Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXIX (2000), pp. 457-472

Some remarks about a notion of rearrangement

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Abstract. We consider an extension to n-dimensions of the notion of increasing rearrangement for functions of one variable, and study the behaviour with respect to this operation of some classes of integral functionals. Among other applications, we obtain a simple direct proof of the existence and uniqueness of n-dimensional optimal profiles for transitions in a phase-separation model with non-local interaction energy.

Mathematics Subject Classification (2000): 26D10 (Primary), 26E20, 46E35, 49J45

Introduction

In this note we study a notion of rearrangement, already considered [5] and [9], which generalizes to scalar functions on n-dimensional cylinders the familiar notion of increasing rearrangement for functions on the line, and is defined via a suitable measure-preserving rearrangement of superlevel sets (Definitions 1.1 and 2.1). Our main concern is the behaviour with respect to this operation of certain classes of integral functionals. Among others, we recall the following (more or less known) results: integrals of type $\int g(u)$ are preserved by the rearrangement of u (Theorem 2.6), while integrals of type $\int g(|\nabla u|)$ with g convex are decreased (Theorem 2.10), and so are double integrals of type $\iint J(x'-x)\, g(u(x')-u(x))\, dx'dx$ with g convex and J positive (Theorem 2.11). Under some additional assumptions, we can also show that in the last two cases the rearrangement inequality is strict unless u agrees with its rearrangement.

This rearrangement has already been used in [5] and [9] to study the minimizers of

$$\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2$$

on cylinders in $\mathbb{R}^n$ and with suitable constraints at infinity (cf. Corollary 3.1 and following remarks), also in connection with a conjecture of E. De Giorgi on

This research was partially supported by the Max Planck Institute for Mathematics in the Sciences (Leipzig) and MURST research project "Equazioni Differenziali e Calcolo delle Variazioni".

the entire solutions of the semilinear equation $\Delta u = u(u^2-1)$ in $\mathbb{R}^n$. For recent results on this conjecture, and detailed references as well, see for instance [12].

Following a somehow similar direction, in §3 we study the minimizers of

$$\frac{1}{4}\iint J(x'-x)\left(u(x')-u(x)\right)^2 dx'dx \;+\frac{1}{4}\int (1-u^2)^2\,dx,$$

on cylinders in $\mathbb{R}^n$ and with suitable constraints at infinity – here J is a (possibly anisotropic) positive interaction potential on $\mathbb{R}^n$ vanishing at infinity. Functionals of this form model phase separation in systems which can be described by one scalar parameter u, and admit two stable phases $u=\pm 1$; for instance, they appear in equilibrium Statistical Mechanics as free energies of continuum limits of Ising spin systems on lattices (see [2], [3], [7] and references therein). Using rearrangement we obtain a simple direct proof of the existence of optimal profiles for transitions for such a model in the n-dimensional case (Corollary 3.3, cf. [1, Theorem 3.3]), and a new and rather optimal uniqueness result (see Theorem 3.10 and Corollary 3.12). For related results about travelling and stationary waves, we refer the reader to [3], [4], and references therein.

Acknowledgements. I would like to thank Friedemann Brock for sharing his deep knowledge of rearrangement principles: the present note is partly the result of several conversations with him. I also gratefully acknowledge the hospitality and support of the Max Planck Institute for Mathematics in the Sciences in Leipzig, where an early version of this paper was written.

1. - Rearrangement of sets

Let us begin with some notation. In the following we use the words *increasing* and *positive* in the weak sense, that is, to mean non-decreasing and non-negative respectively. Given real numbers a and b, $a\vee b$ and $a\wedge b$ mean $\max\{a,b\}$ and $\min\{a,b\}$, respectively, while $a^+=\max\{a,0\}$ and $a^-=\max\{-a,0\}$ are the positive and negative part of a. As usual, 1_A stands for the characteristic function of the set A. Sets and functions are always assumed Borel measurable, and we often identify functions (and sets) which agree almost everywhere. For every $t\in\mathbb{R}$, the t-superlevel of a real function u is the set E_t of all points x such that $t\le u(x)$. We write $|B|$ for the Lebesgue measure of a set B in some euclidean space; we also write "$A=B$ a.e." (resp. "$A\supset B$ a.e.") to mean that A agrees with (resp. contains) B up to negligible subsets. The distributional gradient of a function u is denoted by Du, unless it is locally summable, in which case it agrees a.e. with the pointwise (approximate) gradient, denoted by ∇u. For functions of one variable we simply write $\dot u$.

Throughout this paper we identify $\mathbb{R}^n$ and the product $\mathbb{R}\times\mathbb{R}^{n-1}$, and write x_1 for the first coordinate of $x \in \mathbb{R}^n$. We fix a non-empty bounded open subset Ω in $\mathbb{R}^{n-1}$, and set $D^+ := [0, +\infty) \times \Omega$, $D^- := (-\infty, 0) \times \Omega$.

Definition 1.1. Let $\mathscr{F}$ be the class of all sets A included in the cylinder $\mathbb{R} \times \Omega$ such that the symmetric difference $A \triangle D^+ = (D^+ \setminus A) \cup (A \setminus D^+)$ has finite measure. The rearrangement of $A \in \mathscr{F}$ is the half-cylinder

$$A^* := [a, +\infty) \times \Omega \quad \text{with } a := \tfrac{|D^+ \setminus A| - |A \setminus D^+|}{|\Omega|}. \tag{1.1}$$

Remark 1.2. For $n = 1$ there is no Ω, D^+ and D^- become the positive and negative half-line, respectively; $\mathscr{F}$ is the class of all $A \subset \mathbb{R}$ such that $|A \triangle D^+|$ is finite, and A^* is the half-line $[a, +\infty)$ with $a := |D^+ \setminus A| - |A \setminus D^+|$. All results mentioned below hold for $n = 1$ too, with some obvious modifications in the statements.

Remark 1.3. The rearrangement of sets in $\mathscr{F}$ is the *unique* map which takes sets A into half-cylinders A^* of the form $[a, +\infty) \times \Omega$ and is measure-preserving, in the sense that $|A \setminus A^*| = |A^* \setminus A|$, or, equivalently,

$$|D^+ \setminus A| - |A \setminus D^+| = |D^+ \setminus A^*| - |A^* \setminus D^+|. \tag{1.2}$$

Notice that $A = A^*$ a.e. if and only if A agrees a.e. with a (right) half-cylinder.

Lemma 1.4. *For every couple of sets $A, B \in \mathscr{F}$ we have*

$$|A \setminus B| \geq |A^* \setminus B^*|, \tag{1.3}$$

and equality holds if and only if either $|A \setminus B| = 0$ or $|B \setminus A| = 0$.

Proof. We can assume that A^* includes B^*, otherwise $|A^* \setminus B^*| = 0$ and there is nothing to prove. We write $A \setminus B$ as disjoint union of $(A \setminus B) \cap D^+ = (D^+ \setminus B) \setminus (D^+ \setminus A)$ and $(A \setminus B) \cap D^- = (A \setminus D^+) \setminus (B \setminus D^+)$, and then

$$\begin{aligned} |A \setminus B| &= \big|(D^+ \setminus B) \setminus (D^+ \setminus A)\big| + \big|(A \setminus D^+) \setminus (B \setminus D^+)\big| \\ &\geq |D^+ \setminus B| - |D^+ \setminus A| + |A \setminus D^+| - |B \setminus D^+|. \end{aligned} \tag{1.4}$$

One readily checks that the last term in (1.4) is the measure of $A^* \setminus B^*$ (cf. (1.1)), and (1.3) is proved. The verification of the rest of the claim is straightforward. □

The rearrangement of sets decreases perimeter. More precisely, let $P(A)$ denote the perimeter in the sense of Caccioppoli of A, relative to the open set $\mathbb{R} \times \Omega$ (namely the total variation $\|D1_A\|$ of the distributional gradient of 1_A on $\mathbb{R} \times \Omega$ if it is a bounded Radon measure and $+\infty$ otherwise – see, e.g., [8, chapter 5]). Thus we have the following.

Lemma 1.5. *For every $A \in \mathscr{F}$ there holds $P(A) \geq P(A^*) = |\Omega|$, and we have equality if and only if $A = A^*$ a.e.*

Proof. It is obvious that $P(A^*) = |\Omega|$. Assuming that $P(A)$ is finite, we have

$$P(A) = \|D1_A\| \geq \|D_1 1_A\|, \tag{1.5}$$

where D_1 denote the (distributional) partial derivative in the direction x_1. For every $y \in \Omega$ we take the one-dimensional slice $A_y := \{t : (t, y) \in A\}$; then A_y has finite perimeter $P(A_y)$ in $\mathbb{R}$ for a.e. $y \in \Omega$, and

$$\|D_1 1_A\| = \int_\Omega P(A_y)\, dy. \tag{1.6}$$

Since A belongs to $\mathscr{F}$, A_y must belong to the one-dimensional equivalent of $\mathscr{F}$ for a.e. y, and then $P(A_y) \geq 1$. Hence (1.5) and (1.6) yield the desired inequality $P(A) \geq |\Omega|$. Moreover $P(A) = |\Omega|$ would imply that all partial derivatives of 1_A except $D_1 1_A$ vanish, and that A_y is a right half-line for a.e. y. Combining these two pieces of information we obtain that A is equal to a half-cylinder a.e., and then $A = A^*$ a.e. □

2. – Rearrangement of functions

The following definition can be found in [5, §2] and [9, §2] (cf. also [10, §II.3] for a related one-dimensional concept).

Definition 2.1. Let $\mathscr{X}$ be the class of all functions $u : \mathbb{R} \times \Omega \to [-1, 1]$ whose t-superlevels belong to $\mathscr{F}$ for every $t \in (-1, 1)$. The increasing rearrangement of $u \in \mathscr{X}$ is the function $u^* : \mathbb{R} \times \Omega \to [-1, 1]$ whose t-superlevels are the rearrangements of the t-superlevels of u for every $t \in (-1, 1)$.

Remark 2.2. By (1.1) the t-superlevel of u^* is the cylinder $[a_u(t), +\infty) \times \Omega$, where

$$a_u(t) := \frac{|D^+ \setminus E_t| - |E_t \setminus D^+|}{|\Omega|} \quad \text{for } -1 < t < 1, \tag{2.1}$$

and E_t is the t-superlevel of u. We call a_u the *distribution function* of u, in analogy with a similar notion introduced by Hardy, Littlewood and Polya.

If we define the distance between two sets as the measure of their symmetric difference, the map $t \mapsto E_t$ is left-continuous for every $u \in \mathscr{X}$, and then the distribution function a_u is increasing and left-continuous. Using this fact one can show that the function u^* is well-defined and can be written as $u^*(x) := w(x_1)$ with $w : \mathbb{R} \to [-1, 1]$ increasing and right-continuous; in fact w is the (left) inverse of a_u where a_u is strictly increasing.

Notice that $u = u^*$ a.e. if and only if u agrees a.e. with an increasing function of the first variable.

REMARK 2.3. Lemma 1.4 implies that the measure of pre-images of intervals of type $B = [t_1, t_2)$ with $-1 < t_1 < t_2 < 1$ is preserved by increasing rearrangement, that is

$$\big|\{x : u(x) \in B\}\big| = \big|\{x : u^*(x) \in B\}\big|.$$

Since the class of all sets B relatively compact in $(-1, 1)$ which satisfy this identity is closed by finite disjoint union, countable increasing union and countable decreasing intersection, it contains all Borel sets relatively compact in $(-1, 1)$.

REMARK 2.4. Since u^* is a bounded increasing function of the first variable, Du^* is a bounded Radon measure. If in addition u is of class $W^{1,1}_{\mathrm{loc}}$, then Du^* also belongs to $L^1(\mathbb{R} \times \Omega)$.

Let us prove the latter claim. Since $u^*(x)$ can be written as $v(x_1)$ with v increasing, all its partial derivatives but $D_1 u^*$ vanish, while $D_1 u^*$ agrees with the product measure $\dot{v} \times \mu$, where $\dot{v}$ is the measure derivative of v and μ is the $(n-1)$-dimensional Lebesgue measure on Ω. Therefore, proving that Du^* belongs to L^1 reduces to show that $\dot{v}(A) = 0$ for every $A \subset \mathbb{R}$ such that $|A| = 0$. Consider now the maximal increasing multifunction $\tilde{v}$ corresponding to v [(1)] and set $B := \tilde{v}(A)$. Then $\dot{v}(A) = |B|$, and, taking into account Lemma 1.5,

$$\dot{v}(A) = |B| = \int_B \frac{P(E_t^*)}{|\Omega|} dt \le \int_B \frac{P(E_t)}{|\Omega|} dt = \frac{1}{|\Omega|} \int_{u^{-1}(B)} |\nabla u| \, dx,$$

where E_t^* and E_t denote the t-superlevels of u^* and u respectively, and the last equality follows from the coarea formula (see [8, §5.5]). Now we notice that $A = \tilde{v}^{-1}(B) = v^{-1}(B)$, and then $A \times \Omega = (u^*)^{-1}(B)$. If A is negligible in $\mathbb{R}$, then $(u^*)^{-1}(B)$ is negligible in $\mathbb{R} \times \Omega$, and so is $u^{-1}(B)$ (by Remark 2.3). Hence $\dot{v}(A) = 0$ by the previous formula.

REMARK 2.5. The class $\mathscr{X}$ contains all functions $u : \mathbb{R} \times \Omega \to [-1, 1]$ such that $u(x)$ tends uniformly to ± 1 as $x_1 \to \pm\infty$. If we set $\bar{u} := +1$ on D^+ and $\bar{u} := -1$ on D^-, then $\mathscr{X}$ also includes the class $\mathscr{X}_p$ of all $u : \mathbb{R} \times \Omega \to [-1, 1]$ such that $u - \bar{u}$ belongs to $L^p(\mathbb{R} \times \Omega)$, with $1 \le p < +\infty$ (notice that each $\mathscr{X}_p$ can be endowed with the complete distance $d(u, v) := \|u - v\|_p$). More generally, a function $u : \mathbb{R} \times \Omega \to [-1, 1]$ belongs to $\mathscr{X}$ if and only if there exists a convex function $g : [-2, 2] \to [0, +\infty]$, null in 0 and strictly positive elsewhere, such that $\int g(u - \bar{u})$ is finite.

We just sketch the proof of the last claim. To construct g for a given $u \in X$, we use the following identity (which can be derived in a similar way as the second formula in the proof of Theorem 2.6 below): for every $v : \mathbb{R} \times \Omega \to (-2, 2)$ and every g as above

$$\int_{\mathbb{R} \times \Omega} g(v(x)) \, dx = \int_0^2 \dot{g}(s) \cdot |\{v \ge s\}| \, ds - \int_{-2}^0 \dot{g}(s) \cdot |\{v \le s\}| \, ds.$$

(1) Namely, the function which associates to each $y \in \mathbb{R}$ all values t between the left and right limit of v at y (thus $\tilde{v}$ agrees with v at all points y where v is continuous, that is, all but countably many).

Let E_t denote the t-superlevel of u for every $t \in (-1,1)$. Then, for $s \in (0,2)$, the s-superlevel of $u-\bar{u}$ is $E_{s-1} \setminus D^+$, while for $s \in (-2,0)$ the s-sublevel of $u-\bar{u}$ is $D^+ \setminus E_{s+1}$. Therefore we take g so that $g(0) := 0$, $\dot{g}(s) := (1 \vee |E_{s-1} \setminus D^+|)^{-1}$ for $s \in (0,2)$, and $\dot{g}(s) := -(1 \vee |D^+ \setminus E_{s+1}|)^{-1}$ for $s \in (-2,0)$. Thus $\dot{g}$ is increasing and smaller than 1 in modulus, and then g is convex, positive, and 1-Lipschitz. Moreover, the formula above (applied to $v := u - \bar{u}$) yields $\int g(u-\bar{u}) \leq 4$. Conversely, given u such that $\int g(u-\bar{u})$ is finite for some g, for $t \in (-1,1)$ we obtain that $D^+ \setminus E_t$ is the $(t-1)$-sublevel of $u-\bar{u}$, while $E_t \setminus D^+$ is the $(t+1)$-superlevel of $u-\bar{u}$, and both must have finite measure by the formula above (applied to $v := u-\bar{u}$); hence u belongs to $\mathscr{X}$.

The properties of rearrangement of sets listed in Remark 1.3 and Lemmas 1.4 and 1.5 are mirrored by the corresponding properties of rearrangement of functions given, respectively, in Theorem 2.6, Proposition 2.7 (and Theorem 2.11), and Theorem 2.10 below.

Theorem 2.6 was already proved in the one-dimensional case in [1, §5], weaker versions of the n-dimensional statement were given in [5, §2], and [9, §2]. The idea of the proof of Proposition 2.7 is taken from the proof of [1, Lemma 5.9]. The first part of Theorem 2.10 was already proved for $g(t) = t^2$ in [5, §2], and generalized to $g(t) = t^p$ in [9, §2]. Some of these results are well-known for radial decreasing rearrangement (see for instance [10, §II.9], [11], and references therein). Notice however that the second part of Theorem 2.10 is stronger than the corresponding statement for radial decreasing rearrangement.

Theorem 2.6. *For every positive lower semicontinuous function g : $[-1,1] \to [0,+\infty]$ and every $u \in \mathscr{X}$ there holds*

$$\int_{\mathbb{R}\times\Omega} g\big(u(x)\big)\,dx = \int_{\mathbb{R}\times\Omega} g\big(u^*(x)\big)\,dx. \tag{2.2}$$

Proof. We can assume that g vanishes at ± 1, otherwise both sides of (2.2) are infinite and there is nothing to prove. Since every lower semicontinuous function g vanishing at ± 1 can be monotonically approximated by smooth functions vanishing in a neighborhood of ± 1, it suffices to prove (2.2) for such functions g only.

If g is of class C^1 and $g(\pm 1) = 0$, then $g(s) = \int_{-1}^s \dot{g} = -\int_s^1 \dot{g}$, and

$$g\big(u(x)\big) = \int_{-1}^1 \dot{g}(t)\,1_{E_t}(x)\,dt = -\int_{-1}^1 \dot{g}(t)\,\big(1 - 1_{E_t}(x)\big)\,dt,$$

Hence

$$\int_{\mathbb{R}\times\Omega} g\big(u(x)\big)dx = \int_{D^-}\Big(\int_{-1}^1 \dot{g}(t)\,1_{E_t}(x)dt\Big)dx - \int_{D^+}\Big(\int_{-1}^1 \dot{g}(t)\,\big(1-1_{E_t}(x)\big)dt\Big)dx.$$

Now we apply Fubini's theorem to the right-hand side [(2)], and integrate with

[(2)] We should verify that the function $\dot{g}(t)\,1_{E_t}(x)$ – and similarly $\dot{g}(t)\,\big(1 - 1_{E_t}(x)\big)$ – is summable on $D^- \times (-1,1)$. This follows from the fact that $\dot{g}(t) = 0$ for $t \leq -1+\delta$ for some $\delta > 0$ (by assumption), while $|E_t \setminus D^+|$ is bounded by $|E_{-1+\delta} \setminus D^+|$ for $t \geq -1+\delta$.

respect to x:

$$\int_{\mathbb{R}\times\Omega} g\big(u(x)\big)\,dx = \int_{-1}^{1} \dot g(t)\,\big(|E_t \setminus D^+| - |D^+ \setminus E_t|\big)\,dt. \tag{2.3}$$

Eventually we apply identity (2.3) to $\int g(u)$ and $\int g(u^*)$, and then (1.2) yields (2.2). □

Proposition 2.7. *Let be given a lower semicontinuous function $g : [-2,2] \to [0,+\infty]$ which is positive, convex, null at 0, and even (i.e., $g(s) = g(-s)$). Then for every $u, v \in \mathscr{X}$ there holds*

$$\int_{\mathbb{R}\times\Omega} g(u - v) \geq \int_{\mathbb{R}\times\Omega} g(u^* - v^*). \tag{2.4}$$

Proof. Since every l.s.c. convex functions can be monotonically approximated by smooth ones, it suffices to prove the claim for g of class C^2. Since $g(0) = \dot g(0) = 0$, for every s, s' such that $s < s'$ there holds

$$g(s' - s) = \int_s^{s'}\int_t^{s'} \ddot g(t' - t)\,dt'dt.$$

Let E_t and F_t denote the t-superlevels of u and v respectively; then one can readily check that the previous identity yields

$$g\big(u(x) - v(x)\big) = \int_{-1}^{1}\int_t^{1} \ddot g(t' - t)\,\big[1_{E_{t'}}(x)\,(1 - 1_{F_t}(x)) + 1_{F_{t'}}(x)\,(1 - 1_{E_t}(x))\big]\,dt'dt,$$

and integrating over all $x \in \mathbb{R}\times\Omega$

$$\int_{\mathbb{R}\times\Omega} g\big(u(x) - v(x)\big)\,dx = \int_{-1}^{1}\int_t^{1} \ddot g(t' - t)\,\big(|E_{t'} \setminus F_t| + |F_{t'} \setminus E_t|\big)\,dt'dt. \tag{2.5}$$

Inequality (2.4) follows by applying identity (2.5) to $\int g(u - v)$ and $\int g(u^* - v^*)$, and taking Lemma 1.4 into account. □

Remark 2.8. If g is of class C^2, we deduce from the proof above (and the second part of Lemma 1.4) that if equality holds in (2.4), then for every $t, t' \in (-1,1)$ we have either $\ddot g(t' - t) = 0$, or $E_{t'} \subset F_t$ a.e., or $E_{t'} \supset F_t$ a.e.[(3)]

Remark 2.9. Taking $g(t) := |t|^p$ with $1 \leq p < \infty$, (2.4) becomes

$$\|u^* - v^*\|_p \leq \|u - v\|_p. \tag{2.6}$$

Hence increasing rearrangement is a 1-Lipschitz mapping of $\mathscr{X}_p$ into itself (cf. Remark 2.5).

[(3)] Here we used the left-continuity in t of E_t and F_t (cf. Remark 2.2) to pass from "almost every" to "every" $t, t' \in (-1,1)$.

Theorem 2.10. *Let be given a function $g : [0, +\infty) \to [0, +\infty)$ which is finite, convex, null at 0 and strictly increasing. Then, for every $u \in \mathscr{X} \cap W^{1,1}_{\mathrm{loc}}$ we have* [(4)]

$$\int_{\mathbb{R}\times\Omega} g\big(|\nabla u|\big) \geq \int_{\mathbb{R}\times\Omega} g\big(|\nabla u^*|\big). \tag{2.7}$$

Moreover, when the right-hand side is finite, equality holds if and only if $u = u^*$ a.e.

Proof. Since g is convex, strictly increasing, and null at 0, for every $s > 0$ there exist $a > 0$ and $b \geq 0$ (both depending on s) such that

$$g(s) = as - b \quad \text{and} \quad g(s') \geq as' - b \quad \text{for all } s' \geq 0. \tag{2.8}$$

We take a and b for $s := |\nabla u^*(x)|$: since u^* has the same gradient at all points where it takes the same value, a and b can be chosen [(5)] so that they only depend on the value of u^*. Now the following chain of inequalities gives (2.7)

$$\begin{aligned}
\int_{\mathbb{R}\times\Omega} g\big(|\nabla u^*|\big)\, dx &= \int_{\mathbb{R}\times\Omega} a(u^*)|\nabla u^*| - b(u^*)\, dx \\
&= \int_{-1}^{1} a(t)\, P(E_t^*)\, dt - \int_{\mathbb{R}\times\Omega} b(u^*)\, dx \\
&\leq \int_{-1}^{1} a(t)\, P(E_t)\, dt - \int_{\mathbb{R}\times\Omega} b(u)\, dx \\
&= \int_{\mathbb{R}\times\Omega} a(u)|\nabla u| - b(u)\, dx \leq \int_{\mathbb{R}\times\Omega} g\big(|\nabla u|\big)\, dx.
\end{aligned} \tag{2.9}$$

The first identity and the last inequality follow from the choice of a and b (cf. (2.8)); we get the second and third identities by applying the coarea formula (see [8, §5.5]) to u^* and u, respectively; the first inequality follows from Lemma 1.5 and Theorem 2.6. Moreover, if the first inequality is not strict, then $P(E_t^*) = P(E_t)$ for a.e. t, and then $u = u^*$ a.e. (cf. Lemma 1.5).

Notice that this proof does not work if both integrals at the second line of (2.9) are infinite. Therefore a slight modification is required: we assume that the integral of $g(|\nabla u^*|)$ is finite, fix a small parameter $\delta > 0$, and choose $0 < s_1 < s_2 < +\infty$ so that the integral of $g(|\nabla u^*|)$ over all x such that $|\nabla u^*| < s_1$ or $|\nabla u^*| > s_2$ is smaller than δ. Then we take a and b as before for the values of u^* such that $s_1 \leq |\nabla u^*| \leq s_2$, and set $a = b = 0$ for the others. Now both integrals at the second line of (2.9) are finite, and we get

(4) By Remark 2.4, ∇u^* is an L^1 function, and then both sides of (2.7) are well-defined.

(5) If g is not differentiable at s, the corresponding values for a and b are not uniquely determined; consequently the choice of a and b may be not Borel measurable with respect to the value of u^*. This can be fixed by a suitable measurable selection theorem, e.g. [6, Theorem III.6].

$\int g(|\nabla u^*|) - \delta \leq \int g(|\nabla u|)$, which yields (2.7) by letting $\delta \to 0$. The rest of the proof can be similarly fixed. □

For the rest of this section we assume that Ω is a parallelepiped spanned by the vectors $v_1, \dots, v_{n-1} \in \mathbb{R}^{n-1}$, namely $\Omega := \big\{ \sum \alpha_i v_i : \alpha_i \in (0,1) \big\}$. We also assume that all functions on $\mathbb{R} \times \Omega$ (and corresponding superlevels) are extended to $\mathbb{R}^n$ by Ω-periodicity, so that $u(x) = u(x + v_i)$ a.e. for every i. For all such functions we consider

$$G(u) := \int_{\mathbb{R}\times\Omega} \int_{\mathbb{R}^n} J(h)\, g\big(u(x+h) - u(x)\big)\, dh\, dx, \tag{2.10}$$

where g is taken as in Proposition 2.7, and J is a positive Borel function on $\mathbb{R}^n$.

THEOREM 2.11. *If G is given in* (2.10)*, for every $u \in \mathscr{X}$ there holds*

$$G(u) \geq G(u^*). \tag{2.11}$$

If in addition we have that g is of class C^2 and $\ddot{g}(0) > 0$, and J is strictly positive in a neighborhood of 0*, then $G(u) = G(u^*) < +\infty$ implies $u = u^*$ a.e.*

PROOF. If we denote by $\tau_h u$ the translated function $u(x+h)$, $G(u)$ can be rewritten as

$$G(u) = \int_{\mathbb{R}^n} J(h) \Big[\int_{\mathbb{R}\times\Omega} g(\tau_h u - u) \Big]\, dh, \tag{2.12}$$

and to prove (2.11) it suffices to apply inequality (2.4) to $\int g(\tau_h u - u)$ for every h, and take into account the identity $(\tau_h u)^* = \tau_h(u^*)$, which follows from the identity $(E-h)^* = E^* - h$.

We prove now the second part of the claim. Let be given u such that $G(u) = G(u^*) < \infty$. Then $\int g(\tau_h u - u) = \int g(\tau_h u^* - u^*)$ for a.e. h such that $J(h) > 0$, and therefore also for every h in the support of J. By Remark 2.8 and the assumptions on g and J, this implies that for every h in a neighborhood U of 0 and every $t \in (-1,1)$, one of the following inclusions must hold: $E_t - h \supset E_t$ a.e. or $E_t - h \subset E_t$ a.e.

Let t be fixed for the time being, and take $h \in U$ so that the first component h_1 is strictly positive. Since the second inclusion implies $E_t^* - h \subset E_t^*$, which cannot hold when $h_1 > 0$, we conclude that $E_t - h \supset E_t$ a.e. Moreover, if we denote by $\tilde{E}_t$ the set of density points of E_t, this inclusion becomes

$$\tilde{E}_t \supset \tilde{E}_t + h. \tag{2.13}$$

Now, let H be the open half-space of all $h \in \mathbb{R}^n$ with $h_1 > 0$, and consider the set of all $h \in H$ such that inclusion (2.13) holds. Since this set contains $U \cap H$, which is a neighborhood of 0 in H, and is closed under summation, then it must agree with H.

Therefore, $\tilde{E}_t$ includes $\tilde{E}_t+H$, which is the open half-space $(a,+\infty)\times\mathbb{R}^{n-1}$, where a is the infimum of h_1 over all $h\in\tilde{E}_t$. On the other hand, $\tilde{E}_t$ is clearly included in $[a,+\infty)\times\mathbb{R}^{n-1}$. Hence $E_t=(a,+\infty)\times\mathbb{R}^{n-1}$ a.e., and then $E_t=E_t^*$ a.e. (cf. Remark 1.3). Since this holds for every $t\in(-1,1)$, we have proved that $u=u^*$ a.e. □

Remark 2.12. The assumption on J in the second part of Theorem 2.11 can be weakened up to requiring that the approximate tangent cone [(6)] at 0 of the set where J is strictly positive is $\mathbb{R}^n$. The following example shows that this assumption is close to optimal.

Example 2.13. Assume that J is supported on the cone

$$C_a:=\bigl\{h\in\mathbb{R}^n:\,|h_1|\geq a|h|\bigr\}$$

for some $a>0$, and let E be the set of all $x=(x_1,\tilde{x})$ such that $x_1\geq f(\tilde{x})$, where $f:\mathbb{R}^{n-1}\to\mathbb{R}$ is a non-constant Ω-periodic function with Lipschitz constant smaller than $a/\sqrt{1-a^2}$. Let u be the function equal to 1 on E and to -1 elsewhere, so that all superlevels of u are equal to E.

The definition of E yields $x-h\in E$ for every $x\in E$ and every $h\in C_a$ with $h_1\leq 0$, i.e., $E-h\subset E$. Conversely $E-h\supset E$ if $h\in C_a$ and $h_1\geq 0$. From Remark 2.8 we deduce that $\int g(\tau_h u-u)=\int g(\tau_h u^*-u^*)$ for all $h\in C_a$ (as in the proof of Theorem 2.11, $\tau_h u$ denotes the translated function $u(x+h)$), and then $G(u)=G(u^*)$ by (2.12). On the other hand u does not agree with u^* because the function f which determines E is not constant.

3. – Some applications

Theorems 2.6 and 2.10 have the following immediate corollary.

Corollary 3.1. *Let be given $p\in(1,\infty)$ and a positive continuous function W on $\mathbb{R}$ which vanishes at ±1 only. Then the infimum of the functional*

$$\int_{\mathbb{R}\times\Omega}|\nabla u|^p+W(u), \tag{3.1}$$

on $\mathscr{X}$ is equal to the infimum on the subclass of rearranged functions, and every minimizer $u\in\mathscr{X}$ agrees a.e. with u^*.

It follows immediately that every minimizer u can be written as $u(x)=w(x_1-h)$ for a.e. x, where h belongs to $\mathbb{R}$ and w minimizes

$$\int_{\mathbb{R}}|\dot{w}|^p+W(w)$$

(6) The approximate tangent cone at y of a set $S\subset\mathbb{R}^n$ is the cone generated by the cluster points of every sequence $\frac{y_n-y}{|y_n-y|}$, where all y_n are density points of S, and converge to y as $n\to\infty$.

among all increasing functions on $\mathbb{R}$ which tend to ± 1 at $\pm\infty$ and satisfy the normalization $w(0) = 0$. The existence of such minimizers can be easily proved by the direct method (cf. the proof of Corollary 3.3 below), and agrees with the unique solution of the following Cauchy problem [7]

$$\dot{w} = \left[\frac{W(w)}{p-1}\right]^{1/p} \quad , \quad w(0) = 0.$$

The fact that the infimum of (3.1) is achieved on the class of rearranged functions was proved in [9, Theorem 1.1], and in [5, Theorem 1] for $p = 2$, but it can also be directly deduced from a sharp lower bound *á la* Modica-Mortola of the integral in (3.1) (see for instance [13]). The second part of Theorem 2.10 allows a simple direct proof of the fact that *every* minimizer of (3.1) must be a function of the first variable only.

Minimizers of (3.1) in $\mathscr{X}$ are the n-dimensional optimal profiles for transition in the Cahn-Hilliard model for phase separation (cf. [13]). An interesting variation of that model is obtained by replacing the first term in (3.1) by a non-local interaction energy, namely setting

$$(3.2)\quad F(u) := \frac{1}{4}\int_{\mathbb{R}\times\Omega}\int_{\mathbb{R}^n} J(h)\,\big(u(x+h)-u(x)\big)^2\,dh\,dx \;+\int_{\mathbb{R}\times\Omega} W\big(u(x)\big)\,dx,$$

where W is taken as above and J is a positive even function on $\mathbb{R}^n$ which satisfies

$$(3.3)\qquad \int_{\mathbb{R}^n} J(h)\,dh = 1 \quad , \quad \int_{\mathbb{R}^n} J(h)\,|h|\,dh < +\infty.$$

For the rest of this paper we assume, as at the end of the previous section, that Ω is a parallelepiped spanned by the vectors $v_1, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$, and that all functions on $\mathbb{R} \times \Omega$ are extended to $\mathbb{R}^n$ by Ω-periodicity (so that the double integral in (3.2) makes sense). The fact that F is not identically $+\infty$ on $\mathscr{X}$ is granted by the second condition in line (3.3) (see [1, §4c] for a more detailed analysis).

Theorems 2.6 and 2.11 immediately yield the following.

Corollary 3.2. *The infimum of F on $\mathscr{X}$ is equal to the infimum on the subclass of rearranged functions. If in addition J is strictly positive in a neighborhood of* 0 *then every minimizer $u \in \mathscr{X}$ agrees a.e. with u^*.*

A simple computation shows that every function u on $\mathbb{R}^n$ which depends only on the first variable, that is, $u(x) = w(x_1)$ for some w defined on $\mathbb{R}$, satisfies

$$(3.4)\qquad F(u) = |\Omega| \cdot \overline{F}(w),$$

[7] Multiply by $\dot{w}$ both sides of the identity $p(p-1)|\dot{w}|^{p-2}\ddot{w} = \dot{W}(w)$, which is the Euler-Lagrange equation of $\int |\dot{w}|^p + W(w)$, and then integrate.

where

$$\overline{F}(w) := \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{J}(h)\big(w(x+h)-w(x)\big)^2\,dh\,dx \;+\int_{\mathbb{R}} W\big(w(x)\big)\,dx, \tag{3.5}$$

and

$$\bar{J}(h) := \int_{\mathbb{R}^{n-1}} J(h,\tilde{h})\,d\tilde{h}. \tag{3.6}$$

In particular, identity (3.4) applies to rearranged functions, and therefore Corollary 3.2 implies that F is minimized on $\mathscr{X}$ by functions of the form $u(x) := w(x_1 - h)$ with $h \in \mathbb{R}$, where w minimizes $\overline{F}$ on the class (cf. Remark 2.2)

$$\overline{\mathscr{X}} := \left\{ w : \mathbb{R} \to [-1,1] \;\middle|\; \begin{array}{l} w \text{ increasing, right-continuous, such that} \\ w(x) \geq 0 \text{ for } x > 0,\ w(x) \leq 0 \text{ for } x < 0, \\ \text{and } \lim w = \pm 1 \text{ at } \pm\infty \end{array} \right\}. \tag{3.7}$$

The existence of minimizers of $\overline{F}$ on $\overline{\mathscr{X}}$ can be easily proved by the direct method (see the proof of [1, Theorem 2.4]). Take a minimizing sequence (w_k); possibly passing to a subsequence, we can assume that they converge a.e. to a function w (8). Obviously w agrees a.e. with an increasing right-continuous function, takes values in $[-1,1]$, satisfies $w(x) \geq 0$ for $x > 0$ and $w(x) \leq 0$ for $x < 0$ (cf. (3.7)), and $\liminf \overline{F}(w_k) \geq \overline{F}(w)$ by Fatou's lemma. Since $\overline{F}(w)$ is finite and W is strictly positive in $(-1,1)$, the limit of w at $\pm\infty$ must be ± 1. Then w belongs to $\overline{\mathscr{X}}$, and minimizes $\overline{F}$.

We have thus proved the following.

Corollary 3.3. *There always exists at least one minimizer w of the one-dimensional functional $\overline{F}$ on the class $\overline{\mathscr{X}}$, and every function u on $\mathbb{R}^n$ of the form $u(x) := w(x_1 + h)$ with $h \in \mathbb{R}$ minimizes the n-dimensional functional F on $\mathscr{X}$. If in addition J is strictly positive in a neighborhood of* 0*, all minimizers of F on $\mathscr{X}$ can be written as above.*

Remark 3.4. In the phase separation model mentioned above, minimizers of F on $\mathscr{X}$ are called instantons, or optimal profiles for transition (with respect to the direction x_1 – notice that F is not isotropic unless J is).

The fact that the infimum of F over $\mathscr{X}$ is achieved on rearranged functions was proved, in a much more complicated way, in [1, §3]. What is new in the previous corollaries is that *every* minimizer of F on $\mathscr{X}$ agrees with its rearrangement, and can therefore be obtained from a minimizer of $\overline{F}$ on $\overline{\mathscr{X}}$.

We remark that rearrangement techniques plays an essential rôle in the proof not only of the monotonicity of minimizers of F, but also of their existence. Indeed, even in dimension $n = 1$, due to the lack of compactness of X,

(8) This is a well-known compactness result for uniformly bounded sequences of increasing functions. Alternatively, one can notice that the sequence (w_k) is bounded in $BV_{\mathrm{loc}}(\mathbb{R})$, and therefore pre-compact in $L^p_{\mathrm{loc}}(\mathbb{R})$ for every $p < \infty$.

the direct method can be hardly applied directly to the minimization of F (cf., however, [1, §4d]).

Remark 3.5. No integrability assumptions on J are required for the first parts of Corollaries 3.2 and 3.3 to hold; in particular (3.3) can be removed. For the second parts of both corollaries, the positivity assumption on J can be weakened as in Remark 2.12, while the integrability assumptions in (3.3) can be replaced by

$$\int_{\mathbb{R}} \bar{J}(h) \cdot (h \wedge h^2)\, dh < +\infty.$$

Indeed this condition is necessary and sufficient to ensure that $\overline{F}$ is not identically $+\infty$ on $\overline{\mathscr{X}}$ (see [1, Theorem 4.6]), or, equivalently, that F is not identically $+\infty$ on X.

Remark 3.6. By Corollary 3.3, every minimizer w of $\overline{F}$ on $\overline{\mathscr{X}}$ also minimizes $\overline{F}$ on the one-dimensional version of $\mathscr{X}$, which contains in particular all functions on $\mathbb{R}$ bounded between -1 and 1 converging to ± 1 at $\pm\infty$ (cf. Remark 2.5). A simple truncation argument shows that the boundedness assumption can be removed (9). Hence, if W is of class $C^1(\mathbb{R})$ and positive, a standard computation (see, for instance, [1, §4a]) shows that w must satisfy the Euler-Lagrange equation

$$\bar{J} * w - w = \dot{W}(w) \quad , \quad \lim_{x \to \pm\infty} w(x) = \pm 1, \tag{3.8}$$

where $*$ is the usual convolution product, and the equality holds almost everywhere.

Existence and uniqueness results for solutions of (3.8) have been established in [7] for a special choice of W, and then generalized in [3] to include also travelling waves for the associated parabolic equation $w_t = \bar{J} * w - w - \dot{W}(w)$. An extension of these results to the n-dimensional case has been given in [4]. Incidentally, Corollary 3.3 provides an alternative simple proof of the existence of solutions of (3.8) (cf. [7, Theorem 1.1], [4, Theorem 2.7]).

Remark 3.7. Together with Corollary 3.3, the uniqueness up to translations of solutions of (3.8) proved in [3, Theorem 4.1], imply the uniqueness up to translations of minimizers of F on $\mathscr{X}$ (cf. Corollary 3.12 below). The latter result requires that J is of class C^1, W is of class C^2, and the interval $(-1, 1)$ splits into three sub-intervals where $t + \dot{W}(t)$ is either increasing or decreasing. Thus the positivity of J in a neighborhood of 0 is only needed in the reduction to the one-dimensional case. The following example shows that it cannot be completely removed.

Example 3.8. If we set $W(t) := (1 - t^2)/2$, the function w given by $w :\equiv 1$ on $[0, +\infty)$ and $w :\equiv -1$ on $(-\infty, 0)$ minimizes $\overline{F}$ on $\overline{\mathscr{X}}$ for every admissible choice of J (see [1, §2.19]). On the other hand, if we take J and u

(9) Replacing w with the truncated function $(w \wedge 1) \vee -1$ reduces the value of $\overline{F}$.

as in Example 2.13, we have $F(u) = F(u^*)$, and u^* minimizes F on $\mathscr{X}$ because $u^*(x) = w(x_1 - h)$ for some h. Hence u minimizes F on $\mathscr{X}$, too, but does not agree with any translation of u^*.

In the rest of this section we briefly sketch an alternative proof of the uniqueness of minimizers of $\overline{F}$ on $\overline{\mathscr{X}}$ which relies on an interesting convexity argument. We will assume that W is of class $C^1(\mathbb{R})$ and $\bar{J}$ is strictly positive in a neighborhood of 0 (cf. Corollaries 3.2 and 3.3). We begin with a simple observation.

Lemma 3.9. *Every increasing solution w of the equation $\bar{J} * u - u = \dot{W}(u)$ is either constant or strictly increasing. Hence every minimizer of $\overline{F}$ on $\overline{\mathscr{X}}$ is strictly increasing.*

Proof. Take a maximal open (possibly unbounded) interval I where w is constant. Equation $\bar{J} * u - u = \dot{W}(u)$ implies that $\bar{J} * w$ is constant on I and then $(\bar{J} * w)' = \bar{J} * \dot{w} = 0$ a.e. in I. Since $\dot{w}$ is a positive measure and $\bar{J}$ is a positive function, $\dot{w}$ must vanish on $I + U$ where U is any interval containing 0 where $\bar{J}$ is strictly positive. Hence w is constant on $I + U$, which contradicts the maximality of I unless $I = \mathbb{R}$. □

Using the distribution function introduced in Remark 2.2, we can rewrite the one-dimensional version of identity (2.3), with u and g replaced by w and W respectively, as follows [10]

$$\int_{\mathbb{R}} W(w(x))\,dx = -\int_{-1}^{1} \dot{W}(t)\,a_w(t)\,dt = \int_{-1}^{1} W(t)\,d\dot{a}_w(t). \tag{3.9}$$

Similarly, identity (2.5) becomes, for every w and v in $\overline{\mathscr{X}}$,

$$\int_{\mathbb{R}} g\big(v(x)-w(x)\big)^2 dx = \int_{-1}^{1}\int_{t}^{1} \ddot{g}(t'-t)\big[(a_w(t)-a_v(t'))^+ + (a_v(t)-a_w(t'))^+\big]\,dt'dt,$$

and replacing $v(x)$ with $w(x+h)$ and $g(t)$ with t^2,

$$\begin{aligned}\int_{\mathbb{R}} &\big(w(x+h) - w(x)\big)^2\,dx \\ &= \int_{-1}^{1}\int_{t}^{1} 2\big[(a_w(t) - a_w(t') + h)^+ + (a_w(t) - a_w(t') - h)^+\big]\,dt'dt.\end{aligned}$$

Now we set $K(y) := \frac{1}{2}\int \bar{J}(h)\big[(-y+h)^+ + (-y-h)^+\big]\,dh$, and integrate the previous identity in $\frac{1}{4}\bar{J}(h)\,dh$

$$\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{J}(h)\,\big(w(x+h)-w(x)\big)^2\,dh\,dx = \int_{-1}^{1}\int_{t}^{1} K\big(a_w(t')-a_w(t)\big)\,dt'dt. \tag{3.10}$$

[10] The first identity in (3.9) was proved only for smooth W vanishing in a neighborhood of ± 1, and indeed the second integral in (3.9) may be not well-defined otherwise. Yet the last integral is always well-defined because both the function W and the measure $\dot{a}_w$ are positive; hence the equality of the first and last term in (3.9) holds for every W of class C^1, and can be proved by increasing approximation.

Formulas (3.9) and (3.10) yield the remarkable identity (cf. [1, Theorem 2.11)

$$\overline{F}(w) = \int_{-1}^{1}\int_{t}^{1} K\big(a_w(t') - a_w(t)\big)\, dt'dt + \int_{-1}^{1} W(t)\, d\dot{a}_w(t). \tag{3.11}$$

Notice that K is convex (being an average of convex functions), and then the right-hand side of (3.11) is a convex functional of a_w. This is the key point in the proof of the following uniqueness result.

THEOREM 3.10. *If W is of class $C^1(\mathbb{R})$ and $\bar{J}$ is strictly positive in a neighborhood of* 0, *then the minimizer of $\overline{F}$ on $\overline{\mathscr{X}}$ is unique.*

PROOF. Consider two functions w_0 and w_1 in $\overline{\mathscr{X}}$, with distribution functions a_0 and a_1, such that $\overline{F}(w_0)$ and $\overline{F}(w_1)$ are both finite. For every $s \in (0,1)$ let w_s be the function in $\overline{\mathscr{X}}$ with distribution function

$$a_s := sa_1 + (1-s)a_0.$$

We compute the second derivative in s of $\overline{F}(w_s)$ using formula (3.11): since $\ddot{K} = \bar{J}$ a.e. (recall that J, and therefore also $\bar{J}$, are even) and the second integral at the right-hand side of (3.11) is linear in a_w, we get

$$\frac{d^2}{ds^2}\overline{F}(w_s) = \int_{-1}^{1}\int_{t}^{1} \bar{J}\big(a_s(t') - a_s(t)\big)\,\big(a_1(t') - a_0(t') - a_1(t) + a_0(t)\big)^2\, dt'dt. \tag{3.12}$$

Since $\bar{J}$ is positive, $\overline{F}(w_s)$ is a convex function of s.

Assume now that w_0 and w_1 minimize $\overline{F}$ on $\overline{\mathscr{X}}$. By Lemma 3.9, they are both strictly increasing, and then a_0 and a_1 are continuous on $(-1,1)$, vanish at 0 (cf. (3.7)) and converge to $\pm\infty$ at ± 1. On the other hand, the function $\overline{F}(w_s)$, being convex in s, must also be constant, that is, the integral at the right-hand side of (3.12) must vanish for every $s \in (0,1)$. Combining this fact and the positivity of $\bar{J}$ in a neighborhood of 0, we deduce that

$$a_1(t') - a_0(t') = a_1(t) - a_0(t)$$

for every $t \in (-1,1)$ and every t' in some neighborhood of t. Hence $a_1(t) - a_0(t)$ is constant in t, and must vanish everywhere because $a_1(0) - a_0(0) = 0$. Thus $a_0 \equiv a_1$, and $w_0 \equiv w_1$. □

REMARK 3.11. Theorem 3.10 holds even if W is continuous and of class C^1 on $(-1,1)$. Under these assumptions a minimizer w of $\overline{F}$ on $\overline{\mathscr{X}}$ satisfies the equation $\bar{J} * w - w = \dot{W}(w)$ only on the set where $-1 < w < 1$, and a modification of the proof of Lemma 3.9 yields that w is strictly increasing on this set (but not elsewhere, cf. Example 3.8). Hence the distribution function a_w is continuous in $(-1,1)$, which is what we need in the proof of Theorem 3.10.

Combining Theorem 3.10 and Corollary 3.3, we obtain the following.

Corollary 3.12. *If W is of class C^1 and J is strictly positive in a neighborhood of 0, then the minimizers of F on $\mathscr{X}$ are unique up to translations.*

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