f, B_i)$, using (3.4) we get

$$\begin{aligned} \operatorname{meas}(f(S_0)) &\lesssim \sum_i r_i^{2-2/p} \left(\int_{B_i} |\nabla^2 f|^p \right)^{1/p} \\ &\le \left(\sum_i r_i^2 \right)^{1-1/p} \left(\sum_i \int_{B_i} |\nabla^2 f|^p \right)^{1/p} \\ &\lesssim \operatorname{meas}(A)^{1-1/p} \left(\int_A |\nabla^2 f|^p \right)^{1/p}, \end{aligned} \tag{3.5}$$

where the second inequality follows by applying Hölder inequality in the form $\sum a_i b_i \le (\sum a_i^q)^{1/q} (\sum b_i^p)^{1/p}$, and the third one holds provided that the balls B_i do not overlap too much—a property that can be obtained by the Besicovitch covering theorem.

To conclude the proof, note that we can choose the open set A so that meas(A) is arbitrarily close to meas(S_0), which is finite, while $\int_A |\nabla^2 f|^p$ is arbitrarily close to $\int_{S_0} |\nabla^2 f|^p$, which is null because $\nabla f = 0$ on S_0 implies $\nabla^2 f = 0$ a.e. on S_0.

3.2. Statement of Sard theorem for $p \le 2$. All versions of Sard theorem I mentioned so far apply to classes of maps that are differentiable at every point, and for which, consequently, the definition of critical set carries no ambiguity. However for $1 \le p \le 2$ the space $W^{2,p}(\mathbb{R}^2)$ embeds in C^0 but not in C^1, and therefore a function f in this space admits a continuous representative which in general is differentiable almost everywhere but not everywhere. Thus for such f we should consider two sets:

$$\begin{aligned} S_0 &:= \big\{ x : \ f \text{ is differentiable at } x \text{ and } \nabla f(x) = 0 \big\} \,, \\ N &:= \big\{ x : \ f \text{ is not differentiable at } x \big\} \,. \end{aligned} \tag{3.6}$$

It turns out that Sard theorem holds in the strongest form (see [4], [8]): *if f is a continuous function of class $W^{2,1}$ then $f(S_0 \cup N)$ is negligible.*

3.3. Outline of the proof. The only information readily available on the set N is that it cannot be too large, and more precisely $\mathscr{H}^1(N) = 0$.[8] Therefore we could obtain that $f(N)$ is negligible if we knew that for every set E in $\mathbb{R}^2$

$$\mathscr{H}^1(E) = 0 \ \Rightarrow \ \mathscr{H}^1(f(E)) = 0 \,. \tag{3.7}$$

This is exactly a particular case of the generalized N-property that I will discuss in the next section (a precise statement is contained in §4.1).

Let me now show how to adapt the proof in §3.1 to obtain that $f(S_0)$ is negligible, too. First of all, notice that this proof, as it is, does not work. The point is that we no longer have estimate (3.2), because for $p \le 2$ the space $W^{2,p}$ does not embeds in C^1, and therefore the value of ∇f at a given point x is not well-defined.

The idea is to replace the term $|\nabla f(x)|$ in (3.2) with

$$\int_B |\nabla f| \, d\mu_B$$

where μ_B is a probability measure supported on B that belongs to the dual of $W^{1,1}$, in the sense that $u \mapsto \int u \, d\mu_B$ is a well-defined bounded functional on $W^{1,1}$, and therefore $u \mapsto \int |u| \, d\mu_B$ is a well-defined continuous seminorm on $W^{1,1}$ (for more details on measures in the dual of $W^{1,1}$ see [20, Section 4.9]). Then we have the following variant of (3.2):[9]

$$\operatorname{osc}(f, B) \lesssim r \int_B |\nabla f| \, d\mu_B + r^2 \fint_B |\nabla^2 f| \,, \tag{3.8}$$

[8] It can be proved that f is differentiable at every point where the gradient ∇f admits an approximate limit (in the L^1-sense). Therefore N is contained in the set of points where this approximate limit does not exists, and since ∇f is of class $W^{1,1}$, this set is negligible with respect to $\mathscr{H}^1$ (see for instance [20, §5.12]).

[9] The proof runs exactly as that of (3.2) provided that the continuous seminorm ϕ used to prove (3.3) is replaced by $\phi(f) := |f(0)| + \int_B |\nabla f| \, d\mu_B$. One has to be careful though, since the constant in (3.8) is affected by the norm of μ_B as an element of the dual of $W^{1,1}$.

Let now S' be the set of all $x \in S_0$ with the following property: there exists a sequence of balls $B = B(x, r_i)$ with $r_i \to 0$ such that on each of these balls we can find a measure μ_B as above, *supported on* $S_0 \cap B$.[10]

With this choice of μ_B the first integral at the right-hand side of (3.8) vanishes, and therefore we get once again estimate (3.4) (with $p = 1$). We can now repeat the rest of the proof in §3.1 as it is, and obtain that $f(S')$ is negligible.

It remains to show that $f(S_0 \setminus S')$ is negligible. We obtain this using (3.7) and

$$\mathscr{H}^1(S_0 \setminus S') = 0\,. \tag{3.9}$$

To prove (3.9), we first need to understand when a point x belongs to S', which in turn implies understanding when the set $S_0 \cap B(x,r)$ can support a probability measure μ_B in the dual of $W^{1,1}$ and how small the dual norm of this measure can be (cf. footnote 10).

So, when does a set E in $\mathbb{R}^2$ support a probability measure μ in the dual of $W^{1,1}$? Intuitively, a *necessary* condition should be that the set E has positive $W^{1,1}$-capacity, or, equivalently, that $\mathscr{H}^1(E) > 0$. It turns out that a *sufficient* condition is that $\mathscr{H}^1_\infty(E) > 0$, where $\mathscr{H}^1_\varepsilon$ are the Hausdorff pre-measures that appear in the definition of Hausdorff measures (see [20, §1.4.1]).[11]

Using this sufficient condition we obtain that x belongs to S' if

$$\limsup_{r\to 0} \frac{\mathscr{H}^1_\infty(S_0 \cap B(x,r))}{r} \geq 1/2\,, \tag{3.10}$$

and therefore for all $x \in S_0 \setminus S'$ the limsup in (3.10) is necessarily strictly smaller than 1, which implies that

$$\limsup_{r\to 0} \frac{\mathscr{H}^1_\infty((S_0 \setminus S') \cap B(x,r))}{r} < 1\,. \tag{3.11}$$

The last step of the proof consists in showing that (3.11) implies (3.9).

3.4. The general case. In [4] we prove the following (but as I said, this is still a work in progress): *Take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ embeds in C^0 (that is, $kp > n$ or $p = 1$ and $k = n$), let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map of class $W^{k,p}$, and define the sets S_0 and N as in* (3.6). *Then*

$$\mathscr{H}^{n/k}(f(S_0 \cup N)) = 0\,. \tag{3.12}$$

Moreover this result is optimal, in the sense that

(i) *the dimension n/k in* (3.12) *cannot be lowered;*

(ii) *if n, k, and p do not satisfy the condition above, then there are maps f on $\mathbb{R}^n$ of class $W^{k,p} \cap C^{k-1}$ for which the Hausdorff dimension of $f(S_0)$ is strictly larger than n/k, and in particular* (3.12) *fails.*

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[10] To be precise, I also require that the norms of these measures as elements of the dual of $W^{1,1}(B)$ are suitably controlled.

[11] This sufficient condition can be obtained by putting together the characterization of measures in the dual of $W^{1,1}$ given in [20, Theorem 4.9.4] and Frostman's lemma [17, Theorem 8.8]. Moreover the dual norm of the measure μ is controlled by the inverse of $\mathscr{H}^1_\infty(E)$ (the smaller the set, the bigger the norm).

To obtain the optimal statement of Sard theorem we should then prove similar estimates for the sets S_h defined in (3.1).

4. Generalized N-property

A map $f:\mathbb{R}^n\to\mathbb{R}^m$ with $m\ge n$ has the Lusin N-property if the following implication holds for every set E contained in $\mathbb{R}^n$:

$$\mathscr{H}^n(E)=0 \;\Rightarrow\; \mathscr{H}^n(f(E))=0\,.$$

This property has been widely studied in the past years, mostly in relation to the area formula. Indeed, the following statement holds: let f be a map which is differentiable (in the approximate sense) at almost every point and has the N-property; then the area formula holds, that is

$$\int_{y\in\mathbb{R}^m}\Bigg[\sum_{x\in f^{-1}(y)\cap E}\varphi(x)\Bigg]\,d\mathscr{H}^n(y)=\int_{x\in E}\varphi(x)\,J_f(x)\,d\mathscr{H}^n(x) \tag{4.1}$$

where φ is any positive Borel function on $\mathbb{R}^n$, E is any Borel subset of $\mathbb{R}^n$, and J_f is the Jacobian of f (defined at every point where f is differentiable).

The proof of this statement is elementary: since f is a.e. differentiable, it has the Lusin approximation property with Lipschitz maps, that is, there exist a sequence of Borel sets F_i and of Lipschitz maps f_i such that the sets F_i cover almost all of $\mathbb{R}^n$ and $f=f_i$ on F_i (see [13, Theorem 3.1.8]). Using the area formula for Lipschitz maps (see [13, Theorem 3.2.5]) we obtain that (4.1) holds when E is contained in the union of all F_i. It remains to show that (4.1) holds when E is contained in the complement of the union of all F_i. Since E is $\mathscr{H}^n$-negligible, the integral at right-hand side of (4.1) vanishes, and to prove that also the integral at the left-hand side vanishes it suffices to show that $f(E)$ is $\mathscr{H}^n$-negligible, which is precisely what the N-property says.

Concerning Sobolev maps, a continuous map $f:\mathbb{R}^n\to\mathbb{R}^m$ of class $W^{1,p}$ has the N-property if $p>n$ (see [16]) and this bound on the summability exponent is sharp (however, *homeomorphisms* of class $W^{1,n}$ also have the N-property; for this and other results on the N-property see for instance the review paper [15]).

In the rest of this section I will focus on a generalization of the N-property that naturally arises when dealing with the Sard theorem for Sobolev maps (see §3.3).

4.1. Generalized N-property. Given a map f between metric spaces and positive numbers α,β, we say that f has the (α,β)-N-property if the following implication holds for every set E contained in the domain of f:

$$\mathscr{H}^\alpha(E)=0 \;\Rightarrow\; \mathscr{H}^\beta(f(E))=0\,.$$

It follows from elementary facts that a Lipschitz map has the (α,α)-N-property for every $\alpha>0$ and, more generally, an Hölder map with exponent γ has the $(\alpha,\alpha/\gamma)$-N-property for every $\alpha>0$.

Concerning Sobolev maps, in [3] we prove the following: *Take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ embeds in C^0 (that is, $kp > n$ or $p = 1$ and $k = n$), and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map of class $W^{k,p}$. Then*

(i) *f has the (α, β)-N-property with $\beta := \frac{\alpha p}{kp+\alpha-n}$ for $\alpha < n - (k-1)p$;*
(ii) *f has the (α, α)-N-property for $\alpha > n - (k-1)p$.*[12]

Moreover this result is sharp, in the sense that

(iii) *the value of β in (i) cannot be lowered;*
(iv) *if we take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ does not embed in C^0, then there are continuous maps $f : \mathbb{R}^n \to \mathbb{R}^m$ of class $W^{k,p}$ that do not have the (α, β)-N-property for any $\alpha > 0$ and $\beta \le m$; in other words, these maps take some sets of dimension arbitrarily close to 0 into sets of dimension m.*

4.2. About the proof. We have two different methods for proving statements (i) and (ii) above. Even though the proof can be achieved by either methods for most k, p, α, β in the range where the N-property holds, yet neither approach covers all cases (or so it seems).

Let me illustrate the first method in the case of the $(1,1)$-N-property for maps $f : \mathbb{R}^2 \to \mathbb{R}^m$ of class $W^{2,1}$. The starting point is the following estimate (the proof is essentially the same as that of estimates (3.2) and (3.8)): for every ball $B = B(x, r)$ there holds

$$\operatorname{osc}(f, B) \lesssim r \fint_B |\nabla f| + r^2 \fint_B |\nabla^2 f| \,. \tag{4.2}$$

We now fix a set E with $\mathscr{H}^1(E) = 0$ and, given $\varepsilon > 0$, we choose a family of balls $B_i = B(x_i, r_i)$ which cover E and satisfy $\sum r_i \le \varepsilon$. Then the sets $f(B_i)$ cover $f(E)$, and we use this cover to estimate the Hausdorff measure of $f(E)$:

$$\mathscr{H}^1(f(E)) \le \sum_i \operatorname{diam}(f(B_i)) \,.$$

Since the diameter of $f(B_i)$ agrees with the oscillation of f on B_i, using (4.2) we obtain

$$\mathscr{H}^1(f(E)) \lesssim \sum_i \frac{1}{r_i} \int_{B_i} |\nabla f| + \sum_i \int_{B_i} |\nabla^2 f| \,. \tag{4.3}$$

We want to show that both sums at the right-hand side of (4.3) tends to 0 as ε tend to 0 (provided the covers $\{B_i\}$ are suitably chosen).

If the balls B_i do not overlap too much (and this can be obtained by Besicovitch covering lemma) we can estimate the second sum by the integral of $|\nabla^2 f|$ over the union A of the balls B_i, and since the area of A tends to 0 as $\varepsilon \to 0$, the same happens to the integral.

[12] The case $\alpha = n - (k-1)p$ is not yet settled, except for $p = 1$ where we know that the (α, α)-N-property holds.

The difficult part is to handle the first sum. First of all we write it as $\int |\nabla f|\, d\mu$ where μ is given by the Lebesgue measure multiplied by the density

$$\rho := \sum_i \frac{1}{r_i} 1_{B_i} ,$$

and then we show that μ belongs to the dual of $W^{1,1}(\mathbb{R}^2)$ in the sense of [20, §4.9] (the key step is to prove that $\mu(B) \lesssim r$ for every ball $B = B(x,r)$). Then the proof is concluded by a careful estimate of the norm of this measure as element of the dual of $W^{1,1}(\mathbb{R}^2)$.

Concerning the second method, let me just hint that it is related to estimates for the *local Hölder exponent* of Sobolev maps. The simplest version of such estimates reads as follows: if α is a real number with $0 < \alpha \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous map of class $W^{1,p}$ with $p > n$, then for $\mathscr{H}^\alpha$-almost every $x \in \mathbb{R}^n$ and every ball $B = B(x,r)$ there holds

$$\operatorname{osc}(f,B) \lesssim r \left(\fint_B |\nabla f|^p \right)^{1/p} = O(r^\gamma) \quad \text{with } \gamma := \frac{p+\alpha-n}{p} . \tag{4.4}$$

The inequality in (4.4) can be proved in the same way as estimate (3.2), and the equality is obtained by applying the following elementary statement with $g := |\nabla f|^p$: given a positive function g in $L^1(\mathbb{R}^n)$ and $0 < \alpha \leq n$, for $\mathscr{H}^\alpha$-almost every $x \in \mathbb{R}^n$ and every ball $B = B(x,r)$ there holds

$$\int_B g = O(r^\alpha)$$

(the estimate applies in the regime $r \to 0$, and it is clearly not uniform in x).

Now, estimate (4.4) says more or less that we can find a sequence of sets such that the restriction of f to each of these sets is Hölder continuous of exponent γ, and the sets cover $\mathbb{R}^n$ except for a residual set which is $\mathscr{H}^\alpha$-negligible. If we neglect this residual set, we immediately obtain that f has the $(\alpha, \alpha/\gamma)$-N-property, and α/γ is exactly the value of β in statement (i) of §4.1 for $k = 1$.

Unfortunately we cannot neglect the residual set, and turning this argument into a real proof requires some work.

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