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Calcul des variations/Calculus of Variations

The calibration method for the Mumford-Shah functional

Giovanni ALBERTI [a], **Guy BOUCHITTÉ** [b], **Gianni DAL MASO** [c]

[a] Dipartimento di Matematica, via Buonarroti 2, 56127 Pisa, Italy
E-mail: alberti@dm.unipi.it

[b] Université de Toulon et du Var, BP 132, 83957 La Garde Cedex, France
E-mail: bouchitte@univ-tln.fr

[c] SISSA, via Beirut 4, 34014 Trieste, Italy
E-mail: dalmaso@sissa.it

Abstract. In this note we adapt the calibration method to functionals of Mumford-Shah type, and provide a criterion (Theorem 1) to verify if a given function is energy minimizing. Among other applications, we use this criterion to show that certain triple-junction configurations are minimizing (Example 3).

Une méthode de calibration pour la fonctionnelle de Mumford-Shah

Résumé. *Dans cette note, nous présentons une condition suffisante d'optimalité pour la minimisation de la fonctionnelle de Mumford-Shah. Cette condition (Théorème 1) résulte d'une variante de la méthode de calibration et permet de montrer notamment que certaines lignes à jonction triple correspondent à des configurations optimales (Example 3).*

Version française abrégée

La fonctionnelle de Mumford-Shah a été introduite dans le contexte de la détection de contours d'images ([8], [7]). Elle s'écrit en dimension n sous la forme:

$$F(u) := \int_{\Omega\setminus Su} |\nabla u|^2 + \alpha \mathscr{H}^{n-1}(Su) + \beta \int_{\Omega\setminus Su} |u-g|^2 \ , \tag{1}$$

où Ω est un ouvert régulier de $\mathbb{R}^n$, $g : \Omega \to [0,1]$ est une donnée (niveau de gris), α et β des constantes positives et $\mathscr{H}^{n-1}$ la mesure de Hausdorff de dimension $(n-1)$; l'inconnue $u : \Omega \to \mathbb{R}$ est supposé à priori de classe C^1 en dehors d'un ensemble singulier Su dont la forme et la position n'est pas donnée. Minimiser F revient donc à optimiser à la fois la function *et* l'ensemble singulier.

Une simplification usuelle du modèle souvent utilisée pour étudier la régularité des minimiseurs de F consiste à enlever le terme $\beta \int |u-g|^2$, ce qui en posant $\alpha = 1$, conduit à:

$$F(u) := \int_{\Omega\setminus Su} |\nabla u|^2 + \mathscr{H}^{n-1}(Su) \ . \tag{2}$$

On s'intéresse alors aux fonctions qui minimisent $F(u)$ lorsque la trace de u est donnée au bord. En considérant différents types de variations, on montre aisément qu'un minimiseur trés régulier u doit satisfaire certains critères: u doit être harmonique sur $\Omega \setminus Su$ avec des dérivées normales nulles de chaque coté de Su et de plus la courbure moyenne de Su doit être égale à la différence des carrés des normes du gradient du u de chaque coté. Des conditions supplémentaires ont été données dans le cas $n = 2$ et nous renvoyons à [7] et [3] pour une description plus précise des conditions d'équilibre et les résultats de régularité correspondants.

Il est trés important de noter que F n'est pas convexe et par conséquent aucun système de conditions nécessaires déduites de variations infinitésimales ne pourra conduire à une condition suffisante d'optimalité. L'objet de cette note est de présenter une condition suffisante (Théorème 1) et quelques applications (Examples 1-4). Les preuves détaillées apparaitront dans un papier ultérieur [1].

Avant de présenter la méthode, nous devons étendre la définition de F à une classe plus large de fonctions: l'espace $SBV(\Omega)$ des fonctions spéciales à variations bornée (voir [2], [3]). Brièvement La function ∇u dans (2) devient le gradient approximatif (défini Lebesque p.p.) et Su représente l'ensemble des points de discontinuités (essentielles) de u, i.e. $Su = \{x \in \Omega; u^+(x) > u^-(x)\}$ où $u^+(x), u^-(x)$ désignent les limites (approximatives) inférieure et supérieure de u. L'appartenance de u à $SBV(\Omega)$ se traduit par le fait que cet ensemble Su est $n-1$ rectifiable et admet en $\mathscr{H}^{n-1}$-presque tout point $x \in Su$, un hyperplan tangent dont nous noterons $\nu_u(x)$ la normale unitaire dirigée de $u^-(x)$ vers $u^+(x)$. Il est bien connu que le problème initial et celui dans la formulation SBV sont complétement équivalents (cf. [4] ou [3], chapter 7).

L'idée fondamentale de notre approche consiste à réécrire F comme une fonction convexe de la nouvelle variable $1_u : \Omega \times \mathbb{R} \to [0, 1]$ déduite en associant à u la fonction caractéristique de son hypographe. Plus précisèment, posons pour tout $u : \Omega \to \mathbb{R}$, $1_u(x, t) := 1$ si $t \le u(x)$, $1_u(x, t) := 0$ si $t > u(x)$; si u est dans $SBV(\Omega)$, il est facile de montrer que la dérivée distributionnelle $D1_u$ est une mesure de Radon bornée $\Omega \times \mathbb{R}$. Notre représentation convexe de l'énergie $F(u)$ fait l'objet du

LEMME 1. – *Soit F définie par (2) et $\mathscr{F}$ l'ensemble des champ Boréliens bornés $\phi = (\phi^x, \phi^t) : \Omega \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ tels que:*
(a) $|\phi^x(x,t)|^2 \le 4\phi^t(x,t)$ *pour* $x \in \Omega$, $t \in \mathbb{R}$,
(b) $\left| \int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \le 1$ *pour* $x \in \Omega$, $t_1, t_2 \in \mathbb{R}$.
Alors on a pour tout $u \in SBV(\Omega)$

$$F(u) = \sup_{\phi \in \mathscr{F}} \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u \ . \tag{3}$$

De plus, lorsque $F(u)$ est finie, l'égalité

$$F(u) = \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u \tag{4}$$

a lieu pour un champ Borélien $\phi \in \mathscr{F}$ si (et seulement si)
(a′) $\phi^x(x, u(x)) = 2\nabla u(x)$ *et* $\phi^t(x, u(x)) = |\nabla u(x)|^2$ *pour p.t.* $x \in \Omega \setminus Su$;
(b′) $\int_{u^-(x)}^{u^+(x)} \phi^x(x,t)\, dt = \nu_u(x)$ *pour* $\mathscr{H}^{n-1}$*-p.t.* $x \in Su$.

On peut maintenant déduire le résultat principal de cette note.

THÉORÈME 1. – *Soit* $u \in SBV(\Omega)$ *et supposons qu'il existe un champ* ϕ *de classe* C^1 *sur* $\overline{\Omega} \times \mathbb{R}$ *à divergence nulle et vérifiant* (a), (b), (a′), (b′). *Alors* u *minimise la fonctionnelle* F *définie par* (2) *sur l'ensemble des fonctions* v *telles que* $v = u$ *sur* $\partial\Omega$.

Remarque 1. – Nous appellerons *calibration pour* u tout champ ϕ vérifiant les conditions du Théorème 1, ceci par analogie avec la méthode bien connue utilisée dans le cas des surfaces minimales (cf. [6]). Il est supposé ici que ϕ est de class C^1 afin de pouvoir intégrer par parties (voir (5)), mais cette hypothèse peut être affaiblie de différentes manières (voir [1] pour plus de détails). Par exemple il suffit que ϕ soit C^1 par morceaux avec des discontinuités purement tangentielles (la condition de divergence nulle sera alors satisfaite au sens des distributions).

Quelques applications significatives du Théorème 1 consistent à exhiber des calibrations dans quelques cas particuliers (voir Examples 1-4 ci-aprés et [1]). Le problème de savoir si de telles calibrations existent pour tout minimiseur est encore ouvert.

1. Introduction

The Mumford-Shah functional was introduced in [8] in the context of a variational approach to edge detection problems (cf., also, [7]), and can be written, in dimension n, as

$$F(u) := \int_{\Omega\setminus Su} |\nabla u|^2 + \alpha \mathscr{H}^{n-1}(Su) + \beta \int_{\Omega\setminus Su} |u-g|^2 \ , \tag{1}$$

where Ω is a regular bounded domain in $\mathbb{R}^n$, $g : \Omega \to [0,1]$ is a given function (input grey level), α and β are positive constants, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (that is, the usual $(n-1)$-dimensional volume in case of subsets of Lipschitz hypersurfaces, the length in the most relevant case $n = 2$); the unknown function $u : \Omega \to \mathbb{R}$ is regular (say, of class C^1) out of a closed singular set Su, whose shape and location are not prescribed. Thus minimizing F means optimizing the function *and* the singular set, which is indeed often regarded as an independent unknown.

A relevant simplification of the previous functional, which occurs in the study of regularity properties of minimizers of F, is obtained by dropping the lower order term $\beta \int |u-g|^2$, and setting for simplicity $\alpha = 1$, i.e.,

$$F(u) := \int_{\Omega\setminus Su} |\nabla u|^2 + \mathscr{H}^{n-1}(Su) \ . \tag{2}$$

In this case, one is interested in minimizers of F with given boundary values, that will be simply called *minimizers*.

Considering different classes of infinitesimal variations, one can show that every minimizer must satisfy certain equilibrium conditions, which could be globally labelled Euler-Lagrange equations for F. For instance, u must be harmonic on $\Omega \setminus Su$ with vanishing normal derivatives on Su, and the mean curvature of Su must be equal to the difference of the squared norms of the traces of ∇u on the two sides of Su (at those points where Su is sufficiently regular). Additional conditions have been derived for the two-dimensional case; we refer the reader to [7] and [3] for a precise description of these equilibrium conditions, and related regularity results.

However, since F is not convex, all conditions which can be derived by infinitesimal variations are necessary for minimality, but never sufficient. The purpose of this note is precisely to present a sufficient condition for minimality (Theorem 1), and give a few applications (Examples 1-4). Detailed proofs and further results will be given in the forthcoming paper [1].

2. Description of the results

Before proceeding, it is convenient to define F on a larger, and more flexible, class of functions, namely the class $SBV(\Omega)$ of special functions of bounded variation ([2], we refer to [3] for further details). In this setting, Su becomes the set of the *essential* discontinuity points of u, while ∇u is the *approximate* gradient of u, which is defined almost everywhere (with respect to Lebesgue measure).

The set Su is no longer closed, but can be covered (up to an $\mathscr{H}^{n-1}$-negligible subset) by countably many hypersurfaces of class C^1, and admits, at $\mathscr{H}^{n-1}$-a.e. $x \in Su$, an *approximate* tangent hyperplane, on both sides of which the function u has *approximate* limits. We denote the larger one by $u^+(x)$ and the smaller one by $u^-(x)$; $\nu_u(x)$ is the normal to Su pointing from the side of $u^-(x)$ to the side of $u^+(x)$. It can be proved that the minimum problems in the original setting and in the SBV setting are completely equivalent (see [4], or [3], chapter 7).

The crucial point of our approach is to rewrite F as a convex functional (of a different variable), as in identity (3) below. For every $u : \Omega \to \mathbb{R}$ we denote by $1_u : \Omega \times \mathbb{R} \to [0,1]$ the characteristic function of the subgraph of u, that is, $1_u(x,t) := 1$ if $t \le u(x)$, and $1_u(x,t) := 0$ if $t > u(x)$. If u belogs to $SBV(\Omega)$, then the subgraph of u has finite perimeter in $\Omega \times \mathbb{R}$, i.e., the distributional derivative $D1_u$ is a bounded Radon measure on $\Omega \times \mathbb{R}$. We begin with a lemma which will be proved in [1].

LEMMA 1. – *Let F be given in* (2), *and $\mathscr{F}$ be the class of all (bounded and Borel measurable) vectorfields $\phi = (\phi^x, \phi^t) : \Omega \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ which satisfy*
(a) $|\phi^x(x,t)|^2 \le 4\phi^t(x,t)$ *for* $x \in \Omega$, $t \in \mathbb{R}$,
(b) $\left| \int_{t_1}^{t_2} \phi^x(x,t)\,dt \right| \le 1$ *for* $x \in \Omega$, $t_1, t_2 \in \mathbb{R}$.
Then, for every $u \in SBV(\Omega)$, we have

$$F(u) = \sup_{\phi \in \mathscr{F}} \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u \ . \tag{3}$$

Moreover the equality

$$F(u) = \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u \tag{4}$$

is achieved if (and only if) ϕ satisfies
(a′) $\phi^x(x,u(x)) = 2\nabla u(x)$ *and* $\phi^t(x,u(x)) = |\nabla u(x)|^2$ *for a.e.* $x \in \Omega \setminus Su$;
(b′) $\int_{u^-(x)}^{u^+(x)} \phi^x(x,t)\,dt = \nu_u(x)$ *for* $\mathscr{H}^{n-1}$*-a.e.* $x \in Su$.

We can now give the main result of this note.

THEOREM 1. – *Let $u \in SBV(\Omega)$ be given, and assume that there exists a C^1 vectorfield ϕ on $\overline{\Omega} \times \mathbb{R}$ which satisfies* (a), (b), (a′), (b′) *above, and is divergence-free. Then u minimizes the functional F in* (2) *among all functions v with the same boundary values.*

Take indeed any v such that $v = u$ on $\partial\Omega$. Then

$$F(v) \ge \int_{\Omega \times \mathbb{R}} \phi \cdot D1_v = \int_{\Omega \times \mathbb{R}} \phi \cdot D1_u = F(u) \ , \tag{5}$$

where the inequality follows from (3), which in turn is implied by (a) and (b), the first equality follows from the fact that ϕ is divergence-free and $v = u$ on $\partial\Omega$, the second equality follows from (4), which is implied by (a′) and (b′).

Remark 1. – The vectorfield ϕ in Theorem 1 is called a *calibration* for u. This term is taken from the theory of minimal surfaces (cf. [6]), where a vectorfield ϕ in $\mathbb{R}^n$ is said to calibrate an oriented hypersurface S if it agrees on S with the normal vectorfield, is divergence-free, and satisfies $|\phi| \leq 1$ everywhere; the existence of a calibration implies that S minimizes the $(n-1)$-dimensional volume among all oriented hypersurfaces with the same boundary (with a one-line proof like (5)).

Remark 2. – In Theorem 1, ϕ was assumed of class C^1 in order to get the first equality in (5), but this assumption can be relaxed in various ways (see [1] for details). For instance, one may consider piecewise C^1 vectorfields, which may be discontinuous along sufficiently regular interfaces. In this case the divergence-free condition must be understood in the distributional sense, i.e., the pointwise divergence vanishes (where defined) and the normal component of ϕ is continuous across the discontinuity surfaces (see the examples below).

Remark 3. – It is not clear if every minimizer must admit a calibration. Notice that minimal surfaces of codimension one always admit a calibration of some sort (as proved in [5]), but similar conclusions fail to be true in higher codimension. A first step towards an existence result is the following remark: replacing 1_u with an arbitrary BV function w on $\Omega \times \mathbb{R}$, the functional defined by the right-hand side of (3) is convex in w, and it turns out that the existence of a calibration for u is roughly equivalent to the fact that 0 belongs to the subdifferential of this functional at 1_u. This condition is necessary and sufficient for minimizing convex functionals. However it is not known whether 1_u is a minimizer of this auxiliary convex funtional in $BV(\Omega \times \mathbb{R})$ whenever u is a minimizer of F in $SBV(\Omega)$.

We conclude with a few applications of Theorem 1 (see [1] for the details).

Example 1. – Let $n = 1$, $\Omega = (0, a)$; it can be easily verified by direct computations that the linear function $u(x) := bx$ minimizes F in (2) with respect to its boundary values if and only if $b^2 a \leq 1$. In this case a calibration is given by the piecewise constant vectorfield ϕ below:

$$\phi(x,t) := \begin{cases} (2b, b^2) & \text{if } \dfrac{bx}{2} \leq t \leq \dfrac{b(x+a)}{2}, \\ (0,0) & \text{elsewhere.} \end{cases} \tag{6}$$

Example 2. – Let Ω an open subset of $\mathbb{R}^n$, n arbitrary. As pointed out to us by A. Chambolle, a harmonic function u on Ω minimizes F in (2) with respect to its boundary values if

$$\Big(\sup_\Omega u - \inf_\Omega u \Big) \cdot \sup_\Omega |\nabla u| \leq 1 \ ,$$

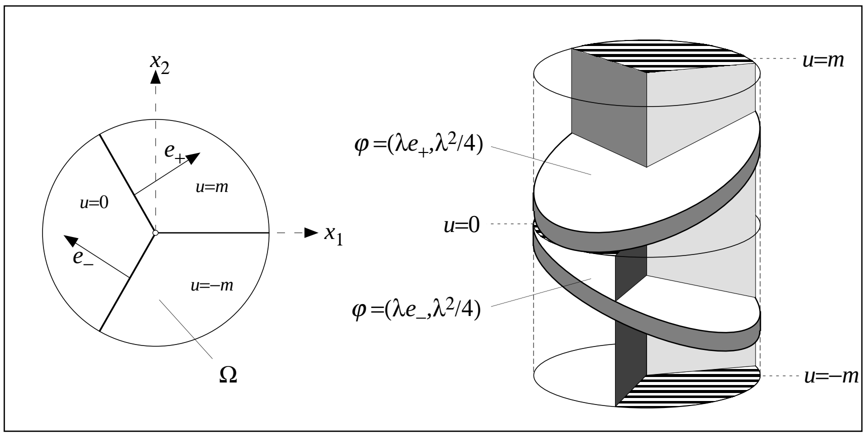
which for $n = 1$ reduces to the constraint $ab^2 \leq 1$ in Example 1. Inspired by the one-dimensional case, we construct the following calibration (cf. (6)):

$$\phi(x,t) := \begin{cases} \big(2\nabla u(x), |\nabla u(x)|^2\big) & \text{if } \dfrac{u(x) + \inf u}{2} \leq t \leq \dfrac{u(x) + \sup u}{2}, \\ (0,0) & \text{elsewhere.} \end{cases}$$

Example 3. – Let $n = 2$, Ω the unit disk centered at the origin, and u be given, in polar coordinates, by $u := m$ for $0 \leq \theta < \frac{2}{3}\pi$, $u := 0$ for $\frac{2}{3}\pi \leq \theta < \frac{4}{3}\pi$, $u := -m$ for $\frac{4}{3}\pi \leq \theta < 2\pi$, and m is a positive real constant. Thus Su is given by three line segments meeting at the origin with equal angles, and since this triple junction is length minimizing, it is natural to conjecture that, for large values of m, u minimizes the functional F in (2) among all functions with the same boundary values.

Indeed we calibrate u for every $m \geq \sqrt{2}$. Inspired by the one-dimensional case described in Example 2, we take $e_\pm := (\pm\sqrt{3}/2, 1/2)$, and $\lambda > 0$ such that $\frac{\lambda}{2} + \frac{1}{\lambda} \leq m$, and we define the calibration by (cf. the figure below)

$$\phi(x,t) := \begin{cases} (\lambda e_+, \lambda^2/4) & \text{if } \frac{\lambda}{4}(1 + x\cdot e_+) \leq t \leq \frac{\lambda}{4}(1 + x \cdot e_+) + \frac{1}{\lambda}, \\ (\lambda e_-, \lambda^2/4) & \text{if } \frac{\lambda}{4}(-1 + x\cdot e_-) - \frac{1}{\lambda} \leq t \leq \frac{\lambda}{4}(-1 + x \cdot e_-), \\ (0,0) & \text{elsewhere.} \end{cases}$$



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