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The calibration method for the Mumford-Shah functional and free-discontinuity problems

[etended version]

Abstract. We present a minimality criterion for the Mumford-Shah functional, and more generally for non convex variational integrals on SBV which couple a surface and a bulk term. This method provides short and easy proofs for several minimality results.

Keywords: Mumford-Shah functional, free-discontinuity problems, special functions of bounded variation, necessary condition for minimality, calibrations, minimal partitions, gradient flow.

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1. Introduction

The Mumford-Shah functional was introduced in [33] and [34] within the context of a variational approach to image segmentation problems (cf. [34] and [29], Chapter 4). In dimension n it can be written as follows

$$F(u) := \int_{\Omega \setminus Su} |\nabla u|^2 \, dx + \alpha \mathscr{H}^{n-1}(Su) + \beta \int_\Omega (u-g)^2 \, dx \ , \tag{1.1}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$, $g : \Omega \to [0,1]$ is a given function (input grey level), α and β are positive (tuning) parameters, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure—namely the usual $(n-1)$-dimensional area in case of subsets of regular hypersurfaces. The unknown function $u : \Omega \to \mathbb{R}$ is regular out of a closed singular set Su, whose shape and location are not prescribed; thus minimizing F means optimizing the function u and the singular set Su.
The existence of minimizers of F in dimension $n = 2$ was proved directly by considering the closed set Su as the main independent variable ([34], see also [17], and [29], Chapter 15), while in general dimension it was obtained by first defining F on the space $SBV(\Omega)$ of special functions with bounded variation (see Sect. 2), where the existence of (weak) minimizers could be proved by semicontinuity and compactness (see [3] and [7], Chapter 5), and then recovering a solution of the

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original problem by a suitable regularity result ([19], see also [7], Chapter 6). Moreover the existence result in [3] applies to a large class of functionals which occur in the modeling of a wide range of phenomena, from image segmentation, to fractures in brittle materials, to nematic liquid crystals (see [7], Sect. 4.6, for a survey), and therefore the Mumford-Shah functional should be regarded as the prototypical example of functional coupling bulk and surface contributions.

On a mathematical level, one of the most relevant features of F is a deep lack of convexity. Hence, not only minimizers may be not unique, but "identifying" them is by no means an easy task, also in terms of efficient algorithms. Clearly, every minimizer u satisfies certain equilibrium conditions which can be obtained by considering different types of infinitesimal variations (see [34] or [7], Sect. 7.4): for instance u must satisfy $\Delta u = \beta(u-g)$ in the complement of the singular set Su, the normal derivative of u on Su vanishes (where Su is a regular surface), while the mean curvature of Su multiplied by α is equal to the difference of the energy densities $|\nabla u|^2 + \beta(u-g)^2$ on the two sides of Su. However, due to the lack of convexity of F, these conditions do not imply minimality—not even local minimality.

In this paper we propose a sufficient condition for minimality, and describe some applications. To explain the idea behind this principle, we restrict our attention to the homogeneous Mumford-Shah functional, which is obtained by setting $\alpha = 1$ and $\beta = 0$ in (1.1), namely

$$F_0(u) := \int_{\Omega\setminus Su} |\nabla u|^2\,dx + \mathscr{H}^{n-1}(Su)\ . \tag{1.2}$$

We assume now that u and Su are sufficiently regular, and denote by u^+ and u^- the traces of u at the two sides of Su (so that $u^+ > u^-$) and by ν_u the unit normal to Su which points from the side of u^- to that of u^+. The complete graph of u, denoted by Γu, is the boundary of the subgraph of u, and is oriented by the inner normal $\nu_{\Gamma u}$. Thus Γu consists of the union of the usual graph of u, where $\nu_{\Gamma u} = (|\nabla u|^2+1)^{-1/2}(\nabla u, -1)$, and an additional "vertical part" given by the union of all segments with endpoints $(x, u^-(x))$ and $(x, u^+(x))$ with $x \in Su$, where $\nu_{\Gamma u} = (\nu_u, 0)$.

Now we look for vectorfields $\phi = (\phi^x, \phi^t)$ on $\Omega\times\mathbb{R}$ such that $F_0(u)$ is larger than or equal to the flux of ϕ through Γu for every u, that is

$$F_0(u) \geq \int_{\Gamma u} \phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n\ . \tag{1.3}$$

Since the right-hand side of (1.3) can be re-written as

$$\int_\Omega \big[\phi^x(x,u)\cdot\nabla u - \phi^t(x,u)\big]\,dx + \int_{Su}\Big[\int_{u^-}^{u^+}\phi^x(x,t)\,dt\Big]\cdot\nu_u\,d\mathscr{H}^{n-1}\ , \tag{1.4}$$

the term $\int|\nabla u|^2$ is larger than the first integral in (1.4) when ϕ satisfies $|\xi|^2 \geq \phi^x(x,t)\cdot\xi - \phi^t(x,t)$ for every $(x,t)\in\Omega\times\mathbb{R}$ and $\xi\in\mathbb{R}^n$, or, equivalently, when

(a) $|\phi^x(x,t)|^2 \leq 4\phi^t(x,t)$ for $x\in\Omega$, $t\in\mathbb{R}$.

On the other hand $\mathscr{H}^{n-1}(Su)$ is larger than the second integral in (1.4) when

(b) $\displaystyle\Big|\int_{t_1}^{t_2}\phi^x(x,t)\,dt\Big| \leq 1$ for $x\in\Omega$, $t_1, t_2\in\mathbb{R}$.

Moreover equality holds in (1.3) for a particular u if

(a') $\phi^x(x,u(x)) = 2\nabla u(x)$ and $\phi^t(x,u(x)) = |\nabla u(x)|^2$ for $x\in\Omega\setminus Su$,

(b') $\displaystyle\int_{u^-(x)}^{u^+(x)}\phi^x(x,t)\,dt = \nu_u(x)$ for $x\in Su$.

Let now be given a function u, and assume that there exists a vectorfield ϕ which is divergence-free and satisfies assumptions (a), (b), (a'), and (b') above. Then for every function v which agrees with u on the boundary of Ω there holds

$$F_0(v) \geq \int_{\Gamma v}\phi\cdot\nu_{\Gamma v}\,d\mathscr{H}^n = \int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n = F_0(u)\ , \tag{1.5}$$

where the first equality follows from the divergence theorem, since ϕ is divergence-free and Γu and Γv have the same boundary. Hence the minimality of a given function u can be proved by constructing such a ϕ.

We call the vectorfield ϕ a *calibration* for u, and indeed our minimality criterion closely resembles the classical principle of calibrations for minimal hypersurfaces (see the survey [30] for further references). However, functionals of Mumford-Shah type are not included in the general theory of calibrations (see [22]) because they cannot be written as integrals over the complete graph of u: the novelty of our approach consists in introducing suitable non-local constraints (namely, condition (b)) to define the class of admissible vectorfields.

In Sect. 3 we expand the idea outlined above and develop the principle of calibrations for minimizers of F and more general functionals (Theorems 3.3 and 3.8) with or without prescribed boundary values. In particular we recover the principle of *paired calibrations* for minimal partitions introduced in [10] and [31] (in fact, a slight generalization of it—cf. Theorem 3.9 and Remark 3.11).

However, even though the principle is relatively simple to understand, to construct a calibration for a given u may be very difficult. As a matter of fact, we do not know of any general method of construction, not even for minimal surfaces. Instead, we have collected in Sects. 4 and 5 many examples of calibrations for F_0 and F, and gathered some helpful remarks and observations. Despite the lack of a general recipe, we can give short and easy proofs of several minimality results, among which we recall the following:

(1) every harmonic function minimizes F_0 when the gradient is sufficiently small (Paragraph 4.6);

(2) a function which is constant on each element of a minimal partition of the domain is a minimizer of F_0 when the values are sufficiently far apart from each other; in particular this applies to the so-called triple junction (Paragraphs 4.8 and 4.9);

(3) every solution of the equation $-\Delta u + \beta(u-g) = 0$ with Neumann boundary conditions is a minimizer of F for large β (Paragraph 5.3);

(4) if g is the characteristic function of a regular set, then $u := g$ minimizes F for large β (Paragraph 5.4).

Notice that (3) and (4) give a strong indication that for initial data with smooth singular sets the gradient flow associated with F_0 in the L^2-metric (defined via time discretization, cf. [14], [25]) leaves the singular set still, at least for small times, and agrees with the heat flow elsewhere. Partial results in this direction are given in Remark 5.7 and Paragraph 5.10; see [24] for a proof in the one-dimensional case (although with a slightly different definition of the gradient flow).

Finally, we wish to recall the so-called "cracktip" conjecture: the function u on the plane given in polar coordinates by $u := \sqrt{2\rho/\pi}\,\sin(\theta/2)$, with $-\pi < \theta \le \pi$, minimizes the homogeneous Mumford-Shah functional F_0 among all functions on the plane which agree with u outside a bounded set. This conjecture has been recently proved in [8], but so far no calibration has been found.

The rest of the paper is organized as follows: in Sect. 2 we recall the basic notation about finite perimeter sets and the space SBV, which is indeed the natural setting for our theory. However, under most regards the unfamiliar reader can just replace the word SBV with "smooth out of a piecewise smooth singular set", and "finite perimeter" with "piecewise smooth boundary". We conclude the section with a rather general form of the divergence theorem, which allows our theory to include also discontinuous calibrations. The proof of this and other technical results is postponed to the Appendix.

Some of the results included here were announced in [2] and [15]. Further applications of the calibration method can be found in [16], [28], [32], [9].

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2. Notation and preliminary results

Throughout this paper, sets and functions are always assumed to be Borel measurable, and we *do not* identify functions which agree almost everywhere. A *vectorfield* on a subset E of $\mathbb{R}^n$ is a map from E into $\mathbb{R}^n$, and its divergence is always intended in the sense of distributions (relative to the interior of its domain). Ω is a (possibly unbounded) open subset of $\mathbb{R}^n$, $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $\mathscr{H}^k$ stands for the k-dimensional Hausdorff measure and $\mathscr{L}^n$ for the n-dimensional Lebesgue measure (but in integrals we write dx instead of $d\mathscr{L}^n$). The characteristic function of a set E is the function 1_E which takes value 1 in E and 0 outside. The restriction of any Borel measure μ to a set E is denoted by $\mu \llcorner E$, while $g \cdot \mu$ is the (vector) measure canonically associated with any μ-summable (vector) function g. A (vector) function f on $\mathbb{R}^n$ has *approximate limit* a at x, and we write $\operatorname{ap}\lim_{y\to x} f(y) = a$, if

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x,r)} |f(y) - a|\, dy = 0 \tag{2.1}$$

(when f is defined on $E \subset \mathbb{R}^n$, $B(x,r)$ must be replaced by $B(x,r) \cap E$). Note that this definition slightly differs from the usual one, given in terms of the density at x of the pre-images of neighbourhoods of a.

We recall now some notation and basic facts about finite perimeter sets, BV and SBV functions; for more precise definitions and a detailed account of the results we refer to [7], Chapters 3 and 4.

The space $BV(\Omega)$ consists of all real functions $u \in L^1(\Omega)$ whose distributional gradient Du is (represented by) a bounded vector measure on Ω; to simplify the notation we denote the integral of an $\mathbb{R}^n$-valued function f with respect to Du by $\int_\Omega f \cdot Du$. We recall that when Ω has Lipschitz boundary, then u admits a *trace* on the boundary (in the approximate sense), still denoted by u.

The *singular set* Su consists of all points where u has no approximate limit; if u is a BV function then Su is an $(n-1)$-dimensional *rectifiable* set, which means that it can be covered, up to an $\mathscr{H}^{n-1}$-negligible subset, by countably many hypersurfaces of class C^1. Thus, for $\mathscr{H}^{n-1}$-almost every $x \in Su$, there exist the approximate normal $\nu_u(x) \in \mathbb{S}^{n-1}$ and the traces $u^+(x)$ and $u^-(x)$ of u on the two sides of Su, namely the approximate limits of the restrictions of u to the two half-spaces defined by the (approximate) tangent hyperplane of Su at x. We arrange so that $u^+(x) > u^-(x)$, and $\nu_u(x)$ is pointing from the side of $u^-(x)$ to that of $u^+(x)$.

The measure Du can be canonically decomposed as the sum of three mutually orthogonal measures: the *Lebesgue part* $\nabla u \cdot \mathscr{L}^n$, where ∇u is the approximate gradient of u, the *jump part* $(u^+ - u^-)\,\nu_u \cdot \mathscr{H}^{n-1} \llcorner Su$, and a remainder, called *Cantor part*, which is singular but does not charge any $\mathscr{H}^{n-1}$-finite set; the space $SBV(\Omega)$ consists of all functions $u \in BV(\Omega)$ whose distributional gradient has no Cantor part. A subset E of Ω has *finite perimeter* (in Ω) if the distributional derivative of its characteristic function 1_E is a bounded vector measure on Ω; the *measure theoretic boundary* $\partial_* E$ is the singular set of 1_E, while the inner normal $\nu_{\partial_* E}$ is the associated normal vectorfield ν_{1_E}. The distributional gradient of 1_E has no Lebesgue and no Cantor part, that is

$$D1_E = \nu_{\partial_* E} \cdot \mathscr{H}^{n-1} \llcorner \partial_* E\ . \tag{2.2}$$

A general form of the divergence theorem

When looking for calibrations for a given function, it is often convenient to consider also vectorfields which are not regular. In doing so, however, we face some technical difficulties.

First of all, the first identity in (1.5) depends on the divergence theorem, and may not hold when ϕ is divergence-free (in the sense of distributions) but not continuous, because the flux of such a vectorfield through a given surface is not well-defined. To solve this problem, we must assume a certain regularity in ϕ.

Definition 2.1. *A vectorfield ϕ on a subset E of $\mathbb{R}^n$ is* approximately regular *if it is bounded, and for every Lipschitz hypersurface M in $\mathbb{R}^n$ there holds*

$$\operatorname*{ap\,lim}_{y\to x}\big[\phi(y)\cdot\nu_M(x)\big]=\phi(x)\cdot\nu_M(x)\quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x\in M\cap E, \tag{2.3}$$

where $\nu_M(x)$ is the normal to M at x.

Remark 2.2. If ϕ is approximately regular, then (2.3) can be extended to every rectifiable set M, ν_M being now understood in the approximate sense. If ϕ admits traces ϕ^+ and ϕ^- on the two sides of M (in the approximate sense), then (2.3) is equivalent to the compatibility condition

$$\phi\cdot\nu_M=\phi^+\cdot\nu_M=\phi^-\cdot\nu_M \quad \mathscr{H}^{n-1}\text{-a.e. in } M\cap E, \tag{2.4}$$

which links the pointwise values of ϕ on M with the values of the traces. In particular, if ϕ is (approximately) continuous $\mathscr{H}^{n-1}$-almost everywhere on E, then it is approximately regular. Similarly, if ϕ is (approximately) continuous on the complement of a rectifiable set S, then ϕ is approximately regular if and only if (2.3) holds for $M:=S$.

Remark 2.3. If ϕ has the special form $\phi:=(0,\dots,0,\psi)$ where $\psi=\psi(x_1,\dots,x_n)$ is a bounded real function which is continuous in the variable x_n, then ϕ is approximately regular. Take indeed a Lipschitz surface M, and let M_0 be the subset of all points $x\in M$ such that the n-th component of $\nu_M(x)$ vanishes. Then equality (2.3) obviously holds for all $x\in M_0$. To prove that it also holds for $\mathscr{H}^{n-1}$-a.e. point in $M\setminus M_0$ it suffices to notice that ψ is approximately continuous in the complement of a set of type $N\times\mathbb{R}$, where N is a $\mathscr{L}^{n-1}$-negligible subset of $\mathbb{R}^{n-1}$. Thus ϕ is approximately continuous at all points except those in $N\times\mathbb{R}$, which form an $\mathscr{H}^{n-1}$-negligible subset of $M\setminus M_0$ by the area formula (see [21], Theorem 3.2.22).

We can now state a refined version of the classical divergence theorem (the proof is postponed to the Appendix).

Lemma 2.4. *Let Ω be an open set in $\mathbb{R}^n$ with Lipschitz boundary and inner normal $\nu_{\partial\Omega}$, ϕ an approximately regular vectorfield on $\overline{\Omega}$, and u a function in $BV(\Omega)$. Assume moreover that* $\operatorname{div}\phi\in L^\infty(\Omega)$ *and $u\phi\in L^1(\partial\Omega,\mathscr{H}^{n-1})$. Then*

$$\int_\Omega \phi\cdot Du=-\int_\Omega u\operatorname{div}\phi\,dx-\int_{\partial\Omega}u\,\phi\cdot\nu_{\partial\Omega}\,d\mathscr{H}^{n-1}\,, \tag{2.5}$$

Notice that $u\phi$ always belongs to $L^1(\partial\Omega,\mathscr{H}^{n-1})$ when $\partial\Omega$ is bounded, because in this case the trace of u on $\partial\Omega$ belongs to $L^1(\partial\Omega,\mathscr{H}^{n-1})$.

Remark. As shown in [36], for a Borel vectorfield ϕ with divergence in L^1 it is possible to give a functional definition of the trace of the normal component of ϕ on any Lipschitz surface M, and precisely as a continuous extension of the trace operator for regular vectorfields. When M is the boundary (of some set Ω), this notion of trace automatically satisfies the divergence theorem (in the sense that formula (2.5) holds for every function u of class C^1_c), but since modifying ϕ in a Lebesgue-negligible set affects neither this trace nor the distributional divergence, one can easily produce examples where the trace does not agree with the normal component of ϕ on M. Thus Lemma 2.4 shows implicitly that this is never the case when ϕ is approximately regular. One may wonder if every vectorfield with divergence in L^∞ agrees, up to a Lebesgue-negligible set, with an approximately regular one. Unfortunately the answer is negative, even for divergence-free vectorfields (cf. [36], example after Proposition 2.1).

Going back to the first identity in (1.5), we remark that verifying that a vectorfield ϕ is divergence-free is relatively easy when ϕ is of class C^1 because the distributional divergence agrees with the classical one, which can be explicitly computed. If ϕ is piecewise C^1, the task is slightly more difficult, and can be carried out in many concrete cases with the help of the following lemma (the proof is postponed to the Appendix).

Lemma 2.5. *Let ϕ be a bounded vectorfield on an open set $\Omega\subset\mathbb{R}^n$, and assume that there exist a closed set S and a function $f\in L^1_{\mathrm{loc}}(\Omega)$ such that $\operatorname{div}\phi=f$ in the sense of distributions on $\Omega\setminus S$. Then the identity $\operatorname{div}\phi=f$ holds also on Ω if S can be written as $S:=S_0\cup S_1$, with S_0 an $\mathscr{H}^{n-1}$-negligible closed set and S_1 a (possibly disconnected) Lipschitz hypersurface, and ϕ satisfies* (2.3) *for $M:=S_1$ and $E:=\Omega$.*

The point of this lemma is roughly the following: since the divergence is a first order differential operator, $\operatorname{div}\phi$ cannot “charge” any set of codimension larger than 1, and therefore S_0 can be safely removed. On the other hand, the part of $\operatorname{div}\phi$ supported on the hypersurface M is given by the difference of the traces (whenever defined) of the normal components of ϕ on the two sides of M, which happens to vanish if (2.3) holds, and then we are allowed to neglect S_1, too.

Remark 2.6. Lemma 2.5 will be often applied in one of the following forms.
(a) Suppose that ϕ is a bounded vectorfield on $\overline{\Omega}$, continuous on $\overline{\Omega}\setminus(S_0\cup S_1)$, and divergence-free on $\Omega\setminus(S_0\cup S_1)$, with S_0 and S_1 given as above. If ϕ satisfies (2.3) with $M=S_1$, then ϕ is approximately regular on $\overline{\Omega}$ and divergence-free on Ω (cf. Remark 2.2).
(b) Let be given finitely many pairwise disjoint Lipschitz open sets Ω_j whose closures cover $\overline{\Omega}$, and approximately regular, divergence-free vectorfields ϕ_j on $\overline{\Omega}_j$. Let ϕ be any vectorfield on $\overline{\Omega}$ which agrees at any point with one of the ϕ_i (hence ϕ is uniquely determined at least on the union of all Ω_i). Then ϕ is approximately regular and divergence-free provided that the vectorfields ϕ_i satisfy the compatibility conditions

$$\phi_i\cdot\nu_{\partial\Omega_i}=\phi_j\cdot\nu_{\partial\Omega_j}\quad \mathscr{H}^{n-1}\text{-a.e. on } \partial\Omega_i\cap\partial\Omega_j,$$

which are equivalent to condition (2.4) for ϕ.

The complete graph of an SBV function

We fix now some notation and state some results which are more specific to this paper. In the following Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and inner unit normal $\nu_{\partial\Omega}$; U is an open subset of $\Omega\times\mathbb{R}$ with Lipschitz boundary whose closure can be written as

$$\overline{U} := \big\{(x,t)\in\overline{\Omega}\times\mathbb{R}:\ \tau_1(x)\le t\le\tau_2(x)\big\}\ , \tag{2.6}$$

where the functions $\tau_1,\tau_2:\overline{\Omega}\to[-\infty,+\infty]$ satisfy $\tau_1<\tau_2$. The letters x and t denote the variables in Ω (or $\mathbb{R}^n$) and $\mathbb{R}$, respectively; ϕ is a bounded vectorfield defined on a subset of $\mathbb{R}^n\times\mathbb{R}$, with components $\phi^x\in\mathbb{R}^n$ and $\phi^t\in\mathbb{R}$. Thus $\mathrm{div}\phi=\mathrm{div}_x\phi^x+\partial_t\phi^t$, where div_x is the (distributional) divergence with respect to x and ∂_t the (distributional) derivative with respect to t.

Definition 2.7. *Given $u\in BV(\Omega)$, 1_u is the characteristic function of the subgraph of u in $\Omega\times\mathbb{R}$, namely $1_u(x,t):=1$ for $t\le u(x)$ and $1_u(x,t):=0$ for $t>u(x)$, while the* complete graph *of u, denoted by Γu, is the measure theoretic boundary of the subgraph of u, i.e., the singular set of 1_u.*

Since the subgraph of u has finite perimeter in $\Omega\times\mathbb{R}$ (see, e.g., [27], Proposition 1.4), the definition of Γu is well-posed. Moreover $D1_u=\nu_{\Gamma_u}\cdot\mathscr{H}^n\llcorner\Gamma u$ (cf. (2.2)), where ν_{Γ_u} is the inner normal of the subgraph of u, and the flux through Γu of a vectorfield ϕ on $\Omega\times\mathbb{R}$ is

$$\int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n=\int_{\Omega\times\mathbb{R}}\phi\cdot D1_u\ . \tag{2.7}$$

An alternative way to compute this flux is given in the following lemma (the proof is postponed to the Appendix).

Lemma 2.8. *Let u be a function in $SBV(\Omega)$ and let ϕ be a vectorfield defined at least on Γu. Then*

$$\begin{aligned}\int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n&=\int_\Omega\big[\phi^x(x,u)\cdot\nabla u-\phi^t(x,u)\big]\,dx\\&\quad+\int_{Su}\Big[\int_{u^-}^{u^+}\phi^x(x,t)\,dt\Big]\cdot\nu_u\,d\mathscr{H}^{n-1}\ ,\end{aligned} \tag{2.8}$$

where u, $u^\pm$, ∇u, and ν_u are computed at x.

Formula (2.8) corresponds to a decomposition of the derivative of 1_u, or, better, to a decomposition of the complete graph Γu as union (up to $\mathscr{H}^n$-negligible sets) of a "regular" part which consists of all $\big(x,u(x)\big)$ such that u is approximately continuous at x and has approximate gradient $\nabla u(x)$, and a "vertical" part which consists of all (x,t) with $x\in Su$ and $t\in(u^-(x),u^+(x))$. Note that for a general BV function there would be an additional subset of Γu, corresponding to the Cantor part of Du.
The following version of the divergence theorem (cf. Lemma 2.4) yields the first equality in (1.5) (the proof is postponed to the Appendix).

Lemma 2.9. *Let be given functions u and v in $BV(\Omega)$ whose complete graphs lie in $\overline{U}$, and an approximately regular vectorfield ϕ on $\overline{U}$ which is divergence-free in U. Then*

$$\begin{aligned}\int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n-\int_{\Gamma v}\phi\cdot\nu_{\Gamma v}\,d\mathscr{H}^n&=\\=\int_{\partial\Omega}\Big[\int_u^v\phi^x(x,t)dt\Big]\cdot\nu_{\partial\Omega}\,d\mathscr{H}^{n-1}\ .\end{aligned} \tag{2.9}$$

3. Calibrations for free-discontinuity problems

In this section we expand the idea explained in the introduction and state the calibration principle for a large class of free-discontinuity problems. We begin with the Mumford-Shah functional, with or without lower order term, and then consider more general functionals, possibly with discontinuous integrands, which also include minimal partition problems.
Throughout this section Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, U is an open subset of $\Omega\times\mathbb{R}$ with Lipschitz boundary which satisfies (2.6), u is a function in $SBV(\Omega)$. The functionals $F(u)$ and $F_0(u)$ are given in (1.1) and (1.2), respectively, with Su and ∇u now defined as in Sect. 2, $\alpha>0$ and $\beta\ge0$ are fixed constants, and g belongs to $L^\infty(\Omega)$. In the following definition we fix some terminology about minimizers of F, or any other functional on SBV.

Definition 3.1. *A function u is an (absolute) minimizer of F if $F(u)\le F(v)$ for all $v\in SBV(\Omega)$, and is a Dirichlet minimizer if $F(u)\le F(v)$ for all $v\in SBV(\Omega)$ with the same trace on $\partial\Omega$ as u; u is a $\overline{U}$-minimizer if $\Gamma u\subset\overline{U}$ and $F(u)\le F(v)$ for all $v\in SBV(\Omega)$ with $\Gamma v\subset\overline{U}$, and is a $\overline{U}$-Dirichlet minimizer if we add the restriction that the competing functions v have the same trace on $\partial\Omega$ as u.*

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Lemma 3.2. *Let ϕ be a vectorfield on $\overline{U}$ which satisfies*

(a) $\phi^t(x,t)\ge\frac14|\phi^x(x,t)|^2-\beta(t-g)^2$ *for $\mathscr{L}^n$-a.e. $x\in\Omega$ and every $t\in[\tau_1,\tau_2]$,*

(b) $\Big|\displaystyle\int_{t_1}^{t_2}\phi^x(x,t)\,dt\Big|\le\alpha$ *for $\mathscr{H}^{n-1}$-a.e. $x\in\Omega$ and every $t_1,t_2\in[\tau_1,\tau_2]$,*

where the functions τ_1 and τ_2 are defined by (2.6) and, like g, are computed at x. Then for every u such that $\Gamma u\subset\overline{U}$ we have

$$F(u)\ge\int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n\ . \tag{3.1}$$

Moreover, equality holds in (3.1) for a given u if and only if

(a') $\phi^x(x,u)=2\nabla u$ *and* $\phi^t(x,u)=|\nabla u|^2-\beta(u-g)^2$ *for $\mathscr{L}^n$-a.e. $x\in\Omega$,*

(b') $\displaystyle\int_{u^-}^{u^+}\phi^x(x,t)\,dt=\alpha\,\nu_u$ *for $\mathscr{H}^{n-1}$-a.e. $x\in Su$,*

where u, $u^\pm$, ∇u, ν_u, and g are computed at x.

Proof. Take u such that $\Gamma u \subset \overline{U}$. We recall that by Lemma 2.8

$$\int_{\Gamma u} \phi \cdot \nu_{\Gamma u}\, d\mathscr{H}^n = \int_\Omega \big[\phi^x(x,u)\cdot\nabla u - \phi^t(x,u)\big]\, dx + \int_{Su} \Big[\int_{u^-}^{u^+} \phi^x(x,t)\, dt\Big]\cdot \nu_u \, d\mathscr{H}^{n-1}\ . \tag{3.2}$$

It is an elementary fact that for every $\xi,\, \eta \in \mathbb{R}^n$ we have $\xi\cdot\eta - \frac14|\xi|^2 \le |\eta|^2$, and equality holds if and only if $\xi = 2\eta$. Hence, setting $\xi := \phi^x(x,u)$ and $\eta := \nabla u$, and taking (a) into account, we obtain that

$$\begin{aligned}\phi^x(x,u)\cdot\nabla u - \phi^t(x,u) &\le \phi^x(x,u)\cdot\nabla u - \frac14|\phi^x(x,u)|^2 + \beta(u-g)^2\\ &\le |\nabla u|^2 + \beta(u-g)^2 \quad \mathscr{L}^n\text{-a.e. on } \Omega,\end{aligned}$$

and consequently

$$\int_\Omega \big[\phi^x(x,u)\cdot\nabla u - \phi^t(x,u)\big]\, dx \le \int_\Omega \big[|\nabla u|^2 + \beta(u-g)^2\big]\, dx\ . \tag{3.3}$$

Moreover, equality holds in (3.3) if and only if $\phi^x(x,u) = 2\nabla u$ and $\phi^t(x,u) = \frac14|\phi^x(x,u)|^2 - \beta(u-g)^2 = |\nabla u|^2 - \beta(u-g)^2$ for a.e. $x\in\Omega$, which is (a').
As for the second integral in the right-hand side of (3.2), condition (b) above implies

$$\Big[\int_{u^-}^{u^+} \phi^x(x,t)\, dt\Big]\cdot\nu_u \le \Big|\int_{u^-}^{u^+} \phi^x(x,t)\, dt\Big| \le \alpha \quad \mathscr{H}^{n-1}\text{-a.e. on } Su,$$

and then

$$\int_{Su} \Big[\int_{u^-}^{u^+} \phi^x(x,t)\, dt\Big]\cdot\nu_u\, d\mathscr{H}^{n-1} \le \alpha\mathscr{H}^{n-1}(Su)\ . \tag{3.4}$$

Moreover it is clear that equality holds in (3.4) if and only if (b') is satisfied. Inequality (3.1) follows now from (3.2), (3.3), and (3.4), as well as the rest of the statement. $\square$

Theorem 3.3. *Let u be a function with complete graph contained in $\overline{U}$, and assume that there exists an approximately regular vectorfield ϕ on $\overline{U}$ which is divergence-free on U and satisfies assumptions* (a), (b), (a'), *and* (b') *of Lemma 3.2. Then u is a Dirichlet $\overline{U}$-minimizer of F. If in addition the normal component of ϕ at the boundary of $\Omega\times\mathbb{R}$ vanishes, that is*

$$\phi^x \cdot \nu_{\partial\Omega} = 0 \quad \mathscr{H}^n\text{-a.e. on } (\partial\Omega\times\mathbb{R})\cap\partial U, \tag{3.5}$$

then u is also an absolute $\overline{U}$-minimizer of F.

Proof. Let v be a function in $SBV(\Omega)$ such that $v = u$ on $\partial\Omega$ and $\Gamma v \subset \overline{U}$. Then

$$F(v) \ge \int_{\Gamma v} \phi\cdot\nu_{\Gamma v}\, d\mathscr{H}^n = \int_{\Gamma u} \phi\cdot\nu_{\Gamma u}\, d\mathscr{H}^n = F(u)\ . \tag{3.6}$$

Here, the first inequality and the last equality follow from Lemma 3.2, while the first equality follows from Lemma 2.9. We have thus proved that u is a Dirichlet $\overline{U}$-minimizer of F. Moreover, assuming (3.5) we obtain that the first equality in (3.6) holds even if the traces of v and u on $\partial\Omega$ differ, which proves that u is an absolute $\overline{U}$-minimizer of F. $\square$

Definition 3.4. *The vectorfield ϕ in the first part of Theorem 3.3 is called a* Dirichlet calibration *for u on $\overline{U}$ (with respect to F). If ϕ satisfies the additional assumption* (3.5)*, then it is an* absolute calibration.

When $U := \Omega\times\mathbb{R}$ we omit to write it. When it is clear from the context, we may also omit to specify the functional, the set U, and whether the calibration is Dirichlet or absolute, and simply say that ϕ is a calibration for u, or that ϕ calibrates u.

Remark 3.5. If ϕ is an absolute calibration for u, then it is also an absolute calibration for every other minimizer. Indeed, if $F(v) = F(u)$, the first inequality in (3.6) must be an equality, and by Lemma 3.2 this means that ϕ satisfies assumptions (a') and (b') for v too. Similarly, if ϕ is a Dirichlet calibration for u, then it is also a Dirichlet calibration for any other Dirichlet minimizer with the same boundary values as u.
This fact can be sometimes used to prove that the minimizer is unique: for instance, if ϕ calibrates a function u with a negligible singular set (i.e., $\mathscr{H}^{n-1}(Su) = 0$), and the inequality in assumption (b) is always strict, then we deduce that assumption (b') can only be satisfied by functions with negligible singular sets, and therefore all minimizers should have this property. But on this class of functions F is strictly convex (for $\beta > 0$, and even for $\beta = 0$ in case of Dirichlet minimizers), and therefore the minimizer must be unique (see Remarks 4.7 and 5.2, and Paragraphs 5.3 and 5.4).

Remark 3.6. For $\alpha = 1$ and $\beta = 0$, i.e., for the homogeneous Mumford-Shah functional F_0 in (1.2), assumptions (a), (b), (a'), and (b') in Lemma 3.2 become

(a) $\phi^t(x,t) \ge \frac14|\phi^x(x,t)|^2$ for $\mathscr{L}^n$-a.e. $x\in\Omega$ and every $t\in[\tau_1,\tau_2]$,

(b) $\Big|\int_{t_1}^{t_2} \phi^x(x,t)\, dt\Big| \le 1$ for $\mathscr{H}^{n-1}$-a.e. $x\in\Omega$ and every $t_1, t_2\in[\tau_1,\tau_2]$,

(a') $\phi^x(x,u) = 2\nabla u$ and $\phi^t(x,u) = |\nabla u|^2$ for $\mathscr{L}^n$-a.e. $x\in\Omega$,

(b') $\int_{u^-}^{u^+} \phi^x(x,t)\, dt = \nu_u$ for $\mathscr{H}^{n-1}$-a.e. $x\in Su$.

Remark. It must be noticed that, given a boundary value w in $L^1(\partial\Omega,\mathscr{H}^{n-1})$, the Dirichlet problem

$$\min\Big\{F_0(u)\,:\, u\in SBV(\Omega),\ u = w\ \mathscr{H}^{n-1}\text{-a.e. on } \partial\Omega,\ \Gamma u\subset\overline{U}\Big\} \tag{3.7}$$

may not have a solution, even for a very regular w, due to a lack of continuity of the trace operator on $SBV(\Omega)$. Therefore problem (3.7) is usually replaced by the relaxed problem

$$\min\Big\{F_0(u)+\mathscr{H}^{n-1}\big(\{x\in\partial\Omega:\, u(x)\neq w(x)\}\big)\,:\, u\in SBV(\Omega),\,\Gamma u\subset\overline{U}\Big\}\,. \quad (3.8)$$

A variant of the standard lower semicontinuity and compactness theorems in $SBV(\Omega)$ (see [37]) shows that the problem (3.8) has always a solution. A calibration for the relaxed problem (3.8) is an approximately regular, divergence-free vectorfield ϕ on $\overline{\Omega}\times\mathbb{R}$ which satisfy conditions (a), (b), (a'), (b') of Remark 3.6 and the following two conditions, which may be understood as extensions of (b) and (b') to the boundary:

(c) $\Big|\int_w^s \phi^x(x,t)\,dt\Big|\leq 1$ for $\mathscr{H}^{n-1}$-a.e. $x\in\partial\Omega$ and every $s\in[\tau_1,\tau_2]$,

(c') $\int_w^u \phi^x(x,t)\,dt=\nu_{\partial\Omega}$ for $\mathscr{H}^{n-1}$-a.e. $x\in\partial\Omega$ with $u(x)\neq w(x)$,

where u, w, τ_1 and τ_2 are computed at x. We leave the verification to the reader.

Calibrations for general functionals

We consider now the functional

$$\Psi(u):=\int_\Omega f(x,u,\nabla u)\,dx+\int_{Su}\psi(x,u^-,u^+,\nu_u)\,d\mathscr{H}^{n-1}\,, \quad (3.9)$$

where $f:\Omega\times\mathbb{R}\times\mathbb{R}^n\to[0,+\infty]$ and $\psi:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\to[0,+\infty]$. We refer to [3] for general conditions on f and ψ which imply the lower semicontinuity of Ψ and guarantee the existence of minimizers (however, it should be clear by now that lower semicontinuity is irrelevant for the calibration method).

Let f^* and $\partial_\xi f$ denote the convex conjugate and the subdifferential of f with respect to the last variable. We recall that the subdifferential of $g:\mathbb{R}^n\to[0,+\infty]$ at $\xi\in\mathbb{R}^n$ is the set of vectors $\eta\in\mathbb{R}^n$ such that $g(\xi)+\eta\cdot(\zeta-\xi)\leq g(\zeta)$ for every $\zeta\in\mathbb{R}^n$; then we have $\xi\cdot\eta-g^*(\eta)\leq g(\xi)$ for every ξ, $\eta\in\mathbb{R}^n$, and equality holds if and only if $\eta\in\partial g(\xi)$. Using these properties we obtain the following generalizations of Lemma 3.2 and Theorem 3.3 (we omit the proofs).

Lemma 3.7. *Let ϕ be a vectorfield on $\overline{U}$ which satisfies*

(a) *$\phi^t(x,t)\geq f^*(x,t,\phi^x(x,t))$ for $\mathscr{L}^n$-a.e. $x\in\Omega$ and every $t\in[\tau_1,\tau_2]$,*

(b) *$\Big[\int_{t_1}^{t_2}\phi^x(x,t)\,dt\Big]\cdot\nu\leq\psi(x,t_1,t_2,\nu)$ for $\mathscr{H}^{n-1}$-a.e. $x\in\Omega$, every $\nu\in\mathbb{S}^{n-1}$ and $t_1<t_2$ in $[\tau_1,\tau_2]$.*

Then for every u with complete graph contained in $\overline{U}$ we have

$$\Psi(u)\geq\int_{\Gamma u}\phi\cdot\nu_{\Gamma u}\,d\mathscr{H}^n\,. \quad (3.10)$$

Moreover, equality holds in (3.10) *for a given u if and only if*

(a') *$\phi^x(x,u)\in\partial_\xi f(x,u,\nabla u)$ and $\phi^t(x,u)=f^*(x,u,\phi^x(x,u))$ for $\mathscr{L}^n$-a.e. $x\in\Omega$,*

(b') *$\Big[\int_{u^-}^{u^+}\phi^x(x,t)\,dt\Big]\cdot\nu_u=\psi(x,u^-,u^+,\nu_u)$ for $\mathscr{H}^{n-1}$-a.e. $x\in Su$,*

where u, $u^\pm$, ∇u, and ν_u are computed at x.

Theorem 3.8. *Let u be a function with complete graph contained in $\overline{U}$, and assume that there exists an approximately regular, divergence-free vectorfield ϕ on $\overline{U}$ which satisfies assumptions* (a), (b), (a'), *and* (b') *of Lemma 3.7. Then u is a Dirichlet $\overline{U}$-minimizer of Ψ. If, in addition, the normal component of ϕ on the boundary of $\Omega\times\mathbb{R}$ vanishes, i.e., if* (3.5) *holds, then u is also an absolute $\overline{U}$-minimizer of Ψ.*

Calibrations for minimal partitions

Through this subsection we fix an integer $m\geq 2$. A *partition* of Ω is an ordered sequence $(A_1,\dots,A_m)$ of pairwise disjoint finite perimeter sets, called *phases*, which cover Ω; for $i\neq j$ the *interface* S_{ij} between the phases A_i and A_j is the intersection of the corresponding measure theoretic boundaries, and is oriented ($\mathscr{H}^{n-1}$-a.e.) by the (approximate) normal ν_{ij} pointing from A_i to A_j. Now we consider functionals of the form

$$\mathscr{F}(A_1,\dots,A_m)=\sum_{i<j}\int_{S_{ij}}\psi_{ij}(x,\nu_{ij})\,d\mathscr{H}^{n-1}\,, \quad (3.11)$$

where $\psi_{ij}:\Omega\times\mathbb{S}^{n-1}\to[0,+\infty]$ (see [6]). A partition $(A_1,\dots,A_m)$ is a Dirichlet minimizer of $\mathscr{F}$ if it minimizes $\mathscr{F}$ among all partitions $(B_1,\dots,B_m)$ such that the characteristic functions of A_i and B_i have the same trace on $\partial\Omega$ for every i.

Theorem 3.9. *Let $(A_1,\dots,A_m)$ be a partition of Ω, and assume that there exist approximately regular vectorfields $\phi_1,\dots,\phi_m$ on $\overline{\Omega}$ with divergences in $L^\infty(\Omega)$ which satisfy*

(c) *$\mathrm{div}\phi_i\geq\mathrm{div}\phi_j$ $\mathscr{L}^n$-a.e. in A_i for every $i\neq j$,*

(d) *$\big(\phi_j(x)-\phi_i(x)\big)\cdot\nu\leq\psi_{ij}(x,\nu)$ for $\mathscr{H}^{n-1}$-a.e. $x\in\Omega$ and every $\nu\in\mathbb{S}^{n-1}$ and $i<j$,*

(d') *$\big(\phi_j(x)-\phi_i(x)\big)\cdot\nu_{ij}(x)=\psi_{ij}(x,\nu_{ij}(x))$ for $\mathscr{H}^{n-1}$-a.e. $x\in S_{ij}$ and every $i<j$.*

Then $(A_1,\dots,A_m)$ is a Dirichlet minimizer of $\mathscr{F}$.

Proof. We choose real numbers $a_1<\dots<a_m$ and associate to every partition $(A_1,\dots,A_m)$ the function u which agrees with a_i on each A_i. Thus u belongs to $SBV(\Omega)$, Su is the union of the interfaces S_{ij}, and $\nu_u=\nu_{ij}$ $\mathscr{H}^{n-1}$-a.e. on S_{ij} for every $i<j$. Now we define a functional Ψ of type (3.9) by setting

$$\begin{aligned} f(x,t,\xi)&:=\begin{cases}0 & \text{if } \xi=0 \text{ and } t\in\{a_1,\dots,a_m\},\\ +\infty & \text{otherwise,}\end{cases}\\ \psi(x,t_1,t_2,\nu)&:=\begin{cases}\psi_{ij}(x,\nu) & \text{if } t_1=a_i \text{ and } t_2=a_j \text{ for some } i<j,\\ +\infty & \text{otherwise.}\end{cases}\end{aligned} \quad (3.12)$$

One easily checks that $\Psi(u)$ is finite only if u is the function associated to some partition $(A_1, \dots, A_m)$, and in this case $\Psi(u) = \mathscr{F}(A_1, \dots, A_m)$. Hence $(A_1, \dots, A_m)$ is a Dirichlet minimizer for $\mathscr{F}$ if (and only if) the associated function u is a Dirichlet minimizer for Ψ, therefore it suffices to construct a calibration ϕ for u in the sense of Theorem 3.8.
We define ϕ on $\overline{\Omega} \times \mathbb{R}$ as follows: we take smooth non-negative functions σ_i with support included in (a_i, a_{i+1}) and integral equal to 1, and set

$$\phi^x(x,t) := \sigma_i(t)\big(\phi_{i+1}(x) - \phi_i(x)\big) \quad \text{for } x \in \overline{\Omega},\ a_i \le t \le a_{i+1},$$

then we choose ϕ^t so that ϕ is divergence free, that is,

$$\begin{cases} \phi^t(x, a_i) := 0 & \text{for } x \in A_i, \\ \partial_t \phi^t(x,t) := \sigma_i(t)\big(\mathrm{div}\phi_i(x) - \mathrm{div}\phi_{i+1}(x)\big) & \text{for } x \in \Omega,\ a_i \le t \le a_{i+1}; \end{cases}$$

the definition is completed by setting $\phi(x,t) := \phi(x, a_1)$ for $t < a_1$ and $\phi(x,t) := \phi(x, a_m)$ for $t > a_m$.
Thus ϕ is divergence-free in $\Omega \times \mathbb{R}$ by construction, and one can easily check that it satisfies assumptions (a), (a'), (b), (b') of Lemma 3.7, and precisely

(a) $\phi^t(x, a_i) \ge 0$ for $\mathscr{L}^n$-a.e. $x \in \Omega$ and every i,

(b) $\Big[\int_{a_i}^{a_j} \phi^x(x,t)\,dt\Big] \cdot \nu \le \psi_{ij}(x,\nu)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \Omega$ and every $\nu \in \mathbb{S}^{n-1}$ and $i < j$,

(a') $\phi^t(x, a_i) = 0$ for $\mathscr{L}^n$-a.e. $x \in A_i$ and every i,

(b') $\Big[\int_{a_i}^{a_j} \phi^x(x,t)\,dt\Big] \cdot \nu_{ij}(x) = \psi_{ij}(x, \nu_{ij}(x))$ for $\mathscr{H}^{n-1}$-a.e. $x \in S_{ij}$ and every $i < j$.

Moreover, since each ϕ_i is approximately regular on $\overline{\Omega}$, $(\phi^x, 0)$ is approximately regular on $\overline{\Omega} \times \mathbb{R}$, and the same holds for $(0, \phi^t)$ by Remark 2.3. Hence ϕ is approximately regular, too. □

Remark 3.10. In the proof of Theorem 3.9, to every m-uple $\phi_1, \dots, \phi_m$ satisfying assumptions (c), (d), (d') we have associated a calibration ϕ for the functional Ψ defined by (3.12). We remark that this connection works in both ways, and to every calibration ϕ for Ψ we can associate $\phi_1, \dots, \phi_m$ as in the statement of Theorem 3.9 by taking $\phi_i(x) := \int_{a_1}^{a_i} \phi^x(x,t)\,dt$. Thus the existence of such a familiy of vectorfields not only implies, but is equivalent to the existence of a calibration for Ψ.

Remark 3.11. A particularly relevant example of functional of type (3.11) is the "interface size", which is obtained by taking $\psi_{ij} \equiv 1$ for all $i < j$. In this case assumptions (d) and (d') above reduce to

(d) $|\phi_j(x) - \phi_i(x)| \le 1$ for every $x \in \Omega$ and every $i < j$,

(d') $\phi_j(x) - \phi_i(x) = \nu_{ij}(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in S_{ij}$ and every $i < j$.

Calibrations of this type have already been introduced in [10] and [31] as "paired calibrations" (see also [11], [12]). More precisely, a paired calibration for a partition $(A_1, \dots, A_m)$ is an ordered m-uple of divergence-free vectorfields $\phi_1, \dots, \phi_m$ on $\overline{\Omega}$ which satisfy assumptions (d) and (d') above. Notice that the assumption that the vectorfields ϕ_i are divergence-free is stronger than (c), and in fact Theorem 3.9 allows in principle for a larger class of calibrations.
Among other applications, in [31] it is shown that in any dimension n the partition of a regular simplex in $\mathbb{R}^n$ given by the $n+1$ simplices spanned by one face and the centre is a Dirichlet minimizer of the interface size (and the paired calibration consists simply of $n+1$ *constant* vectorfields which are orthogonal to the corresponding faces). For $n = 3$ this statement was first shown in [39] with a (relatively) long proof. In [10] it is shown that, unlike what happens in dimension 3, the partition of a hypercube in $\mathbb{R}^n$, $n \ge 4$, given by the $2n$ simplices spanned by one face and the centre is a Dirichlet minimizer of the interface size. In both papers the theory is also extended to cover more general functionals.

Additional remarks

We conclude this section with some sparse remarks about the calibration methods; we refer for simplicity to the functional F_0 in (1.2), and calibrations are defined as in Remark 3.6 for Dirichlet minimizers.

3.12. The *SBV* compactness theorem. Let $\mathscr{G}$ be the class of all vectorfields ϕ on $\Omega \times \mathbb{R}$, not necessarily bounded, which satisfy assumptions (a) and (b) of Remark 3.6. One can easily verify that for every u in $SBV(\Omega)$ there exists $\phi \in \mathscr{G}$ which satisfies assumptions (a') and (b') for u, so that equality holds in (3.1), that is, $F_0(u) = \int_{\Gamma u} \phi \cdot \nu_{\Gamma u}\, d\mathscr{H}^n$. Starting from this one, it is possible to construct vectorfields $\phi \in \mathscr{G}$ of class C^1_c such that the value of the right-hand side is arbitrarily close to $F_0(u)$, and therefore (cf.(2.7))

$$F_0(u) = \sup_{\phi \in \mathscr{G} \cap C^1_c} \int_{\Gamma u} \phi \cdot \nu_{\Gamma u}\, d\mathscr{H}^n = \sup_{\phi \in \mathscr{G} \cap C^1_c} \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u\ . \tag{3.13}$$

Moreover, for any function u which is in $BV(\Omega)$ but not in $SBV(\Omega)$, the last two terms in (3.13) are equal to $+\infty$. Since every integral of the form $\int \phi \cdot D1_u$, with ϕ of class C^1_c on $\Omega \times \mathbb{R}$, is continuous in u with respect to the weak* topology of $BV(\Omega)$, formula (3.13) shows that the Mumford-Shah functional F_0, extended to $+\infty$ in $BV \setminus SBV$, is weak* lower semicontinuous.
Under suitable hypotheses on f and ψ, a similar argument applies to the general functionals in (3.9) too, providing an alternative proof of the well-known compactness and semicontinuity results in SBV due to L. Ambrosio (see [3], or [7], Sects. 4.1 and 5.4).

3.13. Existence of calibrations. We briefly discuss here the following basic question: does every minimizer admit a calibration? In fact, in this section we have given different versions of a sufficient condition for minimality, but we do not know if it is actually fulfilled by any minimizer, and, what is even more relevant to applications, how to verify it, that is, how to construct a calibration.
Now let X be the class of all real functions $v \in L^1_{\mathrm{loc}}(\Omega \times \mathbb{R})$ whose gradient is a bounded measure, $\mathscr{G}$ the class of vectorfields defined in Paragraph 3.13, and for

every $v \in X$

$$G(v) := \sup_{\phi \in \mathscr{G} \cap C^1_c} \int_{\Omega \times \mathbb{R}} \phi \cdot Dv \ . \tag{3.14}$$

Thus $F_0(u) = G(1_u)$ by (3.13), and G, being the supremum of a family of linear functionals, is convex. Therefore, given $u \in SBV(\Omega)$, and denoting by X_u the affine space of all $v \in X$ which agree with 1_u on the boundary of $\Omega \times \mathbb{R}$, the function 1_u minimizes G on X_u if and only if the subdifferential of G at 1_u contains the zero element. Now, since the family of vectorfields $\mathscr{G} \cap C^1_c$ is convex, the subdifferentials of G at 1_u correspond to the class of all linear functionals $v \mapsto \int \phi \cdot Dv$ (with ϕ in the closure of $\mathscr{G} \cap C^1_c$ in a suitable abstract space) which agree with $G(1_u)$ at 1_u, that is, to the class of all ϕ which satisfy assumptions (a') and (b') of Remark 3.6 for u, while the zero element corresponds to functionals $v \mapsto \int \phi \cdot Dv$ which vanish whenever v vanishes at the boundary, that is, to vectorfields ϕ which are divergence-free.

In other words we have shown that u admits a generalized Dirichlet calibration (in a suitable abstract space, larger than the space of approximately regular vectorfields considered in this paper) if and only if 1_u is a Dirichlet minimizer of G. However, the point we want to make clear is that the fact that u is a Dirichlet minimizer of F implies that 1_u is a Dirichlet minimizer of G if and only if the infima of F and G (under the corresponding boundary constraints) agree; this has been proved in [13] for the one-dimensional case $n = 1$, but is not known in higher dimension.

The relations with minimal surfaces. A vectorfield ϕ is said to calibrate an oriented hypersurface S (with boundary) if it satisfies $|\phi| \le 1$ everywhere, it agrees on S with the normal vectorfield, and is divergence-free; the existence of a calibration implies that S minimizes the area among all oriented hypersurfaces with the same boundary, and the proof is just one line, like (1.5). In particular, the first assumption ensures the the flux of ϕ through any hypersurface is less than the area, while the second one ensures that it agrees with the area for S. So the first two assumptions play a rôle akin to that of assumptions (a), (b), (a'), and (b') in Lemma 3.2.

Notice that the area functional admits a natural extension from regular oriented surfaces (with fixed boundary) to the affine space of all normal currents (with same boundary), which is nothing else but the mass. Since the mass is convex, minimum points are characterized by the fact that the subdifferential of the mass contains the zero element, which turns out to be equivalent to the existence of a calibration with Borel coefficients (cf. [22], Proposition 4.10(3)). Thus a minimal surface admits a calibration if and only if it also minimizes the area (the mass) among all normal currents with same boundary, that is, if the infima of the area functional on the two classes coincide. Notice that this is always true for hypersurfaces (and curves) but may be no longer true in higher codimension (cf. [38] and [40]).

4. Applications to minimizers of F_0

In this section we give some examples of Dirichlet minimizers of the homogeneous Mumford-Shah functional F_0. We begin with a few remarks which may be useful when constructing calibrations.

Remark 4.1. By a simple truncation argument, to prove that a function $u : \Omega \to [m, M]$ is a (Dirichlet) minimizer for F_0 it suffices to show that $F_0(u) \le F_0(v)$ for all competitors v such that $m \le v \le M$. Thus it is enough to show that u is a Dirichlet $\overline{U}$-minimizer, with $U := \Omega \times (m, M)$. In the following we often tacitly assume this principle, and construct calibrations in $\overline{\Omega} \times [m, M]$ instead of $\overline{\Omega} \times \mathbb{R}$. The same conclusion holds for F if g satisfies $m \le g \le M$, too, but may fail for a functional of the general form (3.9), due to lack of suitable truncations.

Remark. Regarding the previous remark, notice that a calibration ϕ on $\overline{\Omega} \times [m, M]$ can be extended to $\overline{\Omega} \times \mathbb{R}$ in a rather simple way: it suffices to set $\phi(x, t) := \big(0, \phi^t(x, m)\big)$ for $t < m$ and $\phi(x, t) := \big(0, \phi^t(x, M)\big)$ for $t > M$ (cf. Remarks 2.3 and 2.6(b)).

Remark 4.2. We can construct divergence-free vectorfields on an open set $U \subset \Omega \times \mathbb{R}$ using fibrations of U by graphs of harmonic functions. This construction is a particular case of a classical result about extremal fields of scalar functionals (see, e.g., [1], Sect. 4). Given harmonic functions $\{u_\lambda\}$ whose graphs are pairwise disjoint and cover U, for all $(x, t) \in U$ we set

$$\phi(x, t) := \big(2\nabla u_\lambda(x), |\nabla u_\lambda(x)|^2\big) \ , \tag{4.1}$$

where $\lambda = \lambda(x, t)$ is taken so that $t = u_\lambda(x)$. Thus ϕ is a vectorfield on U which, by construction, satisfies assumption (a) of Remark 3.6, and assumption (a') for every u_λ. We prove that ϕ is divergence-free under the additional assumption that the function $u(x, \lambda) := u_\lambda(x)$ is of class C^1 and $\partial_\lambda u(x, \lambda) \ne 0$ for every (x, λ), which implies that the parameter λ can be (locally) chosen so to depend on x and t in a C^1 fashion. Then we get

$$\begin{aligned} \operatorname{div}\phi &= 2\Delta_x u + 2\partial_\lambda \nabla_x u \cdot \nabla_x \lambda + 2\nabla_x u \cdot \partial_\lambda \nabla_x u \, \partial_t \lambda \\ &= 2\partial_\lambda \nabla_x u \cdot (\nabla_x \lambda + \nabla_x u \, \partial_t \lambda) \ . \end{aligned} \tag{4.2}$$

On the other hand, deriving the identity $u(x, \lambda(x, t)) = t$ with respect to x and t we get $\nabla_x u + \partial_\lambda u \, \nabla_x \lambda = 0$ and $\partial_\lambda u \, \partial_t \lambda = 1$, respectively. This implies that the last factor in (4.2) vanishes, and thus ϕ is divergence-free (to make this argument work, we need that $\nabla_x u$ is of class C^1 in (x, λ), which can be derived by the fact that each function u_λ is harmonic).

In Paragraphs 4.3 and 4.6 below we apply this idea by embedding a harmonic function that we intend to calibrate into a family of harmonic functions whose graphs fibrate $U := \Omega \times (m, M)$, and taking ϕ as in (4.2). To show that ϕ is a calibration we will have only to verify assumption (b) of Remark 3.6 (because (b') amounts to nothing).

Remark. For $n = 1$ and $\Omega = (a, b)$, the equation $\operatorname{div}\phi = 0$ on $\Omega \times \mathbb{R}$, coupled with the identity $\phi^t = \frac{1}{4}(\phi^x)^2$, reduces to the first order equation

$$\partial_x \phi^x + \frac{1}{2} \phi^x \, \partial_t \phi^x = 0 \ . \tag{4.3}$$

It easily follows from the method of characteristics that ϕ is a C^1 solution of (4.3) in $\Omega \times \mathbb{R}$ if and only if every level set $\{\phi^x = s\}$ is composed of straight lines with slope $s/2$ (intersected with $\Omega \times \mathbb{R}$). In other words, for $n = 1$ all C^1 divergence-free vectorfields ϕ on $\Omega \times \mathbb{R}$ which satisfy $\phi^t = \frac{1}{4}(\phi^x)^2$ (cf. conditions (a) and (a') in Remark 3.6) are associated with a fibration of $\Omega \times \mathbb{R}$ with graphs of affine—i.e., harmonic—functions as in Remark 4.2.

For the rest of this section, calibrations are always intended as Dirichlet calibrations for F_0, in the sense of Remark 3.6. We begin with a discussion of some one-dimensional examples. Of course, in these examples minimality can be easily checked by direct computations, and there is no need for calibrations. Nevertheless, it is instructive to see what happens, and moreover some one-dimensional constructions can be carried over to higher dimensions.

4.3. Affine function in one dimension. Let Ω be the open interval $(0, a)$ and let u be the linear function $u(x) := \lambda x$, with $\lambda > 0$. It is easy to see that u is a Dirichlet minimizer of F_0 if and only if

$$a\lambda^2 \leq 1 \ . \tag{4.4}$$

In this case a calibration is given by the piecewise constant vectorfield:

$$\phi(x,t) := \begin{cases} (2\lambda, \lambda^2) & \text{if } \dfrac{\lambda}{2}x \leq t \leq \dfrac{\lambda}{2}(x+a), \\ (0,0) & \text{otherwise.} \end{cases} \tag{4.5}$$

Thus ϕ satisfies assumptions (a) and (a') of Remark 3.6, and vanishes outside a stripe of constant height (in grey in Fig. 1 below, on the left) which is arranged so that (b) holds and divϕ vanishes (cf. Remark 2.6(b)), while (b') is trivially satisfied.

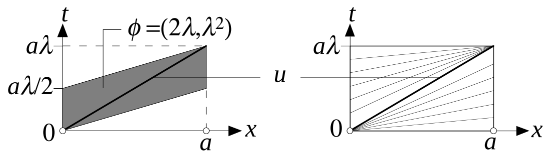

Fig. 1

Another calibration is obtained by fibrating the rectangle $U = (0, a) \times (0, \lambda a)$ with affine functions as shown in Fig. 1, on the right, and applying the construction of Remark 4.2:

$$\phi(x,t) := \begin{cases} \Big(2\dfrac{t}{x}, \Big(\dfrac{t}{x}\Big)^2\Big) & \text{if } 0 \leq t \leq \lambda x, \\ \Big(2\dfrac{\lambda a - t}{a - x}, \Big(\dfrac{\lambda a - t}{a - x}\Big)^2\Big) & \text{if } \lambda x \leq t \leq \lambda a. \end{cases} \tag{4.6}$$

Finally, assumption (b) is satisfied if and only if (4.4) holds.

Remark. If $a\lambda^2 < 1$, then both calibrations described in the previous paragraph satisfy the strict inequality in assumption (b) of Remark 3.6, i.e.,

$$\Big| \int_{t_1}^{t_2} \phi^x(x,t)\, dt \Big| < 1 \quad \text{for every } x \in [0,a] \text{ and every } t_1, t_2 \in \mathbb{R}.$$

By Remark 3.5, this shows that the function $u(x) := \lambda x$ is the unique Dirichlet minimizer of F_0 with $u(0) = 0$ and $u(a) = \lambda a$.

4.4. Step function in one dimension. In Paragraph 4.3, in the limit case $a\lambda^2 = 1$ the linear function $u(x) = \lambda x$ and any step function of the form $u(x) := 0$ for $0 < x < c$ and $u(x) := \lambda a = \sqrt{a}$ for $c < x < a$ (with $0 < c < a$) are both Dirichlet minimizers with the same boundary values. Hence, by Remark 3.5 both vectorfields (4.5) and (4.6) calibrate these step functions when $\lambda := 1/\sqrt{a}$. Furthermore, it is easy to check that they also calibrate any step function u given by $u(x) := 0$ for $0 < x < c$, and $u(x) := h$ for $c < x < a$ with $h \geq \sqrt{a}$.

Remark 4.5. When $a\lambda^2 > 1$ the linear function $u(x) := \lambda x$ is not a Dirichlet minimizer of F_0 (a step function is preferable), but it is still a Dirichlet $\overline{U}$-minimizer, when U is the stripe of all points (x, t) between the graph of $\lambda x - \frac{1}{4\lambda}$ and $\lambda x + \frac{1}{4\lambda}$. A calibration is given by $\phi(x, t) := (2\lambda, \lambda^2)$.
Conversely, when $h < \sqrt{a}$, the step function u in Paragraph 4.4 is no longer a Dirichlet minimizer, but it is Dirichlet $\overline{U}$-minimizer when U is an ε-neighbourhood of the complete graph of u (in grey in Fig. 2) and ε satisfies $\frac{3}{2}\sqrt{\varepsilon} + 2\varepsilon \leq h$. A calibration is given by the piecewise constant vectorfield which vanishes outside the white parallelogram in Fig. 2, and is equal to $\big(\frac{1}{\sqrt{\varepsilon}}, \frac{1}{4\varepsilon}\big)$ inside.

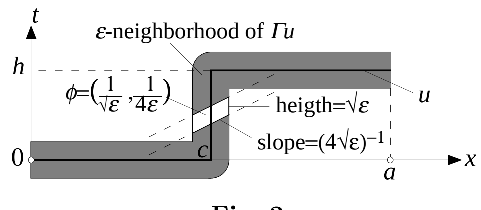

Fig. 2

4.6. Harmonic functions in dimension n. Let u be a harmonic function on Ω. Since u is a Dirichlet minimizer of $\int_\Omega |\nabla u|^2$, it is natural to ask whether it is also a Dirichlet minimizer of F_0. As pointed out by A. Chambolle, this happens when

$$\operatorname{osc} u \cdot \sup |\nabla u| \leq 1 \ , \tag{4.7}$$

where $\operatorname{osc} u$ is the oscillation of u, namely the difference between the supremum M and infimum m of u (over Ω). In the one-dimensional case $n = 1$ this condition reduces to (4.4).
A calibration can be constructed by analogy with (4.5); see Fig. 3, on the left:

$$\phi(x,t) := \begin{cases} \big(2\nabla u(x), |\nabla u(x)|^2\big) & \text{if } \dfrac{1}{2}(u(x)+m) \leq t \leq \dfrac{1}{2}(u(x)+M), \\ (0,0) & \text{otherwise.} \end{cases} \tag{4.8}$$

Another calibration can be obtained, as the one in (4.6), by embedding u in a family of harmonic functions whose graphs fibrate the cylinder $\Omega \times [m, M]$. More precisely we take the functions $m + \lambda(u - m)$ and $M + \lambda(u - M)$ with λ ranging in $[0, 1]$ (see Fig. 3, on the right), and then the construction of Remark 4.2 gives

$$\phi(x,t) := \begin{cases} \left(2\dfrac{t-m}{u(x)-m}\nabla u(x), \left(\dfrac{t-m}{u(x)-m}\right)^2 |\nabla u(x)|^2\right) & \text{if } m \le t \le u(x), \\ \left(2\dfrac{M-t}{M-u(x)}\nabla u(x), \left(\dfrac{M-t}{M-u(x)}\right)^2 |\nabla u(x)|^2\right) & \text{if } u(x) \le t \le M. \end{cases} \tag{4.9}$$

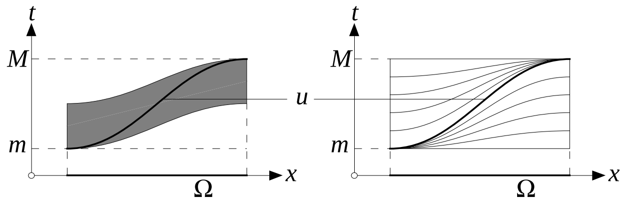

Fig. 3

One easily checks that both vectorfields are divergence-free (see Remarks 2.6(b) and 4.2), and that assumptions (a) and (a') of Remark 3.6 are satisfied; assumption (b') is always trivially satisfied, while assumption (b) holds if an only if (4.7) holds. When (4.7) is not satisfied, u is still a Dirichlet $\overline{U}$-minimizer of F_0, where U is the set of all points $(x, t) \in \Omega \times \mathbb{R}$ which lie between the graph of $u(x) - (4|\nabla u(x)|)^{-1}$ and $u(x) + (4|\nabla u(x)|)^{-1}$; a calibration is given by $\phi(x, t) := \big(2\nabla u(x), |\nabla u(x)|^2\big)$.

Remark 4.7. If inequality (4.7) holds and u is not affine, then the maximum principle implies that $\operatorname{osc} u \cdot |\nabla u(x)| < 1$ for every $x \in \Omega$, and therefore both calibrations constructed in the previous paragraph satisfy the strict inequality in assumption (b) of Remark 3.6. By Remark 3.5, this proves that the harmonic function u is the only Dirichlet minimizer of F_0 with the same boundary values as u.

4.8. Triple junction in the plane. Let $\Omega := B(0, r)$ be the open disk in the plane with radius r and centred at the origin, and let (A_1, A_2, A_3) be the partition of Ω defined as follows: A_i is the set of all $x \in \Omega$ of the form $x = (\rho\cos\theta, \rho\sin\theta)$, with $\frac{2}{3}\pi(i-1) \le \theta < \frac{2}{3}\pi i$. Finally define $u := a_i$ in each A_i, where $a_1 < a_2 < a_3$ are distinct constants.
Thus the singular set of u is given by three line segments $S_{1,2}$, $S_{2,3}$, and $S_{3,1}$ meeting at the origin with equal angles (see Fig. 4, on the left), and it is well-known that this is a minimal network, in the sense that the corresponding partition (A_1, A_2, A_3) is a Dirichlet minimizer of the "interface size" functional (see Remark 3.11). Therefore one would expect that when the values of the constants a_i are sufficiently far apart u is a Dirichlet minimizer of F_0 too, that is, there is no convenience in removing part of the jump and taking a function with non-vanishing gradient.

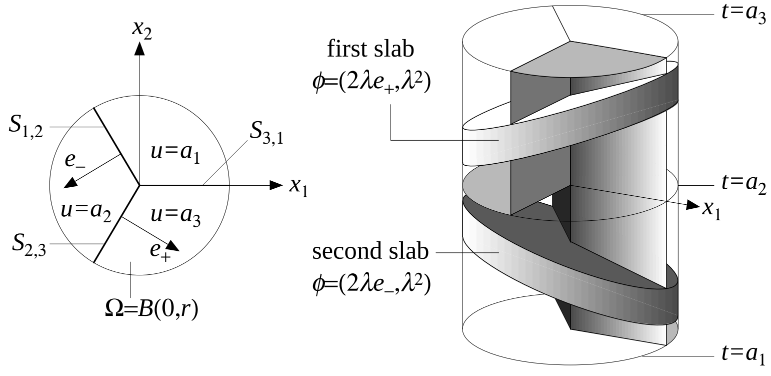

Fig. 4

We prove this statement by calibration. Inspired by the construction described in Paragraphs 4.4 we take $e_\pm := (\pm\sqrt{3}/2, -1/2)$, $\lambda > 0$, and set

$$\phi(x,t) := \begin{cases} (2\lambda e_-, \lambda^2) & \text{if } \left|t - \dfrac{1}{2}(a_1 + a_2) - \dfrac{\lambda}{2} x \cdot e_-\right| < \dfrac{1}{4\lambda}, \\ (2\lambda e_+, \lambda^2) & \text{if } \left|t - \dfrac{1}{2}(a_2 + a_3) - \dfrac{\lambda}{2} x \cdot e_+\right| < \dfrac{1}{4\lambda}, \\ (0,0) & \text{otherwise.} \end{cases} \tag{4.10}$$

Thus ϕ is piecewise constant, satisfies assumption (a) of Remark 3.6 by construction, and vanishes out of two slabs of constant height $\frac{1}{2\lambda}$ (see Fig. 4, on the right). These slabs have been arranged in order to fulfill the following requirements:

(i) one slab is contained in $\Omega \times [a_1, a_2]$ and the other one in $\Omega \times [a_2, a_3]$, so that assumption (a') of Remark 3.6 is satisfied; it is possible to construct such slabs if we can choose λ so that $a_{i+1} - a_i \ge \lambda r + \frac{1}{2\lambda}$, that is, if

$$a_{i+1} - a_i \ge \sqrt{2r}\ ; \tag{4.11}$$

(ii) the compatibility condition (2.4) is satisfied on the boundary of the slabs, so that ϕ is approximately regular and divergence-free (cf. Remark 2.6(b));

(iii) assumption (b') is satisfied for all points x in $S_{1,2}$ and $S_{2,3}$, where ν_u coincides with e_- and e_+ respectively.

Moreover (b') holds also for x in $S_{3,1}$, because $e_- + e_+ = \nu_u$. One also checks that the integral $\int_{t_1}^{t_2} \phi^x(x, t)\, dt$ can be always written as a linear combination $\mu_- e_- + \mu_+ e_+$ with $\mu_\pm$ in $[0, 1]$ (depending on x, t_1, t_2), and since e_+ and e_- span an angle equal to $2\pi/3$, this implies that the integral has modulus not larger than 1. Thus (b) holds, too.

Remark. When $a_2 - a_1$ (or $a_3 - a_2$) is sufficiently small, u is not a Dirichlet minimizer. More precisely, if $a_2 - a_1 \le \frac{1}{3}\sqrt{r}$ (cf. (4.11)), a comparison function v with the same boundary values as u and such that $F(v) < F(u)$ is given, in polar coordinates, by

$$v := \begin{cases} \dfrac{a_1+a_2}{2} & \text{if } 0 \le \theta < \dfrac{4\pi}{3} \text{ and } \rho \le r-d, \\ \dfrac{a_1+a_2}{2} + \dfrac{a_1-a_2}{2}\,\dfrac{\rho-r+d}{d} & \text{if } 0 \le \theta < \dfrac{2\pi}{3} \text{ and } \rho > r-d, \\ \dfrac{a_1+a_2}{2} + \dfrac{a_2-a_1}{2}\,\dfrac{\rho-r+d}{d} & \text{if } \dfrac{2\pi}{3} \le \theta < \dfrac{4\pi}{3} \text{ and } \rho > r-d, \\ a_3 & \text{if } \dfrac{4\pi}{3} \le \theta < 2\pi, \end{cases}$$

where $d := (a_2 - a_1)\sqrt{r}$ (we leave the computations to the reader).

4.9. Minimal partitions in dimension n. One can generalize the example of the triple junction, and conjecture the following: if a partition $(A_1, \dots, A_m)$ of Ω is a Dirichlet minimizer of the "interface size" (see Remark 3.11) and u is a function which takes a constant value a_i on each A_i (with $a_1 < a_2 < \dots < a_m$), then u is a Dirichlet minimizer of F_0 when the values a_i are sufficiently far apart from each other. Unfortunately we can only prove this statement under two additional assumptions:

(i) the partition $(A_1, \dots, A_m)$ is not only minimal, but admits a paired calibration in the sense of [31] and [10], namely there exist approximately regular, divergence-free vectorfields $\phi_1, \dots, \phi_m$ on $\overline{\Omega}$ which satisfy assumptions (d) and (d') in Remark 3.11;

(ii) for $i = 1, \dots, m-1$ there exist Lipschitz functions $\psi_i : \overline{\Omega} \to \mathbb{R}$ which satisfy almost everywhere the first order equation

$$\nabla\psi_i \cdot (\phi_{i+1} - \phi_i) = \frac{1}{2}|\phi_{i+1} - \phi_i|^2 \,. \tag{4.12}$$

Adding, if needed, a constant to ψ_i, we may also assume that

$$\operatorname{osc}\psi_i = 2\|\psi_i\|_\infty \,. \tag{4.13}$$

For $i = 1, \dots, m-1$ we take slabs U_i, included in $\overline{\Omega} \times (a_i, a_{i+1})$, of the form

$$U_i = \left\{(x,t) : \left|t - \frac{1}{2}(a_i + a_{i+1}) - \lambda_i\psi_i(x)\right| < \frac{1}{4\lambda_i}\right\} \,, \tag{4.14}$$

where the constants λ_i will be chosen below. Then we set (cf. (4.10))

$$\phi(x,t) := \begin{cases} \big(2\lambda_i(\phi_{i+1}(x) - \phi_i(x)), \lambda_i^2|\phi_{i+1}(x) - \phi_i(x)|^2\big) & \\ & \text{if } (x,t) \in U_i \text{ for some } i, \\ (0,0) & \text{otherwise.} \end{cases}$$

Taking into account assumption (d') in Remark 3.11 and the definition of the slabs U_i, one can easily check that assumptions (a), (a'), and (b') of Remark 3.6 are satisfied.
Let us check assumption (b). Taken $t_1 \in [a_i, a_{i+1}]$ and $t_2 \in [a_j, a_{j+1}]$ for some i, j, the integral $\int_{t_1}^{t_2} \phi^x(x,t)\,dt$ can be decomposed as the sum of the integrals on the (oriented) intervals $[t_1, a_{i+1}]$, $[a_{i+1}, a_j]$, and $[a_j, t_2]$, and hence it can be written as

$$\mu_1(\phi_{i+1}(x) - \phi_i(x)) + (\phi_j(x) - \phi_{i+1}(x)) + \mu_2(\phi_{j+1}(x) - \phi_j(x))$$

for suitable $\mu_1, \mu_2 \in [0,1]$. But this sum can be reorganized as the difference between $\mu_2\phi_{j+1}(x) + (1-\mu_2)\phi_j(x)$ and $\mu_1\phi_i(x) + (1-\mu_1)\phi_{i+1}(x)$. Therefore its modulus is the distance between two points in the convex hull of the vectors $\phi_1(x), \dots, \phi_m(x)$, which has diameter 1 because of assumption (d) in Remark 3.11, and (b) is proved.
Since the vectorfields ϕ_i are divergence-free and approximately regular by assumption, ϕ is divergence-free and approximately regular within each slab (the approximate regularity of $(\phi^x, 0)$ is immediate, that of $(0, \phi^t)$ follows from Remark 2.3), as well as in the interior of the complement of the union of all slabs. Thus ϕ is divergence-free and approximately regular in $\Omega \times \mathbb{R}$ if (and only if) compatibility condition (2.4) is satisfied on the boundary of each slab (cf. Remark 2.6(b)), which reduces to equation (4.12).
Therefore we have constructed a calibration for u, provided that we can choose λ_i so that the slabs U_i are contained in $\Omega \times (a_i, a_{i+1})$, that is,

$$\frac{a_{i+1} - a_i}{2} \ge \lambda_i\|\psi_i\|_\infty + \frac{1}{4\lambda_i} = \frac{\lambda_i}{2}\operatorname{osc}\psi_i + \frac{1}{4\lambda_i} \,,$$

and this can be done as long as

$$a_{i+1} - a_i \ge \sqrt{2\operatorname{osc}\psi_i} \quad \text{for } i = 1, \dots, m-1. \tag{4.15}$$

Remark 4.10. As noticed in [31], a paired calibration for the partition (A_1, A_2, A_3) described in Paragraph 4.8 is given by the constant vectorfields $\phi_1 := \frac{1}{6}(\sqrt{3}, 3)$, $\phi_2 := \frac{1}{6}(-2\sqrt{3}, 0)$, $\phi_3 := \frac{1}{6}(\sqrt{3}, -3)$, and the linear functions ψ_1 and ψ_2 with gradients $\frac{1}{4}(-\sqrt{3}, -1)$ and $\frac{1}{4}(\sqrt{3}, -1)$ satisfy equation (4.12). Now the construction of Paragraph 4.9 gives exactly the calibration (4.10).

Remark 4.11. The first order equation (4.12) does not always admit solutions. For instance, since the derivative of ψ_i along the integral curves of the vectorfield $\phi_{i+1} - \phi_i$ (i.e., the maximal solutions of the differential equation $\dot\gamma = \phi_{i+1}(\gamma) - \phi_i(\gamma)$) is always positive, when there exists a *nontrivial* closed integral curve within Ω, (4.12) admits no solution. On the other hand, if $\phi_{i+1} - \phi_i$ is C^1 and nowhere vanishing, and all integral curves start and end at the boundary of Ω and intersect a fixed $(n-1)$-dimensional closed manifold M in Ω which is transversal to the vectorfield $\phi_{i+1} - \phi_i$, i.e., a cross-section of the associated flow, then the method of characteristics provides a solution ψ_i to (4.12) of class C^1.

However, such a strong requirement on $\phi_{i+1}-\phi_i$ is far from being necessary: not only there may exist Lipschitz functions ψ_i which satisfy (4.12) almost everywhere even if $\phi_{i+1}-\phi_i$ vanishes somewhere, but for our purposes we can even allow ψ_i to be discontinuous along some integral curve γ: in this case the boundary of the slab U_i in (4.14) is not just the union of the graphs of $\lambda_i\psi_i+\frac{1}{2}(a_1+a_2)+\frac{1}{4\lambda_i}$ and $\lambda_i\psi_i+\frac{1}{2}(a_1+a_2)-\frac{1}{4\lambda_i}$, but there is an additional vertical piece contained in $\gamma\times\mathbb{R}$. Yet the compatibility condition (2.4) is satisfied there, and then ϕ is still divergence-free and approximately regular by Remark 2.6(b).

4.12. Step function in the plane. Let Ω be the rectangle $(-a,a)\times(-b,b)$ in $\mathbb{R}^2$, and let A_1 and A_2 be the sets of all points $x=(x_1,x_2)$ such that $x_1<0$ and $x_1\ge 0$, respectively. The partition (A_1,A_2) is obviously minimal, and a paired calibration is given by the constant vectorfields $\phi_1:=(0,0)$, $\phi_2:=(1,0)$. Thus the linear function ψ_1 with gradient $(\frac{1}{4},0)$ satisfies (4.12), and the construction in Paragraph 4.9 gives a calibration for the step function u which takes the value a_1 on A_1 and a_2 on A_2 provided that $a_2-a_1\ge\sqrt{a}$.
Yet for large values of a this condition is far from optimal, and a better result can be obtained if we consider a different paired calibration for (A_1,A_2). Let $p_+:=(0,b)$ and $p_-:=(0,-b)$, and take $\phi_1:=(0,0)$,

$$\phi_2(x):=\begin{cases}(-\sin\theta_+,\cos\theta_+) & \text{for } x\in B(p_+,b),\\ (\sin\theta_-,-\cos\theta_-) & \text{for } x\in B(p_-,b),\\ (0,0) & \text{otherwise,}\end{cases} \tag{4.16}$$

where $\rho_\pm,\theta_\pm$ are the polar coordinates around the points $p_\pm$; see Fig. 5. A function ψ_1 which satisfies (4.12) almost everywhere is

$$\psi_1(x):=\begin{cases}\dfrac{1}{2}(\theta_++\pi/2)\rho_+ & \text{for } x\in B(p_+,b),\\ \dfrac{1}{2}(\theta_--\pi/2)\rho_- & \text{for } x\in B(p_-,b),\\ 0 & \text{otherwise,}\end{cases}$$

and the construction in Paragraph 4.9, performed with some care because of the discontinuity of ψ_1 along the circles $\partial B(p_\pm,b)$ (cf. Remark 4.11), yields a calibration for u provided that $a_2-a_1\ge\sqrt{\pi b}$. Note that this calibration is defined on the whole stripe $\mathbb{R}\times(-b,b)$ and does not depend on a.

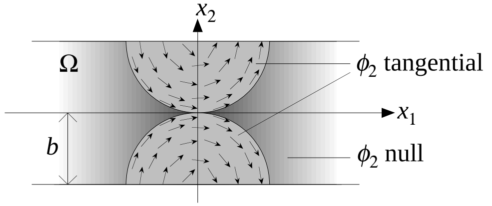

Fig. 5

More on the triple junction. Let us apply again the construction of Paragraph 4.9 to the situation described in Paragraph 4.8, with Ω replaced by an ε-neighbourhood of Su within the ball $B(0,r)$ (in grey in Fig. 6). A paired calibration for the partition (A_2,A_2,A_3) is given by the constant vectorfields $\phi_1:=\frac{1}{6}(\sqrt{3},3)$, $\phi_2:=\frac{1}{6}(-2\sqrt{3},0)$, and $\phi_3:=\frac{1}{6}(\sqrt{3},-3)$, but we can take solutions ψ_i of (4.12) such that $|\psi_i|\le 2\varepsilon$ on $\overline{\Omega}$, independently of the value of r. More precisely, we consider the solution ψ_i of (4.12) which takes the value 0 on the transversal set M_i described in the Fig. 6 for $i=1$ (the construction is similar for $i=2$).

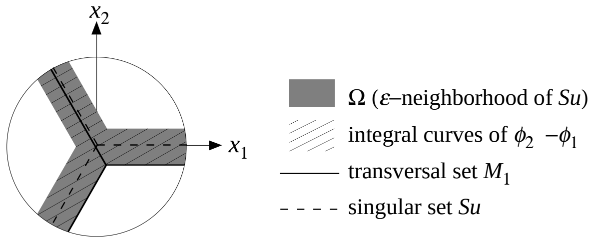

Fig. 6

Therefore, if $a_{i+1}-a_i\ge 2\sqrt{2\varepsilon}$, the construction of Paragraph 4.9 yields a calibration of u on $\overline{\Omega}\times\mathbb{R}$. Moreover this calibration can be extended to a calibration on an ε-neighbourhood of the complete graph of u over the entire $B(0,r)$ by setting it equal to 0 where it is not yet defined. Clearly, this requires that the slabs U_i are contained in $B(0,r)\times(a_i+\varepsilon,a_{i+1}-\varepsilon)$ for $i=1,2$, that is

$$a_{i+1}-a_i\ge 2\varepsilon+2\sqrt{2\varepsilon}\quad\text{for } i=1,2. \tag{4.17}$$

This shows that u is a Dirichlet $\overline{U}$-minimizer of F_0 when U is an ε-neighbourhood of the complete graph of u within $B(0,r)\times\mathbb{R}$ and ε satisfies (4.17). As expected, ε does not depend on the size of the domain, but only on the relative distances of the values a_i.

5. Applications to minimizers of F

In this section we focus on absolute minimizers of the complete Mumford-Shah functional F in (1.1), with $\alpha,\beta>0$, and calibrations will always be intended in the sense of Theorem 3.3 and Definition 3.4. All examples are in dimension n.

5.1. Solutions of the Neumann problem. If we restrict F to functions of class $W^{1,2}$, we obtain the strictly convex and coercive functional $\int_\Omega[|\nabla u|^2+\beta(u-g)^2]\,dx$, and its unique minimizer u is the solution of the Neumann problem

$$\begin{cases}\Delta u=\beta(u-g) & \text{on } \Omega,\\ \partial_\nu u=0 & \text{on } \partial\Omega,\end{cases} \tag{5.1}$$

where ∂_ν denotes the normal derivative. Thus it is natural to ask under which assumptions (on g and β) u is also a minimizer of F on $SBV(\Omega)$. This question

is akin to the minimality of harmonic functions for F_0 discussed in Paragraph 4.6, and following the same ideas we can construct an absolute calibration for u provided that u satisfy condition (5.3) below.
More precisely, we assume that u is of class C^1 up to the boundary (this is always satisfied if $\partial\Omega$ is of class $C^{1,\varepsilon}$ for some $\varepsilon > 0$), we denote the infimum and the supremum of g by m and M respectively, and set

$$A := \Big\{(x,t) \in \overline{\Omega} \times \mathbb{R} : \ \frac{u(x)+m}{2} \le t \le \frac{u(x)+M}{2}\Big\} ,$$

and (cf. (4.8))

$$\phi^x(x,t) := \begin{cases} 2\nabla u(x) & \text{if } (x,t) \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

By the maximum principle $m \le u \le M$ on $\overline{\Omega}$, so that Γu is contained in A, and, independently of the choice of ϕ^t, we can already see that assumption (b) of Lemma 3.2 is satisfied if (cf. (4.7))

$$\operatorname{osc} g \cdot \sup |\nabla u| \le \alpha , \tag{5.3}$$

while assumption (b') is trivially satisfied, and ϕ has vanishing normal component on $\partial\Omega \times \mathbb{R}$ because of the second equation in (5.1). Thus it remains to choose ϕ^t so that (a) and (a') hold, and ϕ is approximately regular and divergence-free.
Assumption (a') sets ϕ^t equal to $|\nabla u|^2 - \beta(u-g)^2$ on the graph of u., while requiring that ϕ is divergence-free in the interior of A yields

$$\partial_t \phi^t = -\operatorname{div}_x \phi^x = -2\Delta u = -2\beta(u-g) .$$

Integrating in t we obtain that ϕ^t is given in A by

$$\begin{aligned} \phi^t &= |\nabla u|^2 - \beta(u-g)^2 - 2\beta(u-g)(t-u) \\ &= |\nabla u|^2 - \beta(t-g)^2 + \beta(t-u)^2 . \end{aligned}$$

Therefore assumption (a) of Lemma 3.2, namely $\phi^t \ge |\nabla u|^2 - \beta(t-g)^2$, is clearly satisfied in A. Note that ϕ is approximately continuous in $\overline{A}$ (this is trivial for the vectorfield $(\phi^x, 0)$, which is continuous, and follows from Remark 2.3 for $(0, \phi^t)$). Moreover, ϕ is divergence-free in the complement of A if we impose that $\partial_t \phi^t = 0$, that is, ϕ^t depends only on x, while the compatibility condition (2.4) on the graphs of $\frac{1}{2}(u+m)$ and $\frac{1}{2}(u+M)$, required in order to have $\operatorname{div}\phi = 0$ on $\Omega \times \mathbb{R}$ (cf. Remark 2.6(b)), yields

$$\phi^t = \begin{cases} -\beta\Big(\dfrac{u+M}{2} - g\Big)^2 + \beta\Big(\dfrac{u-M}{2}\Big)^2 & \text{for } t > \dfrac{u+M}{2}, \\[2ex] -\beta\Big(\dfrac{u+m}{2} - g\Big)^2 + \beta\Big(\dfrac{u-m}{2}\Big)^2 & \text{for } t < \dfrac{u+m}{2}. \end{cases}$$

Finally one can easily check that ϕ is approximately regular also outside A (cf. Remark 2.3) and satisfies condition (a) of Lemma 3.2 as well. Therefore ϕ is an absolute calibration for u, provided that u satisfies (5.3).

Remark 5.2. If inequality (5.3) is strict, the calibration constructed in the previous paragraph satisfies the strict inequality in assumption (b) of Lemma 3.2, and by Remark 3.5 this proves that the solution u of (5.1) is the unique absolute minimizer of F. This is always the case when β is small enough, provided that $\partial\Omega$ is of class $C^{1,\varepsilon}$ for some $\varepsilon > 0$. Indeed, by the maximum principle we have $\|u-g\|_\infty \le \operatorname{osc} g$, and the standard regularity theory for Neumann problems yields $\|\nabla u\|_\infty \le C\,\|\Delta u\|_\infty = C\,\beta\,\|u-g\|_\infty \le C \operatorname{osc} g$, where $C = C(\Omega)$ is a constant depending only on Ω. Therefore (5.3) is satisfied with strict inequality if

$$\beta < \frac{\alpha}{C(\Omega)\,(\operatorname{osc} g)^2} .$$

Remark. The weaker inequality

$$\operatorname{osc} u \cdot \sup |\nabla u| \le \alpha , \tag{5.4}$$

which follows from (5.1) and (5.3) by the maximum principle, is not enough to guarantee the minimality of a solution u of (5.1), not even the Dirichlet minimality. Take indeed $n := 1$, $\Omega := (-a, a)$, and $g(x) := 1$ for $x \ge 0$ $g(x) := -1$ for $x < 0$. Then the solution to (5.1) can be computed explicitly:

$$u(x) := \Big[1 - \frac{\cosh(\gamma(1-|x|/a))}{\cosh\gamma}\Big] g(x) ,$$

where $\gamma := \sqrt{\beta}\,a$. Now we fix α so that (5.4) is satisfied, and precisely

$$\alpha := \operatorname{osc} u \cdot \sup |\nabla u| = 2\sqrt{\beta}\,\tanh\gamma \,\Big[1 - \frac{1}{\cosh\gamma}\Big] ,$$

and take the comparison function $v(x) := \big[1 - 1/\cosh\gamma\big]\, g(x)$; v has the same boundary values as u, and a tedious but straightforward computation gives

$$F(v) = 2\sqrt{\beta}\,\tanh\gamma - \frac{2\sqrt{\beta}}{\cosh^2\gamma}(\sinh\gamma - \gamma) < 2\sqrt{\beta}\,\tanh\gamma = F(u) .$$

Therefore u satisfies condition (5.4) but is not a Dirichlet minimizer of F.

5.3. Solution of the Neumann problem for large β. The construction in Paragraph 5.1 shows that the solution of the Neumann problem (5.1) is an absolute minimizer of F provided that (5.3) holds. However, this condition is far from being necessary. In particular, for large values of the penalization parameter β, the absolute minimizer u of F is close to g, and therefore we expect that discontinuities should not be energetically convenient, at least for sufficiently regular g, and the solution u of (5.1) should be the unique absolute minimizer of F.
We prove this fact by calibration under the assumption that Ω has boundary of class C^2, g is of class $W^{2,p}$ for some $p > n$, and β is larger than a certain β_0, specified in (5.15). Under these assumptions, g belongs to $C^{1,\gamma}(\overline{\Omega})$ for $\gamma := 1-n/p$,

and u belongs to $C^{3,\gamma}(\Omega)\cap C^{1,\delta}(\overline{\Omega})\cap W^{2,p}(\Omega)$ for every $\delta\in(0,1)$ by the standard regularity theory for Neumann problems (see, e.g., [35], Theorems 3.5, 3.16, and 3.17).

Fix a positive constant δ (to be properly chosen later), and take a smooth function $\sigma:\mathbb{R}\to[0,1]$, with support included in $[-2\delta,2\delta]$ and identically equal to 1 in $[-\delta,\delta]$, so that $|\dot\sigma|\le 2/\delta$ (and then $\|\sigma\|_1\le 4\delta$ and $\|\dot\sigma\|_\infty\le 2/\delta$). Set

$$\phi^x(x,t):=2\sigma(t-u(x))\,\nabla u(x)\ . \tag{5.5}$$

To simplify the notation, in the following we simply write σ and ∇u instead of $\sigma(t-u(x))$ and $\nabla u(x)$ (this must be kept into account when deriving), so that (5.5) becomes simply $\phi^x=2\sigma\,\nabla u$.

It follows from (5.1) and (5.5) that ϕ has vanishing normal component at the boundary of $\Omega\times\mathbb{R}$, and $\phi^x=2\nabla u$ on the graph of u. Assumption (a') in Lemma 3.2 prescribes the value of ϕ^t on the graph of u, and precisely

$$\phi^t(x,u):=|\nabla u|^2-\beta(u-g)^2 \quad \text{for all } x\in\overline{\Omega}. \tag{5.6}$$

We impose now that ϕ is divergence-free, which reduces to

$$\begin{aligned}\partial_t\phi^t=-\mathrm{div}_x\phi^x&=-2\sigma\,\Delta u+2\dot\sigma\,|\nabla u|^2\\ &=-2\beta\sigma\,(u-g)+2\dot\sigma\,|\nabla u|^2\ .\end{aligned} \tag{5.7}$$

Identities (5.7) and (5.6) together determine ϕ^t everywhere.

Note that ϕ is approximately regular: this is trivial for the vectorfield $(\phi^x,0)$, which is continuous, and follows from Remark 2.3 for $(0,\phi^t)$ (even though ϕ^t is discontinuous if so is g).

Now we want to verify that assumption (a) of Lemma 3.2 holds, that is,

$$\phi^t\ge\frac14|\phi^x|^2-\beta(t-g)^2\ . \tag{5.8}$$

Since the equality holds by construction on the graph of u, the full inequality is implied by the following inequalities on the derivatives with respect to t of both sides of (5.8):

$$\begin{cases}\partial_t\phi^t\ge\frac12\phi^x\partial_t\phi^x-2\beta(t-g) & \text{for } t>u,\\[2mm] \partial_t\phi^t\le\frac12\phi^x\partial_t\phi^x-2\beta(t-g) & \text{for } t<u.\end{cases} \tag{5.9}$$

Let us consider the first inequality: by (5.5) and (5.7) it becomes

$$-2\beta\sigma\,(u-g)+2\dot\sigma\,|\nabla u|^2\ge2\sigma\dot\sigma\,|\nabla u|^2-2\beta(t-g)\ ,$$

which is equivalent to

$$\beta[(t-g)-\sigma(u-g)]\ge\dot\sigma(\sigma-1)|\nabla u|^2\ . \tag{5.10}$$

When $u<t\le u+\delta$ we have $\sigma=1$, and then (5.10) becomes $t-u\ge0$, which is obviously true. When $t>u+\delta$, we have $(t-g)-\sigma(u-g)\ge\delta-\|u-g\|_\infty$ and $|\dot\sigma(\sigma-1)|\le2/\delta$, and then (5.10) is implied by $\beta(\delta-\|u-g\|_\infty)\ge\frac{2}{\delta}\|\nabla u\|_\infty^2$. This inequality can be rewritten as

$$\delta^2-\delta\,\|u-g\|_\infty-\frac{2}{\beta}\,\|\nabla u\|_\infty^2\ge0\ ,$$

and is satisfied for

$$\delta\ge\|u-g\|_\infty+\sqrt{\frac{2}{\beta}}\,\|\nabla u\|_\infty\ . \tag{5.11}$$

One checks in the same way that (5.11) implies the second inequality in (5.9) too. In other words, assumption (a) of Lemma 3.2 holds if (5.11) holds.

Assumption (b') of Lemma 3.2 is trivially satisfied because Su is empty, while (5.5) and the estimate $\|\sigma\|_1\le4\delta$ imply that assumption (b) of Lemma 3.2 is satisfied with strict inequality if $8\delta\|\nabla u\|_\infty<\alpha$, that is,

$$\delta<\frac{\alpha}{8\|\nabla u\|_\infty}\ . \tag{5.12}$$

Finally, we can find δ that satisfies both (5.11) and (5.12) if

$$\|\nabla u\|_\infty\big(\sqrt{\beta}\,\|u-g\|_\infty+\sqrt{2}\|\nabla u\|_\infty\big)<\frac{\alpha}{8}\sqrt{\beta}\ , \tag{5.13}$$

and by Theorem 3.3 and Remark 3.5 we conclude that, if (5.13) is satisfied, then u is the *unique* absolute minimizer of F.

Thus it remains to show that (5.13) holds for β large enough. Note that u, being a solution of (5.1), depends on β, and there exist positive constants K and $\bar\beta$ (depending on Ω, but not on g and β) such that for every $\beta\ge\bar\beta$ there holds

$$\sqrt{\beta}\,\|u-g\|_\infty+\|\nabla u\|_\infty\le K\|\nabla g\|_{W^{1,p}}\ . \tag{5.14}$$

This estimate can be derived, for instance, from the interpolation inequality (3.1.59) of Theorem 3.1.22 in [26] (one has to replace λ, u, $\mathcal{A}$, and $\mathcal{B}$ with β, $u-g$, Δ, and ∂_ν respectively, and recall that $\Delta u=\beta(u-g)$).

Estimate (5.14) shows that (5.13) holds for

$$\beta>\beta_0:=\max\big\{\bar\beta,\,2^7\alpha^{-2}K^4\|\nabla g\|_{W^{1,p}}^4\big\}\ . \tag{5.15}$$

5.4. Characteristic functions of regular sets. If $g:=1_E$ is the characteristic function of a sufficiently regular compact subset E of Ω, then it is natural to conjecture that for large values of β the minimizer of F is g itself. We prove this statement by calibration under the assumption that the boundary of E is of class $C^{1,1}$ and $\beta>\beta_0$, where β_0 is defined in (5.22). Under these assumptions we also prove that the minimizer is unique.

As in the previous paragraph, we first construct ϕ^x. To this end, we take a Lipschitz vectorfield ψ on $\overline{\Omega}$ which agrees on ∂E with the inner normal of ∂E, is supported on a neighbourhood of ∂E which is relatively compact in Ω, and satisfies $|\psi| \le 1$ everywhere. For instance, we can use that ∂E is locally a graph, which yields a trivial extension of the normal vectorfield on a small neighbourhood of each point, and then use a partition of unity to paste together these different extensions. Now we set

$$\phi^x(x,t) := \sigma(t)\,\psi(x) \quad \text{for all } x \in \overline{\Omega},\ t \in \mathbb{R}, \tag{5.16}$$

where $\sigma : \mathbb{R} \to [0, 2\alpha]$ is a function of class C^1, supported in $[0,1]$, with integral equal to α, and such that $|\dot\sigma(t)| \le 16\alpha$ for $t \in [0,1]$, $\sigma(t) := t^2$ for $t \in [0, 1/8]$, $\sigma(t) := (1-t)^2$ for $t \in [7/8, 1]$.

We see that, independently of the choice of ϕ^t, the vectorfield ϕ has vanishing normal component at the boundary of Ω, and satisfies assumptions (b) and (b') of Lemma 3.2. Since ϕ^x vanishes for $t = 0$ and for $t = 1$, and therefore on the graph of g, requiring that ϕ satisfies assumption (a') yields

$$\phi^t(x, g(x)) := 0 \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega, \tag{5.17}$$

while requiring that ϕ is divergence-free yields (cf. (5.16))

$$\partial_t \phi^t = -\mathrm{div}_x \phi^x = -\sigma\, \mathrm{div}_x \psi\ . \tag{5.18}$$

Conditions (5.17) and (5.18) together determine ϕ^t.

Note that ϕ is approximately regular: this is trivial for the vectorfield $(\phi^x, 0)$, which is continuous, and follows from Remark 2.3 for $(0, \phi^t)$ (even though ϕ^t is discontinuous on $\partial E \times \mathbb{R}$, and where $\mathrm{div}_x\psi$ is discontinuous).

To show that ϕ is an absolute calibration it remains thus to verify assumption (a) of Lemma 3.2, namely

$$\phi^t \ge \frac{1}{4}|\phi^x|^2 - \beta(t-g)^2 = \frac{1}{4}\sigma^2|\psi|^2 - \beta(t-g)^2\ . \tag{5.19}$$

Since the equality holds by construction on the graph of g (cf. (5.17)), it is enough that $\partial_t\phi^t$ satisfies the inequality

$$\partial_t\phi^t := -\sigma\,\mathrm{div}_x\psi > \frac{1}{2}\sigma\dot\sigma|\psi|^2 - 2\beta(t-g) \quad \text{for } t > g(x), \tag{5.20}$$

and the opposite inequality for $t < g(x)$. Inequality (5.20) is clearly satisfied for $t > 1$, since $\sigma(t) = 0$. If $g(x) = 0$ and $0 < t \le 1$, (5.20) is implied by

$$-\sigma\|\mathrm{div}_x\psi\|_\infty > \frac{1}{2}\sigma|\dot\sigma| - 2\beta t\ . \tag{5.21}$$

In turn, (5.21) reduces for $0 < t < \frac{1}{8}$ to

$$-t^2\|\mathrm{div}_x\psi\|_\infty > t^3 - 2\beta t\ ,$$

which is satisfied for $\beta > \frac{1}{16}\|\mathrm{div}_x\psi\|_\infty + \frac{1}{128}$, while, for $\frac{1}{8} \le t \le 1$, (5.21) follows from

$$-2\alpha\|\mathrm{div}_x\psi\|_\infty > 16\alpha^2 - \frac{1}{4}\beta\ ,$$

which is satisfied for $\beta > 8\alpha\|\mathrm{div}_x\psi\|_\infty + 64\alpha^2$. Therefore (5.20) holds for

$$\beta > \beta_0 := \max\left\{\frac{1}{16}\|\mathrm{div}_x\psi\|_\infty + \frac{1}{128},\ 16\alpha\|\mathrm{div}_x\psi\|_\infty + 64\alpha^2\right\}\ . \tag{5.22}$$

The same condition implies also the opposite inequality for $t < g(x)$. This concludes the proof that ϕ calibrates g.

To prove that g is the unique minimizer of F, we first notice that the strict inequality in (5.20) implies the strict inequality in (5.19) for $t > g(x)$, and of course we have the strict inequality for $t < g(x)$, too. In other words, the inequality in assumption (a) of Lemma 3.2 is strict for all $t \ne g(x)$. Now, if u is another minimizer, ϕ must calibrate u, too (cf. Remark 3.5), and in particular it must satisfy assumption (a') of Lemma 3.2 for u, which means that the inequality in assumption (a) is an equality for $t = u(x)$. Therefore we conclude that $u(x) = g(x)$ for $\mathscr{L}^n$-a.e. x in Ω.

Remark 5.5. If $g := 1_E$ is the characteristic function of a set E relatively compact in Ω and $u := g$ minimizes F, then the set E minimizes in particular $\mathscr{F}(A) := F(1_A) = \alpha\mathscr{H}^{n-1}(\partial_* A) + \beta|A \triangle E|$ among all sets A with finite perimeter in Ω. Hence the regularity theory for minimal perimeters yields that, in dimension $n \le 7$, E must be of class $C^{1,\gamma}$ for every $\gamma < 1$, while in dimension two it must be of class $C^{1,1}$ (see, e.g., [5], Theorem 4.7.4). Thus the regularity on g required in the previous paragraph is optimal in dimension two, and close to optimal for $3 \le n \le 7$.

Remark. If g is the characteristic function of a set E of class $C^{1,1}$ which is not relatively compact in Ω, then the result of Paragraph 5.4 can be generalized as follows: $u := g$ is a minimizer of F for large values of β provided that ∂E is orthogonal to $\partial\Omega$ (which is assumed to be sufficiently smooth). An absolute calibration can be constructed as in the previous paragraph, one has only to choose ψ so that it is tangent to the boundary of Ω.

Notice that this orthogonality requirement is necessary: indeed it is easily proved that, given a minimizer of the functional $\alpha\mathscr{H}^{n-1}(\Omega \cap \partial_* A) + \beta|A \triangle E|$ among all finite perimeter sets A in Ω, its boundary is orthogonal to $\partial\Omega$.

We conclude this section with some remarks on the gradient flow associated with the (homogeneous) Mumford-Shah functional.

5.6. Gradient flow for the Mumford-Shah functional. A gradient flow for F_0 with respect to the L^2-metric can be defined in a variational way by time discretization, following the *minimizing movements* approach developed in [18], [4], [14], [25]. Given an initial datum $u_0 \in L^2(\Omega)$ and a discretization step $\delta > 0$, we set $u_{\delta,0} := u_0$ and define inductively $u_{\delta,j}$ for $j = 1, 2, \ldots$ as *any* minimizer of

$$F_0(u) + \frac{1}{\delta}\int_\Omega (u - u_{\delta,j-1})^2\,dx \tag{5.23}$$

among all functions u in $SBV(\Omega)$—with or without prescribed boundary values, according to the boundary condition (Dirichlet or Neumann) imposed on the flow. Then we define $u_\delta : \Omega \times [0,+\infty) \to \mathbb{R}$ by $u_\delta(x,t) := u_{\delta,j}$ for $t := j\delta$, and by linear interpolation for $t \in \big(j\delta,(j+1)\delta\big)$, and call *gradient flow with initial datum* $u(x,0) = u_0(x)$ any possible limit of u_δ as $\delta \to 0$ along any sequence. Note that the flow may be not unique, as even u_δ is not uniquely defined.

Remark 5.7. When the initial datum $u_0 := 1_E$ is the characteristic function of a compact subset of Ω with boundary of class $C^{1,1}$, the gradient flow for F_0 with Neumann (or Dirichlet) boundary conditions is unique, and agrees with u_0 for all t, that is, $u(x,t) := u_0(x)$ on $\Omega \times [0,+\infty)$. This follows immediately from Paragraph 5.4.

Remark 5.8. If u_0 belongs to $W^{1,2}(\Omega)$, and the minimization of (5.23) is restricted a priori to the functions u in $W^{1,2}(\Omega)$, then $F_0(u)$ is just the usual Dirichlet integral, and it can be proved (cf. [4], Example 2.1) that the gradient flow is unique and agrees with the solution of the heat equation

$$\partial_t u = \Delta u \quad \text{on } \Omega \times (0,+\infty)$$

with initial datum $u(x,0) = u_0(x)$ and boundary conditions—Neumann or Dirichlet—according to the boundary conditions imposed in the minimization of (5.23).

The previous remark and the result of Paragraph 5.3 suggest that for a smooth initial datum u_0, the gradient flow associated with F_0 is just the solution of the heat equation. To prove this, however, we need some additional information on the minima of F for large β.

5.9. Improved estimates on the solution of (5.1). Under the regularity assumptions on Ω and g given in Paragraph 5.3, if $\Delta g \in L^\infty(\Omega)$ and $\partial_\nu g = 0$ on $\partial\Omega$, then the solution u to the Neumann problem (5.1) satisfies

$$\|\Delta u\|_\infty \le \|\Delta g\|_\infty \tag{5.24}$$

and an improved version of estimate (5.14)

$$\beta\|u-g\|_\infty + \|\nabla u\|_\infty \le K\|\Delta g\|_\infty\ , \tag{5.25}$$

with K depending on Ω, but not on g and β. In particular condition (5.13) of Paragraph 5.3 holds for

$$\beta > \beta_0 := \max\big\{1,\, 2^7\alpha^{-2}K^4\|\Delta g\|_\infty^4\big\}\ , \tag{5.26}$$

and in that case u is the unique absolute minimizer of F.
To prove (5.24) and (5.25), we first notice that the function $v := g+\varepsilon$ is a super-solution of (5.1) as long as $\varepsilon \ge \beta^{-1}\|\Delta g\|_\infty$, in the sense that

$$\begin{cases} \Delta v \le \beta(v-g) & \text{on } \Omega, \\ \partial_\nu v \le 0 & \text{on } \partial\Omega. \end{cases}$$

Thus u is (a.e.) smaller than $g+\varepsilon$ on Ω. Similarly, $g-\varepsilon$ is a sub-solution, and then

$$\|u-g\|_\infty \le \frac{1}{\beta}\|\Delta g\|_\infty\ , \tag{5.27}$$

which, in view of (5.1), implies (5.24).
Now, u solves the equation $\Delta u = f$ with Neumann boundary conditions and $f := \beta(u-g)$, and well-known estimates (cf. [35], Theorem 3.16) give $\|\nabla u\|_\infty \le K\|f\|_\infty$ for a suitable constant K depending on Ω, but not on f. Together with (5.27), this implies (5.25).

5.10. Gradient flow with smooth initial datum. Assume that Ω has boundary of class C^2, $u_0 \in W^{2,p}(\Omega)$ for some $p > n$, $\Delta u_0 \in L^\infty(\Omega)$, and $\partial_\nu u_0 = 0$ on $\partial\Omega$. Then the gradient flow for F_0 with Neumann boundary conditions and initial datum $u(x,0) = u_0(x)$ constructed in Paragraph 5.6 is unique, and agrees with the solution of the heat equation.
In virtue of Remark 5.8, this claim is an immediate consequence of the following fact: when

$$\delta < \delta_0 := \Big[\max\big\{1,\, 2^7\alpha^{-2}K^4\|\Delta u_0\|_\infty^4\big\}\Big]^{-1}\ ,$$

then, for every integer j, *every* minimizer of (5.23) belongs to $W^{1,2}(\Omega)$. In other words, the solution of the Neumann problem (5.1) with $\beta := 1/\delta$ and $g := u_{\delta,j-1}$ is the unique minimizer of (5.23).
To prove this fact, it suffices to verify that the assumptions of Paragraph 5.9 are satisfied for every j, and precisely: $u_{\delta,j-1} \in W^{2,p}(\Omega)$, $\Delta u_{\delta,j-1} \in L^\infty(\Omega)$, $\partial_\nu u_{\delta,j-1} = 0$ on $\partial\Omega$, and inequality (5.26) holds with $\beta := 1/\delta$ and $g := u_{\delta,j-1}$. The last requirement follows from the choice of δ and the chain of inequalities

$$\|\Delta u_0\|_\infty := \|\Delta u_{\delta,0}\|_\infty \ge \|\Delta u_{\delta,1}\|_\infty \ge \|\Delta u_{\delta,2}\|_\infty \ge \dots\ ,$$

which are implied by (5.24). The $W^{2,p}$ regularity of $u_{\delta,j}$ follows from the corresponding regularity of $u_{\delta,j-1}$, as remarked at the beginning of Paragraph 5.3.

Remark. If we drop the assumption $\partial_\nu u_0 = 0$ on $\partial\Omega$ in Paragraph 5.10, the proof breaks down because we can no longer estimate $\|\Delta u_{\delta,j-1}\|_\infty$ by $\|\Delta u_0\|_\infty$, but we do not know if the conclusion on the gradient flow still holds.

Remark 5.11. The conclusion of the previous paragraph also holds for the gradient flow with Dirichlet boundary conditions. More precisely, if Ω has boundary of class C^2 and u_0 is of class $W^{2,p}(\Omega)$, with $\Delta u_0 \in L^\infty(\Omega)$, then the gradient flow for F_0 with initial datum $u(x,0) = u_0(x)$ on Ω and Dirichlet boundary condition $u(x,t) = u_0(x)$ on $\partial\Omega \times [0,+\infty)$ constructed in Paragraph 5.6 is unique, and agrees with the solution of the heat equation (with same initial datum and boundary conditions). The proof is essentially the same as in the Neumann case, and relies on suitable estimates for the Dirichlet problem (see the following aragraph).

Solution of the Dirichlet problem for large β. Assume that Ω and g satisfy the regularity assumptions of Paragraph 5.3, Δg belongs to $L^\infty(\Omega)$, and g_0 is a function in $W^{2,p}(\Omega)$, and consider the solution u to the Dirichlet problem

$$\begin{cases} \Delta u = \beta(u-g) & \text{on } \Omega, \\ u = g_0 & \text{on } \partial\Omega. \end{cases} \tag{5.28}$$

Then u belongs to $C^{3,\gamma}(\Omega) \cap C^{1,\delta}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for every $\delta \in (0,1)$ (see, e.g., [35], Theorems 3.5, 3.16, and 3.17). We claim that if $g_0 = g$ on $\partial\Omega$ and β is sufficiently large, then u is the unique Dirichlet minimizer of F with boundary value g_0.
This claim can be proved by the same calibration constructed in Paragraph 5.3, provided that estimate (5.14) is suitably replaced. To this end, we notice that $v := g + \varepsilon$ is a super-solution of (5.28) for $\varepsilon \geq \beta^{-1}\|\Delta g\|_\infty$, in the sense that

$$\begin{cases} \Delta v \leq \beta(v-g) & \text{on } \Omega, \\ v \geq g_0 & \text{on } \partial\Omega \end{cases}$$

(we use here that $g = g_0$ on $\partial\Omega$), and, similarly, $g - \varepsilon$ is a sub-solution. As in the previous paragraph, we deduce that $\beta\|u-g\| \leq \|\Delta g\|_\infty$, and hence (cf. (5.24))

$$\|\Delta u\|_\infty \leq \|\Delta g\|_\infty \ .$$

Let us consider now the function $w := u - g_0$; since it solves $\Delta w = f - \Delta g_0$ with Dirichlet boundary conditions $w = 0$ on $\partial\Omega$, and $f := \beta(u-g)$, well-known estimates for solutions of Dirichlet problems (see, e.g., [35], Theorem 3.16) give $\|\nabla w\|_\infty \leq K(\|f\|_\infty + \|\nabla g_0\|_{C^{0,\gamma}})$. Together with the estimate on $\|u-g\|_\infty$, this implies (cf. (5.25))

$$\beta\|u-g\|_\infty + \|\nabla u\|_\infty \leq K\big(\|\Delta g\|_\infty + \|\nabla g_0\|_{C^{0,\gamma}}\big) \ ,$$

with a possibly different K. In particular, condition (5.13) of Paragraph 5.3 is satisfied for

$$\beta > \beta_0 := \max\big\{1,\ 2^7\alpha^{-2}K^4(\|\Delta g\|_\infty + \|\nabla g_0\|_{C^{0,\gamma}})^4\big\} \ , \tag{5.29}$$

and in that case u is the unique Dirichlet minimizer of F with boundary values $u = g_0$.

6. Appendix

In this section we prove some technical lemmas stated in Sect. 2. We follow the notation of that section.

Lemma 6.1. *Let Ω be an open subset of $\mathbb{R}^n$ whose boundary is the graph of a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, and let ϕ be a bounded vectorfield on $\overline{\Omega}$ which has bounded support and satisfies condition* (2.3) *with $M := \partial\Omega$. Then there exists a sequence of vectors y_j such that $y_j \to 0$, $\partial\Omega + y_j \subset \Omega$ for every j, and*

$$\lim_{j\to\infty} \phi(x+y_j)\cdot\nu_{\partial\Omega}(x) = \phi(x)\cdot\nu_{\partial\Omega}(x) \quad \textit{for } \mathscr{H}^{n-1}\textit{-a.e. } x \in \partial\Omega. \tag{6.1}$$

Proof. Let S be the set of all vectors $y \in \mathbb{R}^n$ such that $\partial\Omega + y \subset \Omega$. For every $r > 0$, let $S_r := S \cap B(0,r)$, and consider the double integral

$$\int_{S_r}\Big[\int_{\partial\Omega}\big|(\phi(x+y)-\phi(x))\cdot\nu_{\partial\Omega}(x)\big|\, d\mathscr{H}^{n-1}(x)\Big]\frac{dy}{r^n} \ . \tag{6.2}$$

If we invert the order of integration, condition (2.3) means that the inner integral (over S_r) tends to 0 as $r \to 0$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial\Omega$. Then (6.2) converges to 0 by the dominated convergence theorem (recall that ϕ is bounded and has bounded support).
Since $\partial\Omega$ is the graph of a Lipschitz function, the set S contains an open cone with vertex in 0. Then the measure of S_r is larger than ar^n for some fixed $a > 0$, and therefore we can choose $y_r \in S_r$ so that the value of the inner integral in (6.2) is smaller than the double integral divided by a, and then converges to 0 as $r \to 0$.
In other words, $\phi(x+y_r)\cdot\nu_{\partial\Omega}(x)$ converge to $\phi(x)\cdot\nu_{\partial\Omega}(x)$ in the space $L^1(\partial\Omega, \mathscr{H}^{n-1})$, and then it suffices to choose a subsequence y_j which yields pointwise convergence for $\mathscr{H}^{n-1}$-a.e. $x \in \partial\Omega$. □

Proof of Lemma 2.4. (Sketch) We divide the proof in several steps.

STEP 1. Assume that ϕ belong to $C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. In this case formula (2.5) is well-known—see, e.g., [23], Theorem 2.10, or [7], formula (3.87) in Theorem 3.87.

STEP 2. Assume that ϕ is an approximately regular vectorfield on $\mathbb{R}^n$ with compact support and that $\mathrm{div}\phi \in L^\infty(\mathbb{R}^n)$. Let $\psi_\varepsilon(x) := \varepsilon^{-n}\psi(x/\varepsilon)$ be a standard, radially symmetric, regularizing kernel of class C^∞_c, and take $\phi_\varepsilon := \phi * \psi_\varepsilon$. Thus formula (2.5) holds for each ϕ_ε by Step 1, and it only remains to check that we can pass to the limit as $\varepsilon \to 0$. The convergence of the first integral in the right-hand side of (2.5) follows from the fact that the functions $\mathrm{div}\phi_\varepsilon$ are bounded in L^∞ and converge to $\mathrm{div}\phi$ a.e. in Ω. Since ϕ is approximately regular, the maps $\phi_\varepsilon\cdot\nu_M$ converge to $\phi\cdot\nu_M$ $\mathscr{H}^{n-1}$-a.e. on any Lipschitz surface M, and then also on any rectifiable set M. In particular this implies the convergence of the second integral in the right-hand side of (2.5). The same argument also applies to the left-hand side, provided that we use the coarea formula (cf. [7], Theorem 3.40, or [21], Theorem 4.5.9(13)) to re-write that integral as

$$\int_\Omega \phi_\varepsilon\cdot Du = \int_{\mathbb{R}}\Big[\int_{M_t}\phi\cdot\nu_{M_t}\, d\mathscr{H}^{n-1}\Big]\,dt \ ,$$

where M_t is the measure theoretic boundary in Ω of the sublevel $\{u < t\}$.

STEP 3. If ϕ is a compactly supported, approximately regular vectorfield on a neighbourhood of $\overline{\Omega}$ with $\mathrm{div}\phi$ in L^∞, we reduce to Step 2 using a suitable cut-off function.

STEP 4. Assume that Ω is the subgraph of a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, and ϕ is a compactly supported, approximately regular vectorfield on $\overline{\Omega}$ with $\mathrm{div}\phi \in L^\infty(\Omega)$. We take a sequence of vectors y_j as in Lemma 6.1, and set $\phi_j(x) := \phi(x+y_j)$, $u_j(x) := u(x+y_j)$. By Step 3, formula (2.5) holds with ϕ and u replaced by ϕ_j and u_j, and it remains to check that we can pass to the limit as

$j \to +\infty$. The convergence is immediate for all integrals in (2.5) but the last one. In this case, it suffices to notice that the functions $\phi_j \cdot \nu_{\partial\Omega}$ are uniformly bounded and converge to $\phi \cdot \nu_{\partial\Omega}$ $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$ (by the choice of the vectors y_j), while the traces of u_j on $\partial\Omega$ converge to the trace of u in $L^1(\partial\Omega, \mathscr{H}^{n-1})$ (because the functions u_j converge to u in variation, or, alternatively, because the L^1-norm of the difference of the traces is controlled, up to a constant which does not depend on j, by $|Du|(\Omega \setminus (\overline{\Omega} - y_j))$, which clearly tends to zero).

STEP 5. To prove the general case, we use a locally finite partition of unity consisting of compactly supported smooth functions to reduce to Step 4. □

Proof of Lemma 2.5. We first prove that $\mathrm{div}\phi = f$ on $\Omega \setminus S_0$. Since the problem is local, it is enough to show that $\mathrm{div}\phi = f$ on every bounded open set U with $\overline{U} \subset \Omega \setminus S_0$ and such that $U \setminus S_1$ has two connected components U^+ and U^- with Lipschitz boundary. As ϕ is approximately regular on $\overline{U}^{\pm}$, we can apply formula (2.5) with Ω replaced by $U^{\pm}$ and $u \in C_c^\infty(U)$: as the integrals on $U \cap S_1$ cancel out, we are left with $\int_U \phi \cdot \nabla u \, dx = -\int_U f u \, dx$. Since u is arbitrary, we deduce that $\mathrm{div}\phi = f$ on U.
We prove now that $\mathrm{div}\phi = f$ on Ω. Since $\mathscr{H}^{n-1}(S_0) = 0$, the $(1,1)$-capacity of S_0 is zero (see [20], Sect. 5.6.3), and therefore there exists a sequence of functions σ_j in $C^\infty(\overline{\Omega})$ such that $0 \le \sigma_j \le 1$ in Ω, and $\sigma_j = 0$ in a neighbourhood of S_0, and $\sigma_j \to 1$ strongly in $W^{1,1}(\Omega)$.
Take now an arbitrary function $u \in C_c^\infty(\Omega)$. Then the functions $\sigma_j u$ belong to $C_c^\infty(\Omega \setminus S_0)$ and, since $\mathrm{div}\phi = f$ on $\Omega \setminus S_0$, we have $\int_\Omega \phi \cdot \nabla(\sigma_j u) \, dx = -\int_\Omega f \cdot (\sigma_j u) \, dx$. Moreover the functions $\sigma_j u$ converge to u strongly in $W^{1,1}(\Omega)$, and therefore $\int_\Omega \phi \cdot \nabla u \, dx = -\int_\Omega f u \, dx$, which concludes the proof. □

Proof of Lemma 2.8. By a monotone class argument it is enough to prove (2.8) for ϕ of the form $\phi(x,t) := \rho(t)\,\psi(x)$, with $\rho : \mathbb{R} \to \mathbb{R}$ and $\psi = (\psi^x, \psi^t) : \Omega \to \mathbb{R}^n \times \mathbb{R}$ of class C_c^∞. Let σ be the primitive of ρ vanishing at $-\infty$. Then we have

$$\begin{aligned} \int_{\Omega\times\mathbb{R}} \phi \cdot D1_u &= -\int_{\Omega\times\mathbb{R}} \mathrm{div}\phi \, 1_u \, dx \\ &= -\int_\Omega \Big[\int_{-\infty}^u (\rho \, \mathrm{div}_x \psi^x + \dot\rho \psi^t) \, dt \Big] \, dx \\ &= -\int_\Omega \big[\sigma(u) \, \mathrm{div}_x \psi^x + \rho(u) \psi^t \big] \, dx \, . \end{aligned} \tag{6.3}$$

As u belongs to $SBV(\Omega)$, the chain-rule for BV-functions (see, e.g., [7], Theorem 3.96) gives

$$D[\sigma(u)] = \rho(u) \nabla u \cdot \mathscr{L}^n + \big[\sigma(u^+) - \sigma(u^-)\big] \nu_u \cdot \mathscr{H}^{n-1} \llcorner Su \, .$$

Therefore (6.3) implies

$$\begin{aligned} \int_{\Omega\times\mathbb{R}} \phi \cdot D1_u = &\int_\Omega \big[\rho(u) \psi^x \cdot \nabla u - \rho(u) \psi^t \big] \, dx \\ &+ \int_{Su} \big[\sigma(u^+) - \sigma(u^-) \big] \psi^x \cdot \nu_u \, d\mathscr{H}^{n-1} \, , \end{aligned}$$

which, together with (2.7), gives (2.8) in the case $\phi(x,t) := \rho(t)\,\psi(x)$. □

Proof of Lemma 2.9. We set $w := 1_u - 1_v$ on $\Omega \times \mathbb{R}$. Then w belongs to $BV(\Omega \times \mathbb{R})$ and $Dw = \nu_{\Gamma u} \cdot \mathscr{H}^n \llcorner \Gamma u - \nu_{\Gamma v} \cdot \mathscr{H}^n \llcorner \Gamma v$.
Let us consider the inner trace of w on ∂U. First of all we decompose ∂U as the disjoint union of $(\Omega \times \mathbb{R}) \cap \partial U$ and $(\partial\Omega \times \mathbb{R}) \cap \partial U$. For every C^∞ vectorfield ψ on $\Omega \times \mathbb{R}$ with compact support we apply formula (2.5) of Lemma 2.4 with Ω and ϕ replaced by U and ψ, respectively, and we obtain

$$\begin{aligned} -\int_U w \, \mathrm{div}\psi \, dx &= \int_U \psi \cdot Dw + \int_{\partial U} w \, \psi \cdot \nu_{\partial U} \, d\mathscr{H}^n \\ &= \int_{\Gamma u \cap U} \psi \cdot \nu_{\Gamma u} \, d\mathscr{H}^n - \int_{\Gamma v \cap U} \psi \cdot \nu_{\Gamma v} \, d\mathscr{H}^n \\ &\quad + \int_{\partial U} w \, \psi \cdot \nu_{\partial U} \, d\mathscr{H}^n \, . \end{aligned} \tag{6.4}$$

On the other hand, by the definition of distributional derivative we have also

$$\begin{aligned} -\int_{\Omega\times\mathbb{R}} w \, \mathrm{div}\psi \, dx &= \int_{\Omega\times\mathbb{R}} \psi \cdot Dw \\ &= \int_{\Gamma u} \psi \cdot \nu_{\Gamma u} \, d\mathscr{H}^n - \int_{\Gamma v} \psi \cdot \nu_{\Gamma v} \, d\mathscr{H}^n \, . \end{aligned} \tag{6.5}$$

Due to the particular structure of U and the assumption on the complete graphs of u and v, the function w vanishes a.e. on $(\Omega \times \mathbb{R}) \setminus U$. This fact, together with (6.4) and (6.5), implies that the inner trace of w on $(\Omega \times \mathbb{R}) \cap \partial U$ satisfies

$$w \, \nu_{\partial U} = 1_{\Gamma u} \nu_{\Gamma u} - 1_{\Gamma v} \nu_{\Gamma v} \qquad \mathscr{H}^n\text{-a.e. on } (\Omega \times \mathbb{R}) \cap \partial U \, .$$

Therefore w belongs to $L^1((\Omega \times \mathbb{R}) \cap \partial U, \mathscr{H}^n)$ and

$$\int_{(\Omega\times\mathbb{R})\cap\partial U} w \, \phi \cdot \nu_{\partial U} \, d\mathscr{H}^n = \int_{\Gamma u \cap \partial U} \phi \cdot \nu_{\Gamma u} \, d\mathscr{H}^n - \int_{\Gamma v \cap \partial U} \phi \cdot \nu_{\Gamma v} \, d\mathscr{H}^n \, . \tag{6.6}$$

Now, the trace of w on $\partial\Omega \times \mathbb{R}$ is the difference of the characteristic functions of the traces of u and v on $\partial\Omega$, and therefore it belongs to $L^1(\partial\Omega \times \mathbb{R}, \mathscr{H}^n)$ and vanishes $\mathscr{H}^n$-a.e. on $(\partial\Omega \times \mathbb{R}) \setminus \partial U$. As $\nu_{\partial U} = (\nu_{\partial\Omega}, 0)$ on $(\partial\Omega \times \mathbb{R}) \cap \partial U$, this implies

$$\begin{aligned} \int_{(\partial\Omega\times\mathbb{R})\cap\partial U} w \, \phi \cdot \nu_{\partial U} \, d\mathscr{H}^n &= \int_{\partial\Omega\times\mathbb{R}} w \, \phi^x \cdot \nu_{\partial\Omega} \, d\mathscr{H}^n \\ &= \int_{\partial\Omega} \Big[\int_v^u \phi^x(x,t) dt \Big] \cdot \nu_{\partial\Omega} \, d\mathscr{H}^{n-1} \, . \end{aligned} \tag{6.7}$$

Since the inner trace of w on ∂U belongs to $L^1(\partial U, \mathscr{H}^n)$, we apply formula (2.5) of Lemma 2.4 with Ω and u replaced by U and w, respectively, and get

$$\int_{\Gamma u \cap U} \phi \cdot \nu_{\Gamma u} \, d\mathscr{H}^n - \int_{\Gamma v \cap U} \phi \cdot \nu_{\Gamma v} \, d\mathscr{H}^n = -\int_{\partial U} w \, \phi \cdot \nu_{\partial U} \, d\mathscr{H}^n \, . \tag{6.8}$$

Identity (2.9) follows now from (6.6), (6.7), and (6.8). □

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