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On maps whose distributional Jacobian is a measure

Plan of the Talk

1. Motivations
2. Basic settings
3. Functions with bounded variations and finite perimeter sets
4. Jacobian and distributional Jacobian
5. Jacobian of maps with values in spheres
6. Construction of maps with prescribed topological singularity

1. Motivations

Construction of l a r g e classes of maps for which the Jacobian is well-defined (in some weak sense)

For $u : \mathbb{R}^n \to \mathbb{R}^n$ the Jacobian is just $\det(\nabla u)$

"large" means large enough to have good compactness properties (in the spirit of Sobolev functions)

We essentially works with distributional definitions....

The problem originated from questions in the Calculus of Variations....

2. Basic settings

- Maps and sets are Borel measurable
- "Measure" means measure on Borel sets usually bounded, possibly vector-valued
- Let $u : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$. We say that the (distributional) partial derivative $D_i u = \frac{\partial u}{\partial x_i}$ belongs to L^p if there exists $g_i \in L^p(\mathbb{R}^d)$ s.t.

$u \in L^1_{loc}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} D_i\varphi \cdot u \, dx = -\int_{\mathbb{R}^n} \varphi \cdot g_i \, dx$$

$\forall \varphi \in C^\infty_c(\mathbb{R}^n)$. We denote g_i by $D_i u$

$u \in L^p(\mathbb{R}^n)$

- Sobolev space $W^{1,p}(\mathbb{R}^d) := \{u : D_i u \in L^p \ \forall i\}$

Sobolev functions are a.e. differentiable!

3. Functions with bounded variation (in the sense of distributions)

distributional derivative

Definition. $BV_d(\mathbb{R}^n) := \{ u : Du \text{ is a measure} \}$

$u \in L^1(\mathbb{R}^n)$

$\Updownarrow$

$D_i u$ is a measure for $i=1,\dots,n$

$\Updownarrow$

$\exists$ real-valued meas. μ_i s.t.

$$\int_{\mathbb{R}^n} D_i\varphi \; u \, dx = -\int_{\mathbb{R}^n} \varphi \, d\mu_i \quad \forall \varphi \dots$$

Relation to the classical definition

- Let $u : \mathbb{R} \to \mathbb{R}$ be a function with bounded variation (in the classical sense).
  Then there exists a finite measure μ such that
  $$u(x) = \mu\big((-\infty, x]\big) + \mu(-\infty)$$
  for every x where u is continuos.
  Then $u \in BV_d(\mathbb{R})$ and $Du = \mu$

- Viceversa, if $u \in BV_d(\mathbb{R})$ then there exists $\tilde{u}$ with B.V. in the classical sense such that $u = \tilde{u}$ a.e.

Finite perimeter sets

Def. $E \subset \mathbb{R}^n$ is a finite perimeter set if $1_E \in BV$

characteristic function of E

Remark.

If E is regular then $D1_E = \nu \cdot \mathcal{H}^{n-1} \llcorner \partial E$

inner normal to ∂E

restriction of $(n-1)$-dimensional Hausdorff meas. to the set E

Theorem (De Giorgi - Federer)

If E has finite perimeter then

$$D1_E = \nu \cdot \mathcal{H}^{n-1} \llcorner \partial_* E$$

where $\partial_* E$ is the measure theoretic boundary of E

(set of points where E has density $\frac{1}{2}$)

$\partial_* E$ is rectifiable (and even better!)

($E \subset \bigcup E_k$ s.t. $\mathcal{H}^{n-1}(E_0)=0$, E_k $\mathcal{C}^1$-hypersurf. for $k \geq 1$)

ν is a suitably defined inner normal

4. Jacobian (of smooth maps)

Definition. If $u : \mathbb{R}^n \to \mathbb{R}^n$: $\quad Ju := \det(Du)$.

If $u : \mathbb{R}^3 \to \mathbb{R}^2$: $\quad Ju := Du_1 \times Du_2$.

In general, for $u = (u_1, \dots, u_k) : \mathbb{R}^n \to \mathbb{R}^k$: $\quad Ju := du_1 \wedge \dots \wedge du_k$.

where $du_i := \sum \frac{\partial u_i}{\partial x_j} dx_j$

Remarks.

(i) The Jacobian operator J is well-defined for maps of class $\mathcal{C}^1$ and even $W^{1,k}$.

(ii) Moreover $J : W^{1,k} \to L^1$ is (sequentially) continuous with respect to suitable weak topologies (!)

(iii) What about maps in $W^{1,p}$ with $p < k$?

A fundamental identity for the Jacobian

If $u : \mathbb{R}^2 \to \mathbb{R}^2$ then $Ju = D_1u_1 D_2u_2 - D_1u_2 D_2u_1$
$= D_1(u_1 D_2u_2) - D_2(u_1 D_1u_2)$

In general $Ju = \frac{1}{k} d \left(\sum_{i=1}^{k} (-1)^{i-1} u_i \; \widehat{du_i} \right)$ (*)

$\prod_{j\neq i} du_j$

Definition. The distributional Jacobian of $u \in L^\infty \cap W^{1,k-1}$ is given by (*).

Remarks.

(i) (*) is not the only formula of this type, but is the more "symmetric", and more convenient in certain situations

(ii) Thus $J : L^\infty \cap W^{1,k-1} \to \mathcal{D}'$ is (seq.) continuous in the suitable weak topologies.

(iii) Definition due to J. Ball ('77) and Jerrard & Soner ('02)

4.2

Basic example

Let $u : \mathbb{R}^n \to \mathbb{R}^n$; $u(x) := x/|x|$.

Then $\det(Du(x)) = 0$ a.e.

But $Ju = \alpha_n \cdot \delta_0$ ← Dirac mass at 0

α_n: volume of the unit ball in $\mathbb{R}^n$

Proof: approximate u by $u_\varepsilon(x) := \begin{cases} \frac{x}{\varepsilon} & \text{if } |x| < \varepsilon, \\ \frac{x}{|x|} & \text{if } |x| \geq \varepsilon. \end{cases}$

Then $Ju_\varepsilon = \frac{1}{\varepsilon^n} 1_{B_\varepsilon} \xrightarrow[\varepsilon \to 0]{} \alpha_n \delta_0$.

Theorem (S. Müller '90)

If $u : \mathbb{R}^n \to \mathbb{R}^n$ and Ju is a measure then the absolutely continuous part of Ju is given by $\det(Du)$ times the Lebesgue measure.

4.3

5. Jacobian of maps with values in spheres

Let $u : \mathbb{R}^n \to \mathbb{R}^n$, $\underline{|u|=1 \ \text{a.e.}}$

If $u \in C^1$ (or even $u \in W^{1,n}$)

then $Ju = \det(Du) = 0$ a.e.

But this is not true if $u \in W^{1,n-1}$....

Proposition. If u is smooth outside a finite set $S = \{x_i\}$ then

$$Ju = \alpha_n \sum_i d_i \cdot \delta_{x_i}$$

d_i = degree of the restriction of u to the sphere S_i = $\deg(u; S_i; S^{n-1})$

Theorem. The previous representation for Ju holds even for maps of class $W^{1,n-1}$ in the sense that if Ju is a measure then

$$Ju = \alpha_n \sum_i d_i \, \delta_{x_i}$$

for a suitable choice of $d_i \in \mathbb{Z}$ and $x_i \in \mathbb{R}^n$.

But the points x_i are not all discontinuity points of u. They are the so-called topological singularities ....

Another (more interesting) example

Let $u : \mathbb{R}^3 \to \mathbb{R}^2$, $\underline{|u| = 1 \text{ a.e.}}$

If $u \in C^1$ (or even $u \in W^{1,2}$)

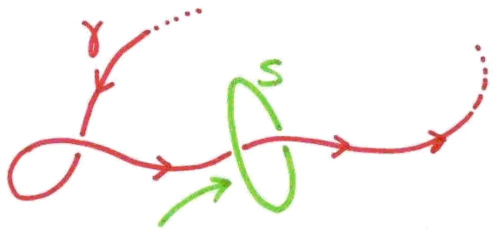
then $Ju = Du_1 \times Du_2 = 0$ a.e.

Proposition. If u is smooth outside a smooth closed curve γ then

$$Ju = \pi \cdot d \cdot \tau_\gamma \cdot \mathcal{H}^1 \llcorner \gamma$$

τ_γ = tangent to γ

d = degree of the restriction of u to the circle S = $\deg(u; S; S^1)$



S: boundary of a disk D that intesects γ in just one point

Theorem. Let $u : \mathbb{R}^3 \to \mathbb{R}^2$ be a map of class $W^{1,1}$ s.t. $|u| = 1$ a.e. and Ju is a measure. Then

$$Ju = \pi \sum_{i=1}^{\infty} d_i \cdot \tau_{\gamma_i} \cdot \mathcal{H}^1 \llcorner \gamma_i$$

where γ_i is a suitable family of closed rectifiable curves such that $\sum \text{length}(\gamma_i) < +\infty$ and d_i are integers.

Is it possible to give a pointwise interpretation of the "singularities" γ_i ?

General situation

Let $u : \mathbb{R}^n \to \mathbb{R}^k$, $k \le n$, $\underline{|u| = 1 \text{ a.e.}}$

Proposition. If u is smooth outside a connected $(n-k)$-dimensional surface M without boundary then

volume of the unit ball in $\mathbb{R}^k$ = α_k; tangent $(n-k)$-vector to M = τ_M

$$\underset{\sim\sim}{Ju} = \alpha_k \cdot d \cdot \tau_M \cdot \mathcal{H}^{n-k} \llcorner M$$

we apply to Ju some Hodge operator ....

d = degree of the restriction of u to ∂D where D is a k-dim. disk that intersect M in one point

Theorem (Jerrard-Soner '02, A.-Baldo-Orlandi '03)

Let $u : \mathbb{R}^n \to \mathbb{R}^k$ be a map of class $W^{1,k-1}$ s.t. $|u| = 1$ a.e. and Ju is a measure. Then

$$\underset{\sim\sim}{Ju} = \alpha_k \cdot \sigma \cdot \tau_M \cdot \mathcal{H}^{n-k} \llcorner M$$

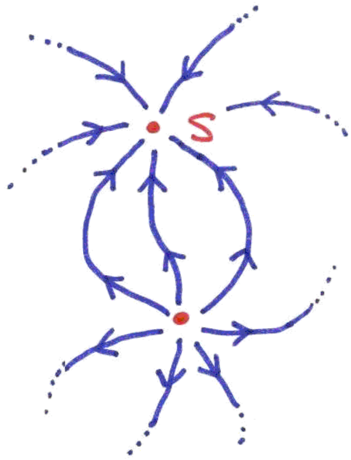
where M is an $(n-k)$-dimensional rectifiable set, τ_M is an $(n-k)$-vector orienting M, and σ an integer-valued multiplicity function.

In fact, there holds more: M, τ_M, σ define an integral current without boundary.

Again: is there a pointwise characterization of M and σ?

Idea of the proof.

An example of map $u: \mathbb{R}^2 \to \mathbb{R}^2$ smooth outside a finite set S and satisfying $|u|=1$ suggests that points in S are the boundary of any level curve of u.



This holds for every $u: \mathbb{R}^n \to \mathbb{R}^k$ s.t. $|u|=1$ a.e. and $u \in W^{1,k-1}$:
In some sense Ju is the boundary of a generic level surface of u.

The right framework is the theory of currents

Then the "rectifiability" of Ju follows from the boundary rectifiability theorem of Federer and Fleming.

Maps with prescribed topological singularities

Problem. Given an $(n-k)$-dim. surface $M \subset \mathbb{R}^n$, connected and without boundary, and $d \in \mathbb{Z}$, find $u: \mathbb{R}^n \to \mathbb{R}^k$, $|u|=1$, u smooth outside M s.t.

$$\deg(u, S, S^{k-1}) = d$$

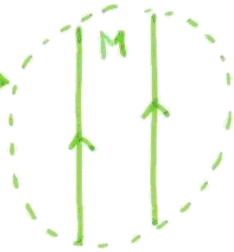
where S is the boundary of a k-dim. disk D s.t. $\#(D \cap M) = 1$.

This is equivalent to requiring

$$Ju = \alpha_k \cdot d \cdot \tau_M \cdot \mathcal{H}^{n-k} \llcorner M$$

Answers (A.-Baldo-Orlandi, '03)

$K=1$ (maps with values in $\{\pm 1\}$): NO



$K=2$ (maps with values in S^1): YES

$K>2$: YES if one allows an extra singularity outside M (with dimension $< n-k$).

Remarks.

(i) The result for $k\geq 2$ can be proved even when M is an $(n-k)$-dim. integral current without boundary.

(ii) The positive answer for $k=2$ implies that every smooth $(n-2)$-dim surface M in $\mathbb{R}^d$ without boundary is a boundary of a smooth hypersurface and a complete intersection

Sketch of construction (for $K=2$, $d=1$)

- M is an $(n-2)$-dim. connected surf. without bdry in $\mathbb{R}^n$.
- We want to construct a 1-form ω on $\mathbb{R}^n\setminus M$ s.t.

$$\int_\gamma \omega = \text{link}(M,\gamma) = \text{linking number of } M \text{ and } \gamma$$

- Indeed ω is the different. of $u : \mathbb{R}^n\setminus M \to \mathbb{R}/\mathbb{Z} = S^1$. And clearly this u satisfies our requirements.
- Notice that

$$\text{link}(M,\gamma) = \deg(\varphi, M\times\gamma, S^{n-1}) \qquad \varphi(x,y) = \frac{x-y}{|x-y|}$$

$$= \frac{1}{\alpha_K}\int_{M\times\gamma} d\varphi \quad \leftarrow (n-1)\text{-form depending on } x,y$$

$$= \int_{x\in\gamma} \boxed{\left(\frac{1}{\alpha_K}\int_{y\in M} d\varphi\right)} \quad \leftarrow 1\text{-form depending on } x$$

- Then we set $\omega(x) =$

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