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Some applications of GMT

There are many applications of GMT.
I chose few one close to PDE's and the Calculus of Variations

I Lecture
- BV functions (quick survey)
- SBV functions and Ambrosio's compactness theorem

II
- Approximation of minimal perimeters by scalar Ginzburg–Landau functionals — Γ-convergence

III & IV
- Coarea formula, oriented coarea formula
- Jacobian (of Lipschitz maps)
- Distributional Jacobian (of Sobolev maps)
- Some applications / essential singularities

Currents are needed here!

Why BV functions....

GENERAL OBSERVATION:

$$u \text{ minimizes } F(u) := \int_\Omega f(x, u, \nabla u)\, dx$$

with Dirichlet boundary condition $u = g$ on $\partial\Omega$

$\Downarrow$ (and $\Uparrow$ if F is convex)

u solves the PDE : $\boxed{\operatorname{div}\left(\frac{\partial f}{\partial p}(x,u,\nabla u)\right) = \frac{\partial f}{\partial u}(x,u,\nabla u)}$ E.L. eq. of F

with the Dirichlet bdry cond. $u = g$ on $\partial\Omega$

Example: u minimizes $F(u) := \int \frac{1}{2}|\nabla u|^2 + fu$ with $u=g$ on $\partial\Omega$

$\Updownarrow$

u solves $\begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

DIRECT METHODS IN CALC. VAR.

Prove the existence of minimizers of F semicontinuity and compactness arguments (and deduce the existence of solutions for the Euler–Lagrange eq. associated to F).

Dirichlet (principle), Hilbert, Tonelli.

In other words, we want to argue as follows:

Let F be a lower semicontinuous function (al) on a topological space X, and assume that the sublevels of F (the sets $\{u \in X : F(u) \le c\}$) are sequentially (pre)compact.
Then every minimizing sequence (u_n) (that is (u_n) satisfies $F(u_n) \to \inf_{u \in X} F(u)$) admits a converging subsequence (u_{n_k}) and the limit is a minimizer of F.

Example

$$F(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + fu \quad \text{is lower semicontinuous}$$

on $X := \{ u \in W^{1,2}(\Omega) : u = g \text{ on } \partial\Omega \}$

w.r.t. the weak topology.

Sublevels of F are bounded and weak-closed in $W^{1,2}$, and therefore sequentially compact

We need to use Sobolev spaces such as $W^{1,2}$!

Question:

What about $F(u) := \underbrace{\int_\Omega \sqrt{1 + |\nabla u|^2}}_{\text{area of the graph of } u}$?

BV functions

Def.: $BV(\Omega) := \{ u \in L^1(\Omega) : \underset{\substack{\uparrow \\ \text{distributional gradient}}}{Du} \text{ is a bounded } \overset{\substack{\text{valued in } \mathbb{R}^n \\ \downarrow}}{\text{measure}} \}$

in other words there exists real measures μ_i s.t.

$$\boxed{\int_\Omega \frac{\partial \phi}{\partial x_i} u = -\int_\Omega \phi \, d\mu_i \quad \text{for } i=1,\dots,n \quad \phi \in C^1_c(\Omega)}$$

and $Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = (\mu_1, \dots, \mu_n)$

$$\|u\|_{BV} := \|u\|_1 + \|Du\|$$

total variation of Du (as a vector measure)

Basic Compactness

If $\|u_n\|_{BV} \le C < +\infty$

then there exists a subsq. (u_{n_k}) converging to $u \in BV$ in the following sense:

$$\begin{cases} u_n \to u & \text{in } L^1(\Omega) \\ Du_n \to Du & \text{in the weak* topology of measures} \end{cases}$$

(This is called weak (weak*) convergence in BV)

PROOF: ......

Useful tools

(Ω bounded regular domain)

- $BV(\Omega) \hookrightarrow L^p(\Omega)$ for every $p \le \frac{n}{n-1}$

  (the immersion is compact for $p < \frac{n}{n-1}$)

- $T : BV(\Omega) \to L^1(\partial\Omega)$ Trace operator

  $(T : u \longmapsto u_{|\partial\Omega})$

- $\|u - \bar{u}\|_p \le C_{p,n} \|Du\| \quad \forall p \le \frac{n}{n-1}$

  $\bar{u}$: average of u on Ω

  Poincaré inequality

- $\left.\begin{array}{l} \|Du\| + |\bar{u}| \\ \|Du\| + \|Tu\|_{L^1(\partial\Omega)} \\ \vdots \end{array}\right\}$ are equivalent to the standard norm of BV

Set of finite perimeter

Def. $E \subset \Omega$ has finite perimeter in Ω if $1_E \in BV(\Omega)$ and we set
$$Per_\Omega(E) := \| D1_E \|$$

Motivation: if ∂E is of class C^1, or even Lipschitz it is easy to see that

$$1_E \in BV(\Omega) \quad \text{and} \quad D1_E = \nu_{\partial E} \cdot \mathcal{H}^{n-1} \llcorner (\partial E \cap \Omega)$$

inner normal — $(n-1)$-dimensional Hausdorff measure — boundary in Ω

Hence
$$Per_\Omega(E) := \| D1_E \| = \mathcal{H}^{n-1}(\partial E \cap \Omega)$$

Basic Compactness (and semicontinuity)

E_n seq. of finite perimeter sets in Ω

If $Per_\Omega(E_n) \le c < +\infty$

Then E_n converge up to subseq. to E in the L^1 sense $\left(1_{E_n} \to 1_E \text{ in } L^1 \iff \mathcal{L}^n(E_n \Delta E) \to 0\right)$

Moreover $\quad Per_\Omega(E) \le \liminf_n Per_\Omega(E_n)$

Corollary: existence of sets with minimal perimeter....

Fine structure of BV functions

- $u \in BV(\Omega)$ is approximately differentiable (in the L^1-sense) at $\mathcal{L}^n$-a.e. $x \in \Omega$

  there exists $\nabla u(x)$ s.t.
  $$\lim_{r \to 0} \fint_{|h| \le r} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|}{|h|} \, d\mathcal{L}^n(h) = 0$$

- u admits an approximate limit (in the L^1-sense) $\tilde{u}(x)$ at every x except a singular set S_u which is $\mathcal{H}^{n-1}$ σ-finite

- S_u is rectifiable, (countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable)

  that is, it can be covered by countably many C^1-hypersurfaces (except an $\mathcal{H}^{n-1}$ negligible subset).

  In particular S_u admits a normal ν_{S_u}

- For $\mathcal{H}^{n-1}$ a.e. $x \in S_u$, there exists the approximate limits $u^+(x)$ and $u^-(x)$ on the two sides of S_u

- $$Du = \underbrace{\nabla u \cdot \mathcal{L}^n}_{\text{a.c. part of } Du} + \underbrace{(u^+ - u^-)\, \nu_{S_u} \cdot \mathcal{H}^{n-1} \llcorner S_u}_{\text{"Jump" part of } Du} + \underbrace{D_c u}_{\text{"Cantor" part}}$$

$$\left(D1_E = \nu_{\partial^* E} \cdot \mathcal{H}^{n-1} \llcorner \partial^* E \right)$$

SBV functions

$$Du = \nabla u \cdot \mathcal{L}_n + (u^+ - u^-)\nu_{S_u} \cdot \mathcal{H}^{n-1} \llcorner S_u$$

$$\Updownarrow$$

Def. $$SBV(\Omega) := \{ u \in BV(\Omega) : D_c u = 0 \}$$

SBV compactness (L. Ambrosio)

Let (u_n) be a sequence in $SBV(\Omega)$ s.t.:

- $\|u_n\|_{BV} \leq C < +\infty$
- ∇u_n are equintegrable
  (i.e. $\int f(|\nabla u_n|) \leq C < +\infty$ for some $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{f(t)}{t} \xrightarrow[t \to +\infty]{} +\infty$)
- $\mathcal{H}^{n-1}(S_{u_n}) \leq C < +\infty$.

Then u_n converge, up to subseq., to some $u \in SBV$.
Moreover

- $\nabla u_n \to \nabla u$ weakly in L^1
- $(Du_n)_s \to (Du)_s$ in the sense of measures

$$\| $$
$$(u_n^+ - u_n^-)\, \nu_{S_{u_n}} \cdot \mathcal{H}^{n-1} \llcorner S_{u_n}$$

Application: existence of minimizers for the Mumford-Shah functional

$$F(u) := \int_{\Omega \setminus S_u} |\nabla u|^2 + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_\Omega |u-g|^2$$

where $g : \Omega \to [0,1]$ is given

and u is assumed C^1 out of a closed singular set S_u which is $\mathcal{H}^{n-1}$- finite

not prescribed!

arising in image segmentation problems

(but similar functionals appears in other problems ....)

Direct proof in dimension $n=2$
[Mumford – Shah, Dal Maso – Morel – Solimini]

Proof by direct methods + regularity theory

EXISTENCE OF MINIMIZERS IN SBV (Ω) } L. Ambrosio

Regularity theory: minimizers are C^1 outside a closed singular set S_u } De Giorgi – Carriero – Leaci

Proof of Ambrosio's theorem

for $n=1$ is trivial

u_n converge (up to subseq.) to some $u \in BV$

$$Du_n = \nabla u_n \cdot \mathcal{L}^1 + \sum_{i=1}^N \alpha_i^n \delta_{x_i^n}$$

N independent of n !

$$\begin{array}{ccccc} Du_n & = & \nabla u_n \cdot \mathcal{L}^1 & + & \sum_{i=1}^N \alpha_i^n \delta_{x_i^n} \\ \downarrow \mathcal{M}\text{-}w^* & & \downarrow L^1\text{-}w & & \downarrow \mathcal{M}\text{-}w^* \\ Du & & g \cdot \mathcal{L}^1 & + & \sum_{i=1}^N \alpha_i \delta_{x_i} \end{array}$$

Hence $u \in SBV$, $g = \nabla u$, (and $\nabla u_n \rightharpoonup \nabla u$).

Not as simple in dimension $n > 1$!

Lemma (Chain rule).

If $u \in BV(\Omega)$, $\varphi: \mathbb{R} \to \mathbb{R}$ is C^1 and Lipschitz

then $\varphi(u) \in BV$ [easy ....]

and

$$D(\varphi(u)) = \dot\varphi(\tilde u)\,(\nabla u \cdot \mathcal{L}^n + D_c u) + (\varphi(u^+) - \varphi(u^-))\,\nu_{S_u} \cdot \mathcal{H}^{n-1} \llcorner S_u$$

Characterization of SBV

Given $u \in BV$, then

$$u \in SBV \iff \exists\, g \in L^1 \text{ s.t. } \sup_{0 \le \varphi \le 1} \|D(\varphi(u)) - \dot\varphi(u)\, g \cdot \mathcal{L}^n\| < +\infty$$

And if so, $g = \nabla u$.

Proof: .....

Lemma (the theorem follows...)

Let (u_n) be bounded in BV, $u_n \in SBV$ and $u_n \to u \in BV$;

Assume $\nabla u_n \rightharpoonup g$ weakly in L^1;

Assume $\mathcal{H}^{n-1}(S_{u_n}) \le C < +\infty$.

Then $u \in SBV$ and $g = \nabla u$.

Proof

It suffices to show that

$$\|D(\varphi(u)) - \dot\varphi(u)\, g \cdot \mathcal{L}^n\| \text{ is bounded for all } 0 \le \varphi \le 1$$

$$\begin{array}{ccc} D(\varphi(u)) & & \dot\varphi(u)\, g \cdot \mathcal{L}^n \\ \uparrow & & \uparrow \\ D\varphi(u_n) & & \dot\varphi(u_n) \cdot \nabla u_n \cdot \mathcal{L}^n \end{array}$$

Hence

$$\begin{aligned} \|D(\varphi(u)) - \dot\varphi(u)\, g \cdot \mathcal{L}^n\| &\le \liminf \|D(\varphi(u_n)) - \dot\varphi(u_n) \nabla u_n \cdot \mathcal{L}^n\| \\ &\le \liminf \|(\varphi(u_n^+) - \varphi(u_n^-))\, \nu_{S_{u_n}} \cdot \mathcal{H}^{n-1} \llcorner S_{u_n}\| \\ &\le \liminf \mathcal{H}^{n-1}(S_{u_n}) \\ &\le C < +\infty. \end{aligned}$$

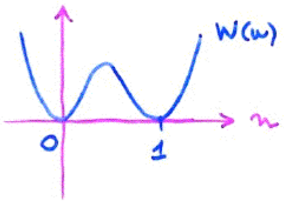
Approximation of perimeter by scalar Ginzburg–Landau functionals

[Problem : asymptotic behaviour as $\varepsilon \to 0$ of minimizers of]

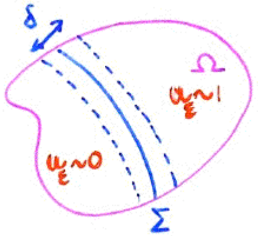
$$F_\varepsilon(u) := \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon^2} W(u)$$

Scalar G.-L. functionals (a.k.a. Cahn-Hilliard) appears in phase separation models....

with prescribed average $\quad \fint_\Omega u = m \,, \quad 0 < m < 1$



Heuristic



Assume u_ε is as in the picture. Then

$$F_\varepsilon(u_\varepsilon) \sim \mathcal{H}^{n-1}(\Sigma)\, \delta \left(\frac{1}{\delta^2} + \frac{1}{\varepsilon^2}\right)$$

$$\sim \frac{1}{\varepsilon} \mathcal{H}^{n-1}(\Sigma)$$

optimization of δ gives $\delta \sim \varepsilon$

We expect $u \cong 1$ or 0 everywhere outside an ε-neighbourhood of a minimal surface Σ.

(chosen among those that separate Ω into a region of volume $m|\Omega|$ and one of vol. $(1-m)|\Omega|$)

you can do better....

Ansatz : $\quad u_\varepsilon(x) := \gamma\left(\frac{\mathrm{dist}_\Sigma(x)}{\varepsilon}\right) \qquad \gamma : \mathbb{R} \to [0,1]$

Then : $\quad F_\varepsilon(u) :\cong \frac{1}{\varepsilon} \mathcal{H}^{n-1}(\Sigma) \int_{-\infty}^{\infty} \dot\gamma^2 + W(\gamma)$

after optimization of γ

$$\cong \frac{1}{\varepsilon}\, \sigma\, \mathcal{H}^{n-1}(\Sigma) \qquad \text{where } \sigma := 2\int_0^1 \sqrt{W(u)}$$

Can we make such statements rigorous?
(possibly without working too hard)

Γ- convergence (in one slide...)

X metric space (later: a space of functions)
$F_\varepsilon : X \to [0,+\infty]$

Def. We say that F_ε Γ-converge to F on X if:

(i) $\forall u \in X \ \forall u_\varepsilon \to u \qquad F(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$

(ii) $\forall u \in X \ \exists u_\varepsilon \to u \qquad F(u) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$

Definition due to DeGiorgi-Franzoni (on topological spaces)
It is actually KURATOWSKI convergence of the epigraphs $E_\varepsilon := \{(u,t) \in X \times [0,+\infty] : F_\varepsilon(u) \le t\}$

Main property:
If u_ε minimizes F_ε on X, then every limit point u of (u_ε) is a minimizer of F

Also:
If $F_\varepsilon \xrightarrow{\Gamma} F$ and $G: X \to [0,+\infty]$ is continuous then $F_\varepsilon + G \xrightarrow{\Gamma} F+G$ (stability under cont. perturbations)

[FINDING THE Γ-LIMIT OF F_ε GIVES INFO ABOUT THE LIMIT OF MINIMIZ. u_ε]

Provided that:
(a) F is not constant !
(b) (u_ε) is pre-compact in X !

Back to the original problem

Theorem (Modica-Mortola (77), Modica (87))
Fix $m \in (0,1)$, let $X := \{u \in L^1(\Omega) : \fint_\Omega u = m\}$
Then

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) & \text{for } u \in W^{1,2} \\ +\infty & \text{elsewhere in } X \end{cases}$$

$$\Big\downarrow \Gamma$$

$$F(u) := \begin{cases} \sigma \|Du\| = \sigma \mathcal{H}^{n-1}(S_u) & \text{for } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{elsewhere in } X. \end{cases}$$

Moreover every sequence (u_ε) s.t. $F_\varepsilon(u_\varepsilon) \le C < \infty$ is precompact in X

Corollary
If u_ε minimizes F_ε, then u_ε converge, (up to subseq.) to $u = 1_E$ where E minimizes the perimeter among all sets in Ω with measure $m|\Omega|$.

Something better can be done ....

Proof

Compactness: $F_\varepsilon(u_\varepsilon) \le C < +\infty \Rightarrow (u_\varepsilon)$ precomp. in L^1

$$F_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u)$$

$$\ge \int_\Omega 2\sqrt{W(u)}\, |\nabla u|$$

$a^2 + b^2 \ge 2ab$
with $a := \sqrt{\varepsilon |\nabla u|^2}$
$b := \sqrt{W(u)/\varepsilon}$

$$= \int_\Omega |D(H(u))| = \| D(H(u)) \|$$

here $H' = 2\sqrt{W}$

Hence: $C \ge F_\varepsilon(u_\varepsilon) \ge \int_\Omega |D(H(u_\varepsilon))| = \| D(H(u_\varepsilon)) \|$

$\Rightarrow$ $H(u_\varepsilon)$ is precompact in L^1

$\Rightarrow$ u_ε is precompact in L^1

Lower bound: $u_\varepsilon \to u \Rightarrow F(u) \le \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)$

$H(u_\varepsilon) \longrightarrow H(u)$ in BV weak*

previous estimate

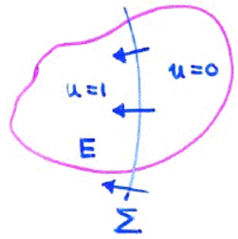
Hence:

$$\liminf F_\varepsilon(u_\varepsilon) \ge \liminf \| D(H(u_\varepsilon)) \|$$

$$\ge \| D(H(u)) \|$$

$$= \sigma\, \mathcal{H}^{n-1}(S_u) = \sigma\, \|Du\|$$

Upper bound: $\forall u\ \exists\, u_\varepsilon \to u,\ F_\varepsilon(u_\varepsilon) \geqslant F(u)$



We can assume $u = 1_E$, E smooth

Let $d_\Sigma(x) :=$ oriented distance from $\partial E = \Sigma$

Set $u_\varepsilon(x) := \gamma\left(\dfrac{d_\Sigma(x)}{\varepsilon}\right)$

where γ solves the ODE $\begin{cases} \dot\gamma = 2\sqrt{W(\gamma)} \\ \gamma(0) = 1/2 \end{cases}$

Indeed for such u_ε there holds

$$F_\varepsilon(u_\varepsilon) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u)$$

$$\le \mathcal{H}^{n-1}(\Sigma) \int_{-\infty}^{\infty} \varepsilon\, \dot\gamma^2 + \frac{1}{\varepsilon} W(\gamma)$$

$$= \mathcal{H}^{n-1}(\Sigma) \int_{-\infty}^{\infty} 2\sqrt{W(\gamma)}\, |\dot\gamma|$$

$$= \mathcal{H}^{n-1}(\Sigma) \int_0^1 \dot H(\gamma)\, d\gamma$$

$$= \mathcal{H}^{n-1}(\Sigma) \cdot \sigma$$

Third Lecture

JACOBIANS OF LIPSCHITZ MAPS

Coarea formula $u : \mathbb{R}^n \to \mathbb{R}^k$ Lipschitz, $n \geq k$.

Then for every $A \subset \mathbb{R}^n$

$$\int_{\mathbb{R}^k} \mathcal{H}^{n-k}(u^{-1}(y) \cap A)\, dy = \int_A p(x)\, dx$$

where $p(x) = \sqrt{\det[(Du(x))(Du(x))^t]}$

$$= \sqrt{\sum_M (\det M)^2}$$

$k \times k$ minor of $(\nabla u(x))$

$$= |du_1 \wedge \dots \wedge du_k|$$

components of u

[Recall that $df = \sum_1^n \frac{\partial f}{\partial x_i} dx_i$

$$= |\underbrace{u^\#(dy_1 \wedge \dots \wedge dy_k)}|$$

pull back according to u of $dy_1 \wedge \dots \wedge dy_n$, standard (volume) form on $\mathbb{R}^k$

Def We call k-dimensional Jacobian of $u : \mathbb{R}^n \to \mathbb{R}^k$ the k-form

$$Ju := du_1 \wedge \dots \wedge du_k$$
$$= u^\#(dy_1 \wedge \dots \wedge dy_k)$$

ATTENTION: usually the word "Jacobian" is used for $|Ju|$. This notation (and definition) is not widely used!

Identification of vectors and covectors

Def. $\star : \Lambda^k(\mathbb{R}^n) \to \Lambda_{n-k}(\mathbb{R}^n)$

$$\star : dx_{\underline{i}} \longmapsto \sigma(\underline{i}', \underline{i})\, e_{\underline{i}'}$$

multi-index $(i_1, \dots, i_k)$

sign of the permutation that re-order $(\underline{i}', \underline{i})$

complement to the multiindex $\underline{i}$

$\star$ is a (sort of) Hodge operator, identifying k-covectors and $(n-k)$-vectors.
Hence, if ω is a k-form, then $\star\omega$ is a $(n-k)$-current.

ATTENTION this is not the standard Hodge oper.

$\star$ behaves nicely w.r.t. ∂ and d

$$\star(d\omega) = (-1)^{n-k}\, \partial(\star\omega)$$

Geometrically:

If $\omega = \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k$ is a simple form then $\star\omega$ is a simple vector that spans the $(n-k)$-space $(\mathrm{Ker}\,\omega_1) \cap (\mathrm{Ker}\,\omega_2) \cap \dots \cap (\mathrm{Ker}\,\omega_k)$

More on the coarea formula

If $u : \mathbb{R}^n \to \mathbb{R}^k$ Lipschitz

Then $\displaystyle\int_{\mathbb{R}^k} \mathcal{H}^{n-k}(u^{-1}(y)\cap A)\,dy = \int_A |Ju|\,dx \qquad \forall A \subset \mathbb{R}^n$

Hence $u^{-1}(y)$ is $\mathcal{H}^{n-k}$-locally finite for $\mathcal{H}^k$-a.e. y.

Moreover $u^{-1}(y)$ is $(n-k)$-rectifiable. [Fed] .....

The result is non trivial even for u of class C^1 (cf. Sard's lemma).

easy

Moreover $\tau(x) := \dfrac{\star Ju(x)}{|Ju(x)|}$

is a simple $(n-k)$-vector that orients $Tan(u^{-1}(y), x)$ for $\mathcal{H}^{n-k}$-a.e. $x \in u^{-1}(y)$ and $\mathcal{H}^k$-a.e. y.

Hence we set $\underbrace{N_y = N_y(u)}_{\text{oriented level curve of } u \ldots} = [u^{-1}(y), \tau, 1]$

In particular we have the oriented coarea form.:

$$\int_{\mathbb{R}^k} N_y \, dy = \star Ju \, \mathcal{L}^n$$

What about the boundary of N_y ?

Simple cases

$\boxed{\text{if } n=k}$ then

$u : \mathbb{R}^k \to \mathbb{R}^k$

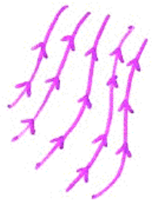
$Ju = (\det(\nabla u(x)))\, dx_1 \wedge \dots \wedge dx_k$

$\star Ju = \det(\nabla u(x)) \cdot \mathcal{L}^k$

$\boxed{\text{if } n=3,\ k=2}$ then

$u : \mathbb{R}^3 \to \mathbb{R}^2, \quad u = (u_1, u_2)$

$\star Ju = \nabla u_1 \times \nabla u_2$ → vector product in $\mathbb{R}^3$



← level curves of u oriented by $\dfrac{\nabla u_1 \times \nabla u_2}{|\nabla u_1 \times \nabla u_2|}$

The oriented coarea formula says that the "diffuse" 1-current $\star Ju = (\nabla u_1 \times \nabla u_2)\cdot \mathcal{L}^3$ can be decomposed as integral of the 1-currents associated to the level curves.

$\boxed{\text{if } k=1}$ then

$u : \mathbb{R}^n \to \mathbb{R}$

$Ju = du = (\nabla u)^*$

$\dfrac{\nabla u}{|\nabla u|}$ orients (as a normal) the level surfaces of u

Beyond Lipschitz maps: Jacobian of Sobolev maps

Recall

$$W^{1,p}(\underset{\subset\,\mathbb{R}^n}{\Omega};\mathbb{R}^k) := \{u\in L^p(\Omega;\mathbb{R}^k) : \underbrace{Du}_{\text{distributional derivative}} \in L^p(\Omega,\mathbb{R}^{k\times n})\}$$

Question: for which p can we define the jacobian of $W^{1,p}$ maps?

Answer: everything is fine for $p \geq k$

$$J : W^{1,p}(\Omega;\mathbb{R}^k) \longrightarrow L^{p/k}(\Omega;\Lambda^k(\mathbb{R}^n))$$

$$u \longmapsto Ju$$

is a continuous operator

But there holds more: J is sequentially continuous from $W^{1,p}$- weak to $L^{p/k}$- weak (for $p>k$)

Important, and non obvious!

What about $W^{1,p}$ with $p<k$?

Fundamental identity

for u of class C^2

$$Ju = du_1\wedge\dots\wedge du_k = \frac{1}{k}\, d\left(\sum_{i=1}^{k}(-1)^{i-1}u_i \bigwedge_{j\neq i} du_j\right)$$

that is to say

$$Ju = u^\#\underbrace{(dy_1\wedge\dots\wedge dy_k)}_{\text{standard volume form on } \mathbb{R}^k} = \frac{1}{k}\, d\Big(\overbrace{u^\#\underbrace{\Big(\sum_{i=1}^{k}(-1)^{i} y_i \bigwedge_{j\neq i} dy_j\Big)}_{(k-1)\text{ form on } \mathbb{R}^k}}^{n-1 \text{ form called: } ju}\Big)$$

Very "popular" in the case $n=k$

$(k-1)$ form on $\mathbb{R}^k$ = standard volume form on every sphere centered at 0.

COROLLARY: $\star Ju$ is a boundary (precisely $\star Ju = (-1)^{n-k+1}\partial(\star ju)$)

DEFINITION (Ball, Brezis-Coron-Lieb, JERRARD-SONER)

distributional Jacobian!

For every u in $W^{1,k-1}(\Omega;\mathbb{R}^k)$ we set

$$Ju := \frac{1}{k}\, d\left(\sum_{i=1}^{k}(-1)^{i-1}u_i \cdot \bigwedge_{j\neq i} du_j\right)$$

J is (sequentially) continuous from $W^{1,k-1}\cap L^\infty$ in $\mathcal{D}'(\Omega,\Lambda^k(\mathbb{R}^n))$

More precisely, if

u_n are uniformly bounded and

$u_n \to u$ weakly in $W^{1,p}$ for some $p>k-1$

strongly in $W^{1,k-1}$

then

$Ju_n \to Ju$ in the sense of distributions

$\star Ju_n \to \star Ju$ in the sense of currents

Remark: If the pointwise and the distributional Jacobian are both defined ($u\in W^{1,k}\cap L^\infty$) then they agree.

Jacobian and the area formula for Sobolev maps

Let $u \in W^{1,k}(\Omega;\mathbb{R}^k)$

Let $\mathcal{D}_u$ be the set of approximate differentiability

Then

$$\int_{\mathbb{R}^k} \mathcal{H}^{n-k}(u^{-1}(y)\cap\mathcal{D}_u)\,dy = \int_{\mathbb{R}^n} |Ju(x)|\,dx \leq \int_{\mathbb{R}^n} |\nabla u|^k < +\infty$$

given a $k\times n$ matrix $M=(m_1,\dots,m_k)$ $|m_1\wedge\dots\wedge m_k| \leq |M|^k$

Moreover $u^{-1}(y)\cap\mathcal{D}_u$ is rectifiable and oriented by $\frac{\star Ju}{|Ju|} =: \tau$ for a.e. y.

Thus $N_y := [u^{-1}(y)\cap\mathcal{D}_u\,;\,\tau\,;\,1]$ is a rectif. current and

$$\int_{\mathbb{R}^k} N_y\,dy = \star Ju$$

Proof: $\mathcal{D}_u$ can be covered by sets where u agrees with a Lipschitz map (countably many...)

Question: DO WE HAVE TO THROW AWAY $\mathcal{D}_u$?

See: Maly-Swanson-Ziemer .....

JACOBIAN OF MAPS VALUED IN SPHERES

from now on we assume $u:\Omega\subset\mathbb{R}^n\to S^{k-1}\subset\mathbb{R}^k$

If $u\in W^{1,k}(\Omega;S^{k-1})$ then $Ju=0$

for a.e. x, $du_1,\dots,du_k$ are linearly dependent because $du:\mathbb{R}^n\to \mathrm{Tan}(S^{k-1},u(x))$ and the latter has dimension $<k$.

This may <u>not</u> be true if $u\in W^{1,p}$ with $k-1\leq p<k$

Fundamental example:

$u:\mathbb{R}^k\to S^{k-1}$; $u(x) := \frac{x}{|x|}$ (then $u\in W^{1,p}$ $\forall p<k$)

then $\star Ju := \alpha_k\,\delta_0$ ← Dirac mass at 0.

(α_k:) volume of unit ball in $\mathbb{R}^k$.

Proof

Approximate u by $u_\varepsilon := \begin{cases} \frac{x}{\varepsilon} & \text{if } |x|<\varepsilon \\ \frac{x}{|x|} & \text{if } |x|\geq\varepsilon \end{cases}$

Then $\star Ju_\varepsilon(x) = \frac{1}{\varepsilon^k}\,1_{B(0,\varepsilon)}$

and $\star Ju_\varepsilon \to \star Ju$ in the sense of currents....

Important $\underbrace{Ju}_{\text{distributional definition}} \neq \underbrace{du_1\wedge\dots\wedge du_k}_{\text{pointwise definition} =0 \text{ in this case}}$!

Jacobian of maps with "nice" singularities

CASE $n=k$

Given $u : \mathbb{R}^k \to S^{k-1}$ smooth outside a finite singular set $S = \{x_i\}$, then

$$\star Ju = \alpha_k \sum_i d_i \delta_{x_i}$$

$$d_i := \deg(u, \partial B_i, S^{k-1})$$

where B_i is a ball that contains x_i only ... it does not matter which ball!

Proof: Of course, $\star Ju$ is supported in $S = \{x_i\}$!

Let ρ be a smooth radial function centered at some x_i, null on the others

$$\int_{\mathbb{R}^k} \star Ju \ \rho \, dx = \int_{\mathbb{R}^k} (\rho\, Ju) \cdot e_1 \wedge \dots \wedge e_k$$

$$= \int_{\mathbb{R}^k} - d\rho \wedge \tfrac{1}{k} ju$$

$$= \int_0^\infty - \dot\rho(t) \left(\int_{tS^{k-1}} \tfrac{1}{k} ju \right) dt$$

acting on τ tangent to tS^{k-1}

sphere centered at x_i with radius t

$$= \int_0^\infty - \dot\rho(t) \tfrac{1}{k} d_i \operatorname{vol}(S^{k-1})$$

$$= \rho(0)\, d_i\, \omega_k$$

Jacobian of maps with nice singularities

CASE $n > k$

Given $u : \mathbb{R}^n \to S^{k-1}$ smooth outside an ORIENTED manifold M without boundary and with codimension k in $\mathbb{R}^n$ then

orientation of M.

$$\star Ju = \alpha_k [M, \tau_M, \theta]$$

where the multiplicity $\theta(x)$ at $x \in M$ is given by

$$\theta(x) := \deg(u, \partial E, S^{k-1})$$

$\leftarrow E$ $n-k$ dimensional disc (suitably oriented) transversal to M intersecting M at x only

θ is locally constant (cf. $\star Ju$ has no bdry).

No proof will be given for this formula ..... (it follows from the case $n=k$ by slicing techniques).

COMMENT

We might say that $\star Ju$ captures the "essential" singularity of u!

$\star Ju$ is sometimes called "topological singularity" of u [Riviere, Pakzad, ...]

What can be said about Jacobians of generic Sobolev maps?

THEOR. Let $u \in W^{1,k-1}(\Omega; S^{k-1})$

- Then

$$\star Ju = \alpha_k \partial N$$

where **N** is an $(n-k+1)$-rectifiable current with integer multiplicity.

- More precisely

$$\star Ju = (-1)^{n-k+1} \alpha_k \partial N_y \quad \text{for a.e. } y \in S^{k-1}$$

COROLLARY

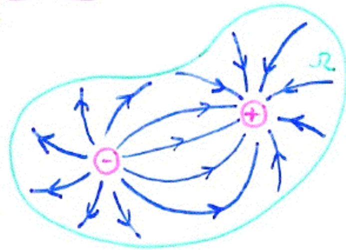
- If $\star Ju$ is a measure, (i.e. has finite mass!) it is also an $(n-k)$-dimensional integral current (up to α_k)
- In particular, for $n=k$ $\star Ju$ is a finite sum of Dirac Masses!

Brezis-Coron-Lieb for $n=k$
Jerrard-Soner for $n<k$

simpler proof: GA.-Baldo-Orlandi (see also Hang-Lin)

Proof

PROOF BY PICTURE!



→ level curves of $u: \Omega \to S^1$

○ singularities of u (where Ju is sitting)

Can we turn this picture into a real proof?

Since $Ju := \frac{1}{k} d(ju) = \frac{1}{k} d(u^{\#}\tilde{\omega})$ $\quad \sum_{1}^{k} (-1)^{i-1} y_i \, \widehat{dy_i}$ (#)

Then $\star Ju = \frac{(-1)^{n-k+1}}{k} \partial(\star ju) = \frac{(-1)^{n-k+1}}{k} \int_{y \in S^{k-1}} \partial N_y \, d\mathcal{H}^{k-1}(y)$

This is not enough.

TO BE PROVED LATER!

We claim indeed that more holds: given $\rho: S^{k-1} \to \mathbb{R}$ such that $\int_{S^{k-1}} \rho = 1$ then

$$Ju = \frac{1}{k} d(ju) = \frac{1}{k} d(u^{\#}\tilde{\omega}) = \frac{1}{k} d(u^{\#}(\rho\tilde{\omega}))$$

Hence

$$\star Ju = \frac{(-1)^{n-k+1}}{k} \int_{y \in S^{k-1}} \rho(y) \, \partial N_y \, d\mathcal{H}^{k-1}(y)$$

and taking a sequence $\rho \to k\alpha_k \delta_{\bar{y}}$

$$\star Ju = \frac{(-1)^{n-k+1}}{k} k\alpha_k \partial N_{\bar{y}} = (-1)^{n-k+1} \alpha_k \partial N_{\bar{y}}$$

It remains to prove the claim

$$d(u^\# \tilde{\omega}) = d(u^\#(\rho\tilde{\omega}))$$

where $\tilde{\omega}$ is the standard volume form on S^{k-1}, and ρ is a function on S^{k-1} s.t. $\fint_{S^{k-1}} \rho = 1$, and $u \in W^{1,k-1}(\Omega, S^{k-1})$

That is, we must show that the following form is null:

$$\begin{aligned} d(u^\#\tilde{\omega}) - d(u^\#(\rho\tilde{\omega})) &= d(u^\#\tilde{\omega} - u^\#(\rho\tilde{\omega})) \\ &= d(u^\#((1-\rho)\tilde{\omega})) \end{aligned}$$

$\langle$ But $(1-\rho)\tilde{\omega}$ is exact, that is $\underset{\text{ON } S^{k-1}}{(1-\rho)\tilde{\omega} = d\alpha}$ $\rangle$

$$\begin{aligned} &= d(u^\#(d\alpha)) \\ &= d(d(u^\#\alpha)) \\ &= 0 \end{aligned}$$

Follows by the fact that $(k-1)$-th cohomology group of S^{k-1} (and of any connected $(k-1)$ dim. manifold) is $\mathbb{R}$, and the identification is given by the integral....

ALTERNATIVE (FALSE!) PROOF:

$$d(u^\#((1-\rho)\tilde{\omega})) = u^\#(d((1-\rho)\tilde{\omega})) = u^\#(0) = 0$$

the differential of a $(k-1)$-form on any $(k-1)$-dimensional manifold must be 0.

Why is it false?

FIRST IDENTITY REQUIRES $u \in W^{1,k}$!

(FEW) APPLICATIONS

Lifting Sobolev maps

Let $u : \Omega \subset \mathbb{R}^n \to S^1$, $u \in W^{1,p}$ $(p \geq 1)$

Is there $\theta : \Omega \to \mathbb{R}$, $\theta \in W^{1,p}$ such that

$$e^{2\pi i\theta} = u \quad ?$$

This is known as "lifting problem", see Demengel, Bourgain – Brezis – Mironescu, others...

Solving this problem implies a solution of the approximation problem:

Let $u : \Omega \to S^1$, $u \in W^{1,p}$

Is there $u_n : \Omega \to S^1$, u_n of class C^1 (or C^∞) s.t. $u_n \to u$ in norm ?

In other words: is $W^{1,p}$ the completion of regular maps?

Answer:

If u can be lifted, then $Ju = 0$.

The converse is true when Ω is simply connected.

In particular there is always lifting (and approximation) for $p \geq 2$.

The case $p > k$ would have been trivial.

THE PROOF IS SIMPLE, AND WILL BE GIVEN....

Preliminary observation

Every $u \in W^{1,p}(\Omega, S^1)$ can be written as

$$u = e^{2\pi i \theta}$$

with $\theta \in W^{1,p}(\Omega, \mathbb{R}/\mathbb{Z})$ ← [Nothing deep: we have just changed the representation of S^1]

More precisely

$$2\pi\, d\theta = \overbrace{u_1 du_2 - u_2 du_1 = -i\bar{u}\, du}^{\text{complex form}}$$

and since $|u| = 1$

$$2\pi\, |\nabla \theta| = |\nabla u|$$

Moreover $2\pi\, d\theta = ju$ hence $\underbrace{\pi\, d(d\theta) = Ju}_{}$

is not 0 because θ is NOT a zero-form (it is not valued in $\mathbb{R}$)

PROOF OF THE THEOREM.

If $u = e^{2\pi i\theta}$ with $\underbrace{\theta \in W^{1,p}(\Omega, \mathbb{R})}_{\text{valued in } \mathbb{R}}$

Then $Ju = \pi\, d(d\theta) = 0$.

On the other hand, $Ju = 0$ implies that the 1-form $d\theta$ is exact, that is $d\theta = d\tilde{\theta}$ for some $\mathbb{R}$-valued $\tilde{\theta}$.

Up to some constant, $\tilde{\theta}$ provides the lifting of u.

Remarks

The problem of Lifting has been studied by Bourgain, Brezis, Mironescu in particular for maps in fractional Sobolev spaces of maps from $\Omega \to S^1$

Note: the Jacobian can be defined even for $u \in H^{1/2}(\Omega, \mathbb{R}^2)$, and turns out to be the only obstruction to lifting [see: Han-Lin, Rivière, ...]

The problem of approximation for maps $u \in W^{1,p}(B, M)$ has been studied by Bethuel

(B: unit ball in $\mathbb{R}^n$; M: k-dimensional manifold)

(characterisation of all p, M such that approx. exists).

Further studies by many others (Brezis-Li, Lin, ...)

For $u \in W^{1,p}(B^n, S^{k-1})$, $k-1 \le p < k$, the Jacobian is the obstruction to approximation....

Example: $W^{1,p}(B^k, S^{k-1})$: ....

ONE FINAL QUESTION

Given M $(n-k)$-dimensional manifold (ORIENTED!) without boundary in $\mathbb{R}^n$ (more generally: $(n-k)$-polyhedral current)

is there $u : \mathbb{R}^n \to S^{k-1}$ smooth outside M such that

$$\star \quad Ju = \alpha_k M \quad ?$$

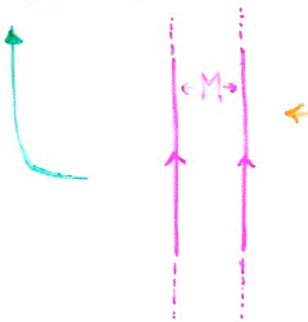
more precisely

ANSWER [Alberti-Baldo-Orlandi]: $\alpha_k [M, \tau_M, 1]$

YES FOR $K = 2$

YES (AND NO) FOR $K > 2$ ← One must allow for an additional singularity (besides M) of dimension $n-k-1$!

NO FOR $K = 1$



← this 1-dimensional oriented manifold in $\mathbb{R}^2$ is not a boundary

Hence cannot be the Jacobian (derivative) of a map $u : \mathbb{R}^2 \to \{\pm 1\}$

Similar question:

GIVEN M $(n-k)$-dimensional integral boundary in $\Omega \subset \mathbb{R}^n$ (or even M = bdry of integral multiplicity rectifiable current), is there $u \in W^{1,k-1}(\Omega, S^{k-1})$ s.t.

$$\star \quad Ju = \alpha_k M$$

ANSWER [A-B-O]: YES FOR $K \geq 2$!

AN INTERESTING GEOMETRIC CONSTRUCTION FOR $K = 2$

Given M smooth, oriented, without boundary in $\mathbb{R}^n$ and dimension $n-2$, we want to find $\theta \in W^{1,1}(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$ smooth outside M such that

(1) $$\star \, d(d\theta) = [M, \tau_M, 1]$$

[The desired map u will be $u := e^{2\pi i \theta}$]

Setting $\underbrace{\omega := d\theta}_{\text{1-form}}$, equation (1) becomes

(2) $$\int_\gamma \omega \cdot \underbrace{\tau_\gamma}_{\text{tangent to } \gamma} = \mathrm{Link}(M, \gamma)$$ ← Linking number of M and γ (measures how many times γ winds around M).

for every curve γ in $\mathbb{R}^n \setminus M$

WE JUST CONSTRUCT ω SATISFYING (2). IT WILL (AUTOMATICALLY!) BE OF THE FORM $\omega = d\theta$.

By definition

$$\mathrm{Link}(M, \gamma) = \deg(\Phi; M \times \gamma, S^{k-1}) =$$

$\Phi(x,y) = \frac{x-y}{|x-y|}$

$$= \int_{M \times \gamma} \Phi^\# \tilde{\omega}$$

standard volume form on S^{k-1}

$$= \int_\gamma \Big[\underbrace{\int_M (\Phi^\# \tilde{\omega})(x,y) \cdot \tau_M(x)}_{(k-1)\text{-form applied to a } (k-2)\text{-vector } = \text{ 1-form!}} \Big] \cdot \tau_\gamma(y)$$

HENCE IT SUFFICES TO SET

$$\omega(y) = \int_M (\Phi^\# \tilde{\omega})(x,y) \cdot \tau_M(x)$$

References (minimal list).

Theory of Sobolev functions (and BV functions):

Ziemer : Weakly Differentiable functions

Evans – Gariepy : Measure Theory and

Giusti : Minimal Surfaces and BV functions

Ambrosio – Fusco – Pallara
Bounded Variations and free Discontinuity problems

← SBV
Mumford Shah functional

<Ambrosio : Survey paper on Proc. III ECM.>

Gamma-convergence, approximation of the area functional by Scalar G.L. functionals:

Braides : Γ-convergence for beginners

<G.A. : Variational models for phase-separation: an approach via Γ-convergence>

Survey paper: see my webpage

DISTRIBUTIONAL JACOBIANS:

Jerrard – Soner : Functions of bounded Higher Variation [INDIANA]

G. A. – Baldo – Orlandi : Maps with prescribed singularites [JEMS]