Real Analysis, Geometric Measure Theory, PDE and Banach Spaces.

University of Warwick
August 17-19, 2007

Giovanni Alberti (PISA)

Structure of null sets in Euclidean space: some results and open problems, I

joint work with
DAVID PREISS and
MARIANNA CSÖRNYEI

PLAN OF THE TALK

- Review of original motivations
- A covering theorem for null sets in the plane
- Applications → Differentiability of Lipschitz functions
  → Tangent field to null sets
  → Laczkovich's problem
- Proof of the covering theorem
- Open problems (extension to higher dimension)

References

G. Alberti, M. Csörnyei, D. Preiss
Structure of null sets in the plane and applications
Proceedings of IV ECM (Stockholm 2004)
European Math. Soc. 2005

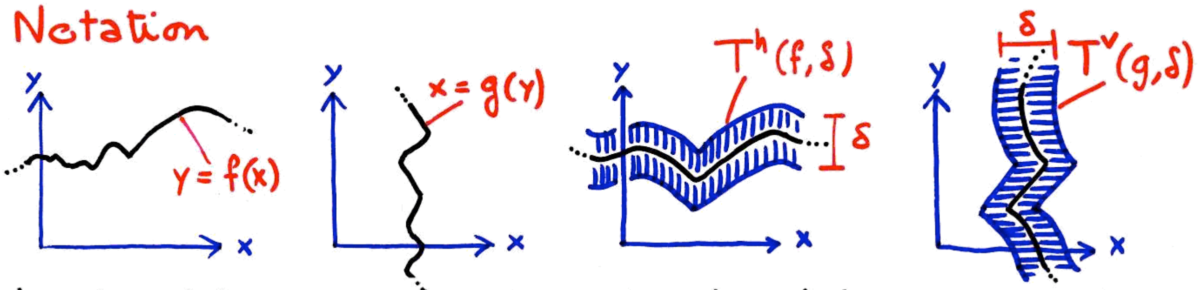
G. Alberti, M. Csörnyei, D. Preiss
Paper in preparation....

1. Motivations

- Differentiability of Lipschitz functions
- Rank-one property of BV functions (and the structure of normal currents)
- Laczkovich's problem

2. A covering theorem for null sets

Notation



horizontal graph | vertical graph | horizontal strip | vertical strip

i.e. graph of a function $y = f(x)$ with Lipschitz constant $\mathrm{Lip}(f) \le 1$

$\inf \{ L : |f(x) - f(y)| \le L |x-y| \ \forall x, y \}$

$T^h(f, \delta) := \{ |y - f(x)| \le \frac{\delta}{2} \}$

$\mathrm{Lip}(f) \le 1$

δ: thickness of the strip

Theorem.

i.e. $\mathscr{L}^2(E) = 0$

Let E be a (compact) null set in $\mathbb{R}^2$.
Then E can be decomposed as $E = E^h \cup E^v$ so that:

(i) $\forall \varepsilon > 0$, E^h can be covered by horiz. strips $T^h_i = T^h(f_i, \delta_i)$ s.t. $\boxed{\sum \delta_i \le \varepsilon}$;

(ii) $\forall \varepsilon > 0$, E^v can be covered by vertical strips $T^v_j = T^v(g_j, \eta_j)$ s.t. $\boxed{\sum \eta_j \le \varepsilon}$.

Remark. This statement can be viewed as a refinement of Fubini's theorem.

3. DIFFERENTIABILITY OF LIPSCHITZ MAPS

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map.

By Rademacher theorem, f is differentiabe a.e. w.r.t. $\mathscr{L}^n$

Is this theorem sharp?

Question (strong version)

Given E null set in $\mathbb{R}^n$ ($\mathscr{L}^n(E)=0$) is there a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is not differentiable at any $x \in E$?

Question (weak version)

Given a positive singular measure μ on $\mathbb{R}^n$ is there a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is not differentiable μ-a.e.?

Remarks

(i) A positive answer to QS implies positive ans. to QW.

(ii) For n fixed, the answer to QS may depend on m.

(iii) For n fixed, the answer to QW does not depend on m.

(iv) What about the distance function

$$f(x) := \text{dist}\,(x, E) \quad ?$$

assume E compact

Note that f is not diff. at $x \in E$ iff x is a porosity point of E ($\exists x_n \to x$, $r_n \to 0$ s.t. $B(x_n, r_n) \cap E = \emptyset$ and $|x_n - x| = O(r_n)$).

The problem is, there exist compact null sets which are not porous at any point.

Even worse, there are singular measures μ s.t. $\mu(E) = 0$ for every porous set E.

Answers (so far...)

$n=1$: the answer to QS (and QW) is positive.

Classical construction.

Cfr. Z. Zahorski, Bull. Soc. Math. France 74 (1946)

$n=2$, $m=1$: the answer to QS may be negative.

D. Preiss, J. Funct. Anal. 91 (1990).

$n=2$, m any : the answer to QS (and QW) is positive.

A.-C.-P.

$n>2$: very little is known...

A construction for $n=1$

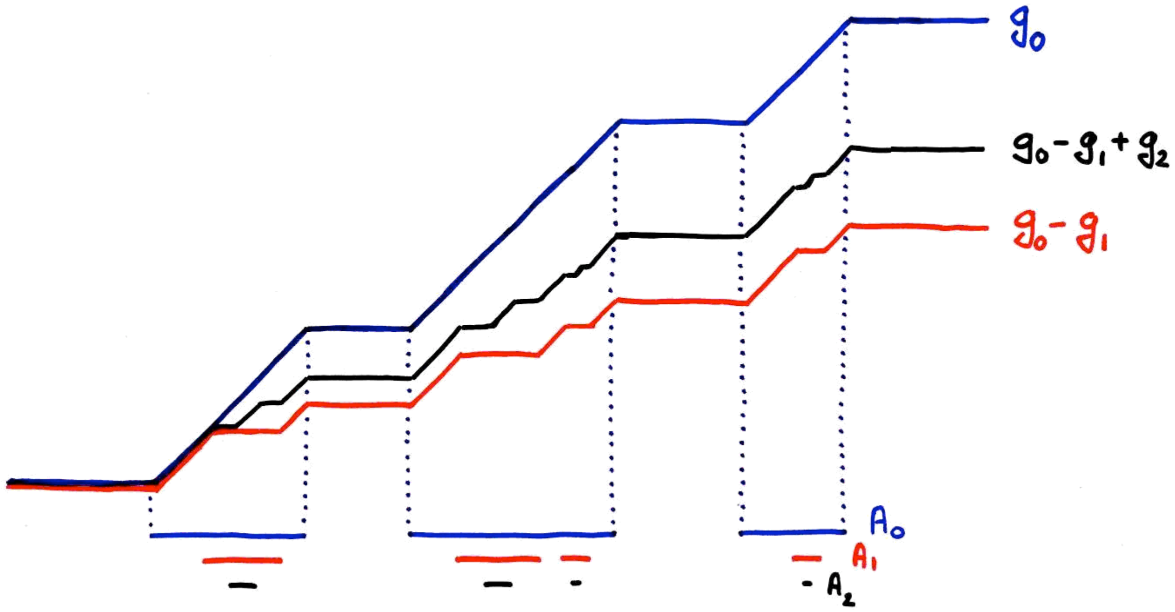
We assume that E is compact.

Since $\mathscr{L}^1(E)=0$, there exist open sets A_n s.t.

- $A_n \searrow E$
- $\mathscr{L}^1(A_n) \le 2^{-n} \mathscr{L}^1(I) \quad \forall I$ conn. comp. of A_{n-1}

For every n, take g_n s.t. $\boxed{g_n' = 1_{A_n}}$
and set

$$f(x) := \sum_{n=0}^{\infty} (-1)^n g_n(x)$$



And in dimension two?

- covering E with discs does not really work...
- instead we cover E with thin strips T_i^h, T_j^v
- for each horizontal strip T_i^h we consider as building block the function g_i^h defined by

$$\frac{\partial g_i}{\partial y} = 1_{T_i^h}$$

- and then....

Tangent field to a null set

Definition

Given a set E in $\mathbb{R}^2$,
given a map $\tau : E \to \{\text{lines in } \mathbb{R}^2\} = G(2,1)$,
we say that τ is tangent to E if
for every curve C of class C^1 in $\mathbb{R}^2$ there holds

$$\tau(x) = \mathrm{Tan}(C,x) \quad \text{for } \underbrace{\text{a.e.}}_{\substack{\text{w.r.t. } \mathcal{H}^1 \\ \text{(length)}}} \ x \in C \cap E$$

Remarks

(i) If E is a $\mathcal{C}^1$ curve then there exists a tangent field τ and $\tau(x) = \mathrm{Tan}(E,x)$ for a.e. x.

(ii) If E is purely unrectifiable (i.e. $\mathcal{H}^1(E \cap \mathcal{C}) = 0$ for every curve $\mathcal{C}$ of class $\mathcal{C}^1$) then every τ is tangent.

(iii) The tangent field τ τ_E (if any exists) is unique up to a purely unrectifiable subset of E.

(iv) If $\mathcal{L}^2(E) > 0$, then no τ is tangent.

Theorem

If $\mathcal{L}^2(E) = 0$, then there exists a tangent field τ.

Auxiliary definition. $E \subset \mathbb{R}^2$, $\mathcal{C} : E \to \{\text{cones in } \mathbb{R}^2 \text{ with center at } 0\}$

Then $\mathcal{C}$ is tangent to E if for every curve C there holds

$$\mathrm{Tan}(C,x) \subset \mathcal{C}(x) \quad \text{for a.e. } x \in C \cap E.$$

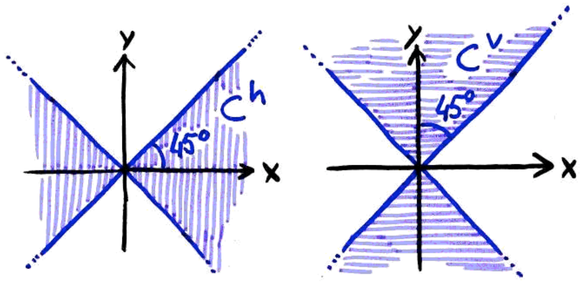
Proof of the theorem.

Step 1

Decompose E as $E^h \cup E^v$

The cone-field $\mathcal{C}(x) := C^h \ \forall x$ is tangent to E^h.

The cone field $\mathcal{C}(x) := C^v \ \forall x$ is tangent to E^v.



Step 2

The cone-field $\mathcal{C}(x) := \begin{cases} C^h & \text{if } x \in E^h \\ C^v & \text{if } x \in E^v \setminus E^h \end{cases}$ is tangent to E.

Step 3

For every angle θ rotate the axes by θ, repeat the construction of Step 2 and obtain another tangent cone-field $\mathcal{C}_\theta(x)$.

Step 4

Set

$$\tau(x) := \bigcap_{\theta \text{ rational}} \mathcal{C}_\theta(x)$$

Then τ is a tangent cone-field and $\tau(x)$ is (contained in) a line for every x.

An application (Rank-one property of BV maps)

Let $\mathcal{F} := \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \mathrm{Lip}(f) < 1 \}$

For every $f \in \mathcal{F}$, let $\mu_f^h := \mathcal{H}^1 \llcorner$ graph of f. $\{y = f(x)\}$

Given P_h probability measure on $\mathcal{F}$ let

$$\mu^h := \int_{\mathcal{F}} \mu_f^h \, dP(f) \qquad \mu(E) := \int_{\mathcal{F}} \mu_f(E) \, dP(f)$$

Let μ^v be constructed in the same way and then 'rotated' by $90°$.

Proposition.

If $\lambda \ll \mu^h$ and $\lambda \ll \mu^v$ then $\lambda \ll \mathcal{L}^2$.

Remark.

This statement is equivalent to the so-called rank-one property for maps with bounded variations

Proof.

Take E s.t. $\mathcal{L}^2(E) = 0$. We claim that $\lambda(E) = 0$.

Indeed, let $E_h := \{ x \in E : \tau_E(x) \in C^h \}$,

$E_v := \{ x \in E : \tau_E(x) \in C^v \}$.

By the definition of τ_E, $\mu_f(E_v) = 0 \quad \forall f \in \mathcal{F}^h$.

Then $\mu^h(E_v) = 0$.

Then $\lambda(E_v) = 0$.

Similarly $\lambda(E_h) = 0$.

Then $\lambda(E) = \lambda(E_h \cup E_v) = 0$.

5. Laczkovich's Problem

Question.

Let E be a set with positive measure in $\mathbb{R}^n$.
Is there a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t.
$f(E)$ contains a ball?

Answers (so far...)

$n=1$: The answer is positive. (And easy!)

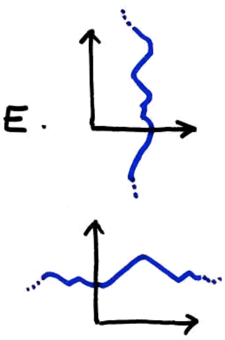
$n=2$: The answer is positive. (But not so easy!)

D. Preiss. Unpublished.
J. Matoušek. Algorithms Combin. 14 (1997).

$n \geq 3$: Nothing is known.

6. Proof of the Covering Theorem

Proposition (Dillworth's lemma)

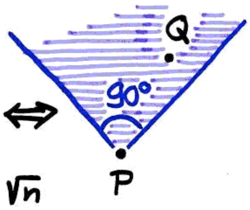


Let E be a finite set in $\mathbb{R}^2$, $n := \#E$.
Then E can be covered by $\sqrt{n}$ vertical graphs and $\sqrt{n}$ horizontal graphs.

Remark.

This is a geometric version of Erdős–Szekeres theorem on monotone subsequences.

Proof of Dillworth's lemma



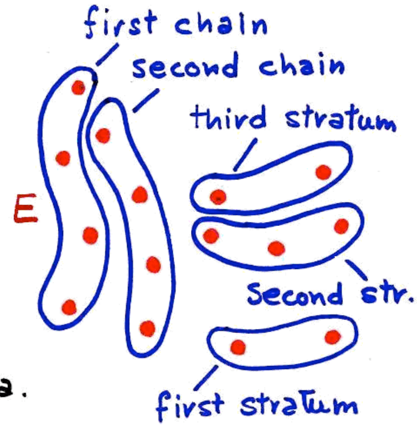
STEP 1. On E consider the partial order $P \preccurlyeq Q \Leftrightarrow$

STEP 2. Remove from E chains with at least $\sqrt{n}$ points as long as it is possible

STEP 3. Remove from what is left the first stratum (set of minima), then the second, etc.

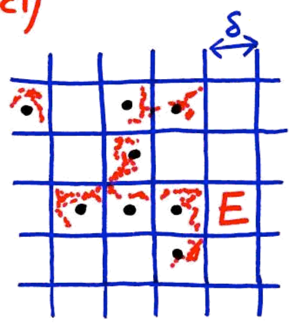
STEP 4. Each chain is contained in a vertical graph. There are at most $\sqrt{n}$ chains.

STEP 5. Each stratum is contained in an horizontal graph. There are at most $\sqrt{n}$ strata.



Proof of the covering theorem (for E compact)

STEP 1. Fix $\delta > 0$. Choose a square grid with size δ. Let E_δ be the set of the centers of the squares that intersect E.



Then $\# E_\delta = O(1/\delta^2)$.

STEP 2. Cover E_δ by $\sqrt{\# E_\delta} = O(1/\delta)$ horiz. graphs f_i and $O(1/\delta)$ vertical graphs g_j using Dillworth's lemma.

STEP 3. Cover E by the strips $T^h(f_i, 2\delta)$ and $T^v(g_j, 2\delta)$. Then

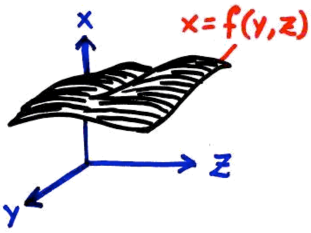
$$\sum_i \delta_i = 2\delta \cdot O(1/\delta) = O(1)$$

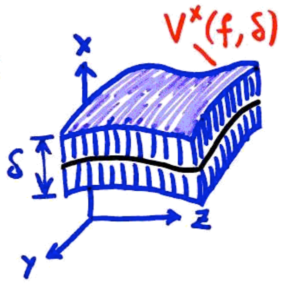
Choose δ so that $O(1) \le \varepsilon$.

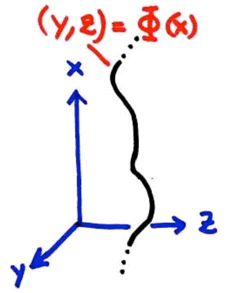
STEP 4. And to conclude....

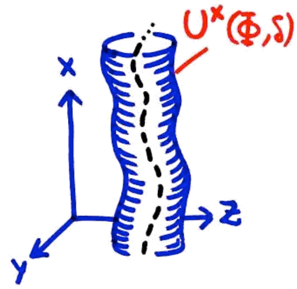
7. Extension to higher dimension ($n=3$)

Notation









Surface of type x and Lip. const. L	Slab of type x and Lip. const. L	Curve of type x and Lip. const. L	Tube of type x and Lip. const. L
graph of $x = f(y,z)$ with $\mathrm{Lip}(f) \le L$	$V^x(f,\delta)$ $\shortparallel$ $\{ \lvert x - f(y,z) \rvert \le \frac{\delta}{2} \}$	graph of the map $(y,z) = \Phi(x)$ with $\mathrm{Lip}(\Phi) \le L$	$U^x(\phi,\delta)$ $\shortparallel$ $\{ \lvert (y,z) - \Phi(x) \rvert \le \frac{\delta}{2} \}$

Similar definitions for surface, curves, etc. of type y and z.

Proposition.

Let E be a null set in $\mathbb{R}^3$. Then $E = E^x \cup F^x$ where

(i) $\forall \varepsilon > 0$, E^x can be covered by slabs $V^x(f_i, \delta_i)$ with Lip. const. 1 so that $\boxed{\sum \delta_i < \varepsilon}$.

(ii) $\forall \varepsilon > 0$, F^x can be covered by tubes $U^x(\Phi_j, \eta_j)$ with Lip. const. 1 so that $\boxed{\sum \eta_j^2 < \varepsilon}$.

But the real question is:

Can we cover E with slabs V_j of type x, y, z Lipschitz constant L, and thickness δ_i so that $\boxed{\sum \delta_i \le \varepsilon}$?

independent of E !

We do not know the answer!

The correct generalization of Dillworth's lemma does not hold!

In [A.C.P] we prove that

Statement A. $\forall L, M < +\infty$, $\exists$ a finite set $E \subset \mathbb{R}^3$ s.t. E cannot be covered by $M \cdot n^{1/3}$ surf. of type x, y, z and Lip. const. L. $\leftarrow n := \# E$

Statement B. $\forall L < +\infty, \delta > 0, \exists E \subset \mathbb{R}^3$ s.t.

$$\sigma_{x,L}(E) \;;\; \sigma_{y,L}(E) \;;\; \sigma_{z,L}(E) \leq \delta n^{2/3}.$$

$$\sigma_{x,L}(E) := \max \left\{ \#(E \cap S) : \begin{array}{l} S \text{ surf. of type } x \\ \text{and Lip. const. } L \end{array} \right\}$$

Statement C. $\forall L < +\infty$, $\exists E \subset \mathbb{R}^3$ s.t.

$$\sigma_{x,L}(E) \;;\; \sigma_{y,L}(E) \;;\; \sigma_{z,L}(E) < n^{2/3}.$$

Statement D. $\forall L < +\infty$, $\exists E \subset \mathbb{R}^3$ s.t.

$$\sigma_{x,L}(E) \cdot \sigma_{y,L}(E) \cdot \sigma_{z,L}(E) < n^2 .$$

Easy example that a l m o s t proves D.

Any product set $E = E_x \times E_y \times E_z$ with $E_x, E_y, E_z \subset \mathbb{R}$ satisfies, for every L,

$$\sigma_{x,L}(E) = \# E_y \cdot \# E_z$$
$$\sigma_{y,L}(E) = \# E_x \cdot \# E_z$$
$$\sigma_{z,L}(E) = \# E_x \cdot \# E_y$$

and therefore

$$\sigma_{x,L}(E) \cdot \sigma_{y,L}(E) \cdot \sigma_{z,L}(E) = (\# E_x)^2 \cdot (\# E_y)^2 \cdot (\# E_z)^2 = (\# E)^2 = n^2$$

The example proving D is obtained as a suitable perturbation of one of these product sets.

Note that this example cannot be used to disprove the conjectured covering theorem for null sets in $\mathbb{R}^3$!

Tangent field to a null set in $\mathbb{R}^3$

Proposition.
Given $E \subset \mathbb{R}^3$, $\mathscr{L}^3(E) = 0$, there exists a map $\tau : E \longrightarrow \{\text{planes in } \mathbb{R}^3\} = G(3,2)$ s.t. for every surface S of class C^1

$$\mathrm{Tan}(S,x) = \tau(x) \quad \text{for } \underset{\text{w.r.t. } \mathscr{H}^2}{\text{a.e.}}\ x \in S \cap E.$$

Open problem.
Can we find $\tau : E \to G(3,2)$ s.t. for every curve C of class C^1

$$\mathrm{Tan}(C,x) \subset \tau(x) \quad \text{for } \underset{\text{w.r.t. } \mathscr{H}^1}{\text{a.e.}}\ x \in C \cap E \quad ?$$

Another open problem

Let $\mathcal{F} := \{\Phi : \mathbb{R} \to \mathbb{R}^2 \text{ s.t. } \mathrm{Lip}(\Phi) < 1\}$.

For every $\phi \in \mathcal{F}$, let $\mu^x_\Phi := \mathscr{H}^1 \llcorner \text{graph of } \Phi$. $\{(y,z) = \Phi(x)\}$

Given P_x prob. meas. on $\mathcal{F}$, let

$$\mu^x := \int_{\mathcal{F}} \mu^x_\Phi \, dP(f).$$

Construct μ^y and μ^z in the same way ....

Question.
If $\lambda \ll \mu^x, \mu^y, \mu^z$, is it true that $\lambda \ll \mathscr{L}^3$?