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Phase Transition with the Line-Tension Effect

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Abstract

We make the connection between the geometric model for capillarity with line tension and the Cahn-Hilliard model of two-phase fluids. To this aim we consider the energies

$$F_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 + \frac{1}{\varepsilon}\int_\Omega W(u) + \lambda \int_{\partial\Omega} V(u)$$

where u is a scalar density function and W and V are double-well potentials. We show that the behavior of F_ε in the limit $\varepsilon \to 0$ and $\lambda \to \infty$ depends on the limit of $\varepsilon \log \lambda$. If this limit is finite and strictly positive, then the singular limit of the energies F_ε lead to a coupled problem of bulk and surface phase transitions, and under certain assumptions agrees with the relaxation of the capillary energy with line tension. These results were announced in [ABS1] and [ABS2].

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1. Introduction

In the classical model for two-phase fluids, it is assumed that every configuration of a fluid in a container $\Omega \subset \mathbb{R}^3$ is described by a mass density u which takes only two values α and β, corresponding to the phases $\mathsf{A} := \{u = \alpha\}$ and $\mathsf{B} := \{u = \beta\} = \Omega \backslash \mathsf{A}$. The energy is located on the interface $\mathscr{S}_{\mathsf{AB}}$ which separates the two phases, with density σ_{AB} (surface tension), and on the contact surfaces $\mathscr{S}_{\mathsf{AW}}$ and $\mathscr{S}_{\mathsf{BW}}$ between the wall of the container and the phases A and B, with density σ_{AW} and σ_{BW} respectively. Then, under some volume constraint, the equilibrium configurations minimize the capillary energy

$$E_0(\mathsf{A}) := \sigma_{\mathsf{AB}}\,|\mathscr{S}_{\mathsf{AB}}| + \sigma_{\mathsf{AW}}\,|\mathscr{S}_{\mathsf{AW}}| + \sigma_{\mathsf{BW}}\,|\mathscr{S}_{\mathsf{BW}}|. \tag{1.1}$$

Here and in the following $|A|$ denote the measure of A, namely the area when A is a surface, and the length when A is a line. Surface energy densities are represented by the letter σ with an index which recall the type of interface under consideration; these coefficients are strictly positive, and clearly do not depend on the particular configuration of the system.

The problem of minimizing (1.1) is the so-called *liquid-drop problem*; the existence of a solution for this minimum problem is assured by the *wetting condition*

$$|\sigma_{\mathsf{AW}} - \sigma_{\mathsf{BW}}| \le \sigma_{\mathsf{AB}}. \tag{1.2}$$

At equilibrium, the interface $\mathscr{S}_{\mathsf{AB}}$ has constant mean curvature, and it meets the wall of the container at a constant *contact angle* θ, which satisfies Young's law (see for instance [RW] or [F])

$$\cos\theta = \frac{\sigma_{\mathsf{AW}} - \sigma_{\mathsf{BW}}}{\sigma_{\mathsf{AB}}}. \tag{1.3}$$

An interesting extension of the previous model is obtained by adding to E_0 an energy concentrated along the line $\mathscr{L}_{\mathrm{c}}$ where $\mathscr{S}_{\mathsf{AB}}$ meets the wall of the container (*contact line*) with density c; this energy density is referred to as line tension (see [RW, WW]). In this model the capillary energy becomes:

$$F_0(\mathsf{A}) := \sigma_{\mathsf{AB}}\,|\mathscr{S}_{\mathsf{AB}}| + \sigma_{\mathsf{AW}}|\mathscr{S}_{\mathsf{AW}}| + \sigma_{\mathsf{BW}}|\mathscr{S}_{\mathsf{BW}}| + \mathrm{c}|\mathscr{L}_{\mathrm{c}}|. \tag{1.4}$$

An alternative way to study two-phase fluids originates from the continuum mechanics approach initiated by Gibbs and revisited by Cahn & Hilliard [CH] in the 60's. The interface $\mathscr{S}_{\mathsf{AB}}$ is now replaced by a thin layer in which the mass density u varies continuously from the value α to the value β, and the energy associated with u is the sum of a Gibbs free energy $\int_\Omega W(u)$, where W is a two-wells potential vanishing at α and β, and a term $\xi \int_\Omega |Du|^2$ which penalizes the non-homogeneity of the fluid. Moreover a boundary contribution $\int_{\partial\Omega} V(u)$ can be added to take into account the interactions between the fluid and the wall of the container.

The coefficient ξ introduces an intrinsic length which is characteristic of the thickness of the interface, and since this length is in general much smaller than the size of the container, it is natural to study the equilibrium of such a fluid in an asymptotic way, i.e., by considering the limits as ε tends to 0 of the minimizers u_ε (subject to a mass constraint $\int_\Omega u_\varepsilon = m$) of the rescaled energies

$$F_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 + \frac{1}{\varepsilon} \int_\Omega W(u) + \lambda \int_{\partial\Omega} V(u), \tag{1.5}$$

where λ represents the order of magnitude of the wall-fluid interactions.

This problem has been studied by several authors, mainly in the case $\lambda = 0$ (that is, when no boundary energy is considered; see for instance [Gu, Mo1]; see [Ba] for multi-phase fluids). In the case $\lambda = 1$, L. Modica [Mo2] established a rigorous connection between the classical model for capillarity E_0 and the Cahn-Hilliard model: the sequence of minimizers u_ε (of F_ε) is pre-compact in $L^1(\Omega)$, each limit point u takes only the values α and β (almost everywhere), and the corresponding phase $\mathsf{A} := \{u = \alpha\}$ solves the liquid-drop problem associated with an energy of type (1.1), where the coefficients σ_{AB}, σ_{AW} and σ_{BW} can be expressed in term of the potentials W and V. We recall that in [Mo2] it was assumed that W, V, λ and Ω do not depend on ε (which means that ε is infinitely smaller than any other parameter of the problem) while in the present work we consider a different behavior for λ, namely that λ tends to infinity as ε tends to zero. Different assumptions have already been discussed in [BS, BDS].

The contribution of this paper is twofold. First we focus on the model for capillarity with line tension associated with the energy F_0. We show that due to a lack of semicontinuity, this functional leads to ill-posed minimum problems. Then we apply the usual relaxation procedure and we compute the relaxed functional $\overline{F}_0$ explicitly.

Our second goal is to establish a rigorous connection between F_0 and the Cahn-Hilliard model. To this end we study the asymptotic behavior of the functionals F_ε in the limit $\varepsilon \to 0$ when λ tends to infinity with a suitable scaling and V is a two-well potential. We show that the limit of F_ε in the sense of Γ-convergence is a functional $F(u)$ which is finite only if u takes values α or β. Thus we can view F as a function of the phase $\mathsf{A} := \{u = \alpha\}$, and it turns out that F agrees with $\overline{F}_0$ for suitable choice of the potentials W and V. Consequently, if u_ε minimizes F_ε subject to the mass constraint $\int_\varepsilon u = m$, and u is a limit point of the sequence (u_ε), then the corresponding phase $\mathsf{A} := \{u = \alpha\}$ minimizes $\overline{F}_0$ subject to a suitable volume constraint.

The relaxation procedure is described in subsection 2.2. We show that the total energy can be properly written by introducing, besides the usual bulk phase $\mathsf{A} \subset \Omega$, an additional variable $\mathsf{A}' \subset \partial\Omega$ which is completely independent of A; A'

and its complement $\mathsf{B}' := \partial\Omega \setminus \mathsf{A}'$ are called *boundary phases*. The total energy Φ_0 of the configuration $(\mathsf{A}, \mathsf{A}')$ is thus given by the sum of three different terms: the classical surface tension on the interface between the bulk phases A and B, a surface density on the wall of the container (which depends on which bulk phase and which boundary phase meet together) and a line density along the line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ which separates the boundary phases A' and B' (*dividing line*).

Thus $\overline{F}_0(\mathsf{A})$ is obtained by taking the minimum of $\Phi_0(\mathsf{A}, \mathsf{A}')$ over all possible A' (see Theorem 2.1). Notice that in general the boundary phase A' where such a minimum is attained differs from the interface $\mathscr{S}_{\mathsf{AW}}$ between A and the wall of the container, and therefore $\overline{F}_0$ is no longer of the form (1.4). In particular it is a nonlocal functional (while Φ_0 is local), and what we called "line tension" is now located on the dividing line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$, which in general does not agree with the contact line $\mathscr{L}_{\mathrm{c}}$; in this case we speak of "dissociation of contact line and dividing line".

In subsection 2.3 we show that a similar situation occurs when we study the asymptotic behavior of F_ε. In order to properly write the limit of the boundary energies, as $\varepsilon \to 0$ we need to introduce besides the usual bulk mass density u an additional variable $v : \partial\Omega \to \mathbb{R}$ called *boundary mass density.* A configuration of the limit problem is represented by a couple (u, v) to which we associate a total energy $\Phi(u, v)$ (see Theorem 2.6). As before, we can recover from Φ a functional which depends only on the bulk density u: the limit $F(u)$ of the functionals F_ε (in the sense of Γ-convergence) is given by the minimum of $\Phi(u, v)$ over all possible v (Corollary 2.7).

Since Φ is finite only when u takes values α and β and v takes values α' and β' (the wells of the potential V), we may regard Φ as a function of $\mathsf{A} := \{u = \alpha\}$ and $\mathsf{A}' := \{v = \alpha'\}$. In subsection 2.4 we encompass Φ_0 and Φ in a more general class of functionals. This leads to different models for capillarity with line tension, and then we need some qualitative comparison; indeed we show that Φ_0 can be always obtained as Φ for a suitable choice of the potentials V and W, while the converse is true only if V and W satisfy certain restrictions.

Sections 3 and 4 are devoted to the proofs of the mathematical results stated in section 2. The main mathematical difficulties arise in the proof of the Γ-convergence result for the functional F_ε. While the limit energy can be evaluated in the bulk as in [Mo1], the characterization of the boundary contribution is more intricate. In particular the two-dimensional part of the boundary contribution is studied by adapting the approach of [Mo2]; for the one-dimensional part we need several steps: first, by localization and slicing arguments we reduce to a problem on a two-dimensional half-disk; then we replace the two-dimensional Dirichlet energy on the half-disk by the $H^{1/2}$ intrinsic norm on the diameter; eventually we are led to a new kind of singular perturbation problem involving a nonlocal term. This problem has its own interest (see Theorem 4.4 and [ABS1]), and brings to the fore the right scaling for λ, namely $\log\lambda \simeq 1/\varepsilon$. Some technical lemmas have been postponed in section 6.

In section 5, we describe the mechanical consequences of our model for line tension in term of equilibrium configurations. We show that the dissociation of contact line and dividing line may occur also at equilibrium, and in that case the contact angle no longer satisfies Young's law but an entirely different condition.

Accordingly, in the quasistatic evolution of such a fluid the contact angle may have discontinuous changes.

2. Description of the Results

We begin by fixing the notation and recalling some standard mathematical results used throughout the paper. Then we discuss the relaxation of the functional F_0 (subsection 2.2) and the asymptotic behavior of the functionals F_ε (subsection 2.3). The comparison between these results is briefly discussed in subsection 2.4.

2.1. Notation

In this paper we consider different domains A with dimension $h = 1, 2, 3$; more precisely, A is always a bounded open set either of $\mathbb{R}^h$ or of a smooth h-dimensional manifold M without boundary, embedded in $\mathbb{R}^3$. We denote by ∂A the boundary of A relative to the ambient manifold; ∂A is always assumed Lipschitz regular.

We denote by $B_r(x)$ the ball with center x and radius r; we write $a \vee b$ and $a \wedge b$ for the maximum and the minimum of a and b respectively.

Unless differently stated A is always endowed with the corresponding h-dimensional Hausdorff measure $\mathscr{H}^h$ (cf. [EG, Chapter 2]). Accordingly, we often write $\int_A f$ instead of $\int_A f \, d\mathscr{H}^h$, and $|A|$ instead of $\mathscr{H}^h(A)$, whereas we never omit an explicit mention of the measure when it differs from $\mathscr{H}^h$. We often use the fact that given a set $B \subset \mathbb{R}^k$ and a Lipschitz function f on B, then $\mathscr{H}^h(f(B)) \leq (\mathrm{Lip}(f))^h \mathscr{H}^h(B)$, where $\mathrm{Lip}(f)$ is the Lipschitz constant of f.

The h-dimensional density of E at a point x is the limit (if it exists) of the ratio $\mathscr{H}^h(E \cap B_r(x))$ over $\omega_h r^h$ as $r \to 0$, where ω_h is the measure of unit ball in $\mathbb{R}^h$. The *essential boundary* of E is the set of all points where E has neither density 1 nor density 0, including all points where the density does not exist. Since the essential boundary agrees with the topological boundary when the latter is Lipschitz regular, we also denote the essential boundary by ∂E.

Throughout the rest of this paper, all the functions and sets are assumed Borel measurable and questions of measurability will never be discussed.

Function Spaces. Let A be an h-dimensional domain and take $u \in L^1_{\mathrm{loc}}(A)$. The derivative of u in the sense of distributions is denoted by Du. As usual $H^1(A)$ is the Sobolev space of all real functions $u \in L^2(A)$ such that Du belongs to $L^2(A)$, and $BV(A)$ is the space of all $u \in L^1(A)$ with bounded variation, that is, such that Du is a bounded Borel measure on A. Notice that when A is an open subset of a manifold $M \subset \mathbb{R}^3$ and $u \in H^1(A)$, then $Du : A \to \mathbb{R}^3$ and $Du(x)$ belongs to the tangent space of M at x for a.e. $x \in A$. If $u \in BV(A)$, then Du is a measure on A which takes values in $\mathbb{R}^3$ and the density of Du with respect to its variation $|Du|$ at x, belongs to the tangent space of M at x for $|Du|$-a.e. $x \in A$. Recall that every bounded set in $BV(A)$ is relatively compact in $L^1(A)$. The letter T denotes the trace operator which maps $H^1(A)$ onto $H^{1/2}(\partial A)$ and $BV(A)$ onto $L^1(\partial A)$.

For further details and results about the theory of BV functions and Sobolev spaces we refer the reader to [EG, Chapters 4 and 5].

Jump set and essential boundary. Let A be an h-dimensional domain and take $u \in L^1_{\mathrm{loc}}(A)$. The *jump set* Su is the complement of the set of Lebesgue points of u, i.e., the set of points where the upper and lower approximate limits of u differ or are not finite. If $u \in BV(A)$ then Su is *rectifiable*: this means that it may be covered by countably many $(h-1)$-dimensional submanifolds of class C^1 except for an $\mathscr{H}^{h-1}$-negligible subset. In particular the dimension of Su does not exceed $h-1$, and if u belongs to $H^1(A)$ then Su is $\mathscr{H}^{h-1}$-negligible (see [EG, Sections 4.8 and 5.9]).

For every $I \subset \mathbb{R}$, we define $BV(A, I)$ as the class of all $u \in BV(A)$ such that $u(x) \in I$ for a.e. $x \in A$. If $I := \{\alpha, \beta\}$, then a function $u : A \to I$ belongs to $BV(A)$ if and only if $\mathscr{H}^{h-1}(Su) < +\infty$, and $(\beta - \alpha)\mathscr{H}^{h-1}(Su)$ agrees with the total variation $\|Du\|$ of the derivative Du (cf. [EG, Section 5.11]). In the particular case $I = \{0, 1\}$, u is the characteristic function of a set E and is denoted by 1_E, and E is called a *set with finite perimeter in* A. Since the essential boundary of E agrees in A with the jump set of 1_E, we deduce that E has a finite perimeter in A if and only if $\mathscr{H}^{h-1}(\partial E \cap A)$ is finite. For this reason the notion of essential boundary fits out purposes more than the topological boundary.

Every rectifiable set S can be endowed with a *(measure theoretic) normal field* ν which enjoys the following property: for every hypersurface M of class C^1 and $\mathscr{H}^{h-1}$-almost every x in $M \cap S$ the vector $\nu(x)$ agrees with a normal unit vector to M at x. Moreover when S is the jump set of a function $u \in BV(A, I)$ with $I := \{\alpha, \beta\}$ we can chose ν so that the measure derivative Du is given by the restriction of the measure $\mathscr{H}^{h-1}$ to the set Su multiplied by the density function $(\beta - \alpha) \cdot \nu$. This choice of ν is unique up to $\mathscr{H}^{h-1}$-negligible sets and is denoted by ν_u; when u is the characteristic function of a finite perimeter set E this normal field is also denoted by ν_E and is called the *(approximate) inner normal to* E.

Eventually we remark that when E has finite perimeter in A the trace of the BV function 1_E on ∂A (which is defined as an element of $L^1(\partial A)$) is the characteristic function of the set $\partial E \cap \partial A$. In this sense, the set $\partial E \cap \partial A$ can be regarded as the trace of E on ∂A.

2.2. *The Relaxation Theorem*

The container is represented by a bounded open set Ω of $\mathbb{R}^3$ with a boundary of class C^1 and the bulk phases are denoted A and B. Since $\mathsf{B} = \Omega \setminus \mathsf{A}$, every configuration is identified by A. In the following $\partial\mathsf{A}$ and $\partial\mathsf{B}$ denote the essential boundaries of A and B, and then the various interfaces involved in the expression of the energies E_0 or F_0 are defined as follows:

$\mathscr{S}_{\mathsf{AB}} := \partial\mathsf{A} \cap \partial\mathsf{B}$ is the surface which separates the phases A and B;

$\mathscr{S}_{\mathsf{AW}} := \partial\mathsf{A} \cap \partial\Omega$ is the surface which separates A from the wall of the container;

$\mathscr{S}_{\mathsf{BW}} := \partial\mathsf{B} \cap \partial\Omega$ is the surface which separates B from the wall of the container.

$\mathscr{L}_{\mathrm{c}} := \partial\mathscr{S}_{\mathsf{AW}}$ is the contact line, i.e., the line which separates $\mathscr{S}_{\mathsf{AW}}$ from $\mathscr{S}_{\mathsf{BW}}$.

In the following the letters $\mathscr{S}$ and $\mathscr{L}$ always denote a surface and a line respectively; consequently, we often denote the area $\mathscr{H}^2(\mathscr{S})$ and the length $\mathscr{H}^1(\mathscr{L})$ simply by $|\mathscr{S}|$ and $|\mathscr{L}|$. The letters in sans-serif A and B will be reserved for the phases.

The admissible configurations of the system belong to the space X of all Borel subsets of Ω. We endow X with the distance $d(\mathsf{A}_1, \mathsf{A}_2) := \mathscr{H}^3(\mathsf{A}_1 \triangle \mathsf{A}_2)$, where $\mathsf{A}_1 \triangle \mathsf{A}_2 := (\mathsf{A}_1 \setminus \mathsf{A}_2) \cup (\mathsf{A}_2 \setminus \mathsf{A}_1)$ is the symmetric difference of A_1 and A_2.

Our first claim is that the functional F_0 defined in (1.4) is not lower semicontinuous on X. The reason can be easily outlined: fix a configuration A and compare its energy with the energy of a new configuration A_δ which is obtained by inserting a layer of phase B with thickness δ between A and the wall (see Figure 1).

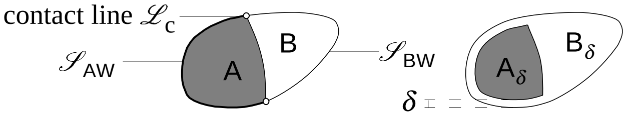

Fig. 1. The configurations A and A_δ.

As δ tends to zero, A_δ converges to A in X, and since A_δ does not touch the wall, the contact line of the new configuration is empty and the interface between the two phases A_δ and B_δ consists roughly speaking in the union $\mathscr{S}_\mathsf{AB} \cup \mathscr{S}_\mathsf{AW}$. Hence

$$F_0(\mathsf{A}) - F_0(\mathsf{A}_\delta) \simeq (\sigma_\mathsf{AW} - \sigma_\mathsf{AB} - \sigma_\mathsf{BW})\,|\mathscr{S}_\mathsf{AW}| + \mathsf{c}|\mathscr{L}_\mathsf{c}|. \tag{2.1}$$

Clearly the right hand side of (2.1) is strictly positive for a suitable choice of A: indeed the area $|\mathscr{S}_\mathsf{AW}|$ is bounded by $|\partial\Omega|$ while the length $|\mathscr{L}_\mathsf{c}|$ can be taken arbitrarily large. Hence for such a configuration there holds $\liminf F_0(\mathsf{A}_\delta) < F_0(\mathsf{A})$.

Let us emphasize that this phenomenon is not related to the particular choice of the topology on the space of configurations X. Since we are interested in minimizing F_0, we can consider only topologies which make F_0 coercive, that is, such that every sequence which is bounded in energy is pre-compact, and it can be easily checked that the choice of any (Hausdorff) topology in this class has no incidence on the lower semicontinuity of F_0. Notice that due to the compact embedding of $BV(\Omega)$ in $L^1(\Omega)$, the metric we imposed on X makes F_0 coercive.

This lack of lower semicontinuity shows that looking for equilibrium configurations on the basis of the model F_0 leads to ill-posed problems. In subsection 5.2 we show that the energy F_0 may admit no minimizer with prescribed volume.

The next natural step is to consider the relaxation of F_0, namely

$$\overline{F}_0(\mathsf{A}) := \inf\Big\{ \liminf_{n\to\infty} F_0(\mathsf{A}_n) : \mathsf{A}_n \to \mathsf{A} \text{ in } X \Big\}. \tag{2.2}$$

First we remark that given a sequence (A_n) which tends to A in X, the trace of A_n on $\partial\Omega$ (i.e., $\mathscr{S}_{\mathsf{A}_n\mathsf{W}}$) converges in X' to a set A' which in general does not agree with the trace of A. This is indeed the case for the sequence (A_δ) defined above (see Figure 1). This consideration suggests that to describe the relaxation of F_0 it is convenient to introduce, besides the usual "bulk" phases A and B, two additional "boundary" phases A' and B'.

Specifically, for every $\mathsf{A} \subset \Omega$ and $\mathsf{A}' \subset \partial\Omega$ we set

$$\begin{aligned} \mathsf{B}' &:= \partial\Omega \setminus \mathsf{A}', & \mathscr{L}_{\mathsf{A}'\mathsf{B}'} &:= \partial\mathsf{A}', \\ \mathscr{S}_{\mathsf{AA}'} &:= \mathscr{S}_{\mathsf{AW}} \cap \mathsf{A}' = \partial\mathsf{A} \cap \mathsf{A}', & \mathscr{S}_{\mathsf{AB}'} &:= \mathscr{S}_{\mathsf{AW}} \cap \mathsf{B}' = \partial\mathsf{A} \cap \mathsf{B}', \\ \mathscr{S}_{\mathsf{BA}'} &:= \mathscr{S}_{\mathsf{BW}} \cap \mathsf{A}' = \partial\mathsf{B} \cap \mathsf{A}', & \mathscr{S}_{\mathsf{BB}'} &:= \mathscr{S}_{\mathsf{BW}} \cap \mathsf{B}' = \partial\mathsf{B} \cap \mathsf{B}'. \end{aligned} \tag{2.3}$$

The line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ separates the phases A' and B', and is called the *dividing line*.

With each configuration $(\mathsf{A}, \mathsf{A}')$ w associate the energy

$$\begin{aligned} \Phi_0(\mathsf{A}, \mathsf{A}') := \sigma_{\mathsf{AB}} \, |\mathscr{S}_{\mathsf{AB}}| + \sigma_{\mathsf{AW}} \, |\mathscr{S}_{\mathsf{AA}'}| + (\sigma_{\mathsf{AB}} + \sigma_{\mathsf{BW}}) \, |\mathscr{S}_{\mathsf{AB}'}| + \\ + \sigma_{\mathsf{BW}} \, |\mathscr{S}_{\mathsf{BB}'}| + (\sigma_{\mathsf{AB}} + \sigma_{\mathsf{BW}}) \, |\mathscr{S}_{\mathsf{BA}'}| + \mathrm{c} |\mathscr{L}_{\mathsf{A}'\mathsf{B}'}|. \end{aligned} \tag{2.4}$$

Therefore F_0 can be written in terms of Φ_0 by

$$F_0(\mathsf{A}) = \Phi_0(\mathsf{A}, \mathscr{S}_{\mathsf{AW}}). \tag{2.5}$$

The space of all admissible configurations is now $X \times X'$, where X is defined above and X' is the space of all Borel subsets of $\partial\Omega$, endowed with the distance $d'(\mathsf{A}'_1, \mathsf{A}'_2) := |\mathsf{A}'_1 \triangle \mathsf{A}'_2|$. Since all coefficients in (2.4) are strictly positive we deduce immediately that the functional Φ_0 is coercive on $X \times X'$ and finite at $(\mathsf{A}, \mathsf{A}')$ if and only if A has finite perimeter in Ω and A' has finite perimeter in $\partial\Omega$. We can now state our relaxation result (see section 3 for the proof):

Theorem 2.1. *The functional Φ_0 is lower semicontinuous on $X \times X'$, and the relaxation of F_0 on X is given by*

$$\overline{F}_0(\mathsf{A}) = \min \big\{ \Phi_0(\mathsf{A}, \mathsf{A}') : \ \mathsf{A}' \in X' \big\}. \tag{2.6}$$

This result is still valid if we replace the space X by the subclass X_v of all $\mathsf{A} \in X$ such that $|\mathsf{A}| = v\}$, where v is a fixed number such that $0 < v < |\Omega|$ (this refinement of Theorem 2.1 requires a slight modification of the proof which we leave to the reader). This remark allows us to consider the minimization of F_0 under the volume constraint $|\mathsf{A}| = v$:

Corollary 2.2. *For every v such that $0 < v < |\Omega|$ there holds*

$$\inf \big\{ F_0(\mathsf{A}) : \ |\mathsf{A}| = v \big\} = \min_{X_v} \overline{F}_0 = \min_{X_v \times X'} \Phi_0. \tag{2.7}$$

Remark 2.3. From (2.5) and (2.6) we conclude that a configuration A minimizes F_0 on X_v if and only if $(\mathsf{A}, \mathscr{S}_{\mathsf{AW}})$ minimizes Φ_0 in $X_v \times X'$. In this case the contact line $\mathscr{L}_{\mathrm{c}}$ coincides with the dividing line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$. In subsection 5.2 we give an example where F_0 has no minimizer on X_v: more precisely, an example where $\mathsf{A}' \neq \mathscr{S}_{\mathsf{AW}}$ for every minimizing configuration $(\mathsf{A}, \mathsf{A}')$ in $X_v \times X'$.

Remark 2.4. When the wetting condition

$$|\sigma_{\mathsf{AW}} - \sigma_{\mathsf{BW}}| \le \sigma_{\mathsf{AB}} \tag{2.8}$$

is not satisfied, the minimum problem $\min\{\Phi_0(\mathsf{A},\mathsf{A}') : \mathsf{A}' \in X'\}$ can be explicitly solved: if $\sigma_{\mathsf{AW}} > \sigma_{\mathsf{BW}} + \sigma_{\mathsf{AB}}$ (the other case is similar) then the minimum is achieved when A' is empty, and (2.6) becomes

$$\overline{F}_0(\mathsf{A}) = \Phi_0(\mathsf{A},\emptyset) = \sigma_{\mathsf{AB}}|\mathscr{S}_{\mathsf{AB}}| + (\sigma_{\mathsf{AB}} + \sigma_{\mathsf{BW}})\,|\mathscr{S}_{\mathsf{AW}}| + \sigma_{\mathsf{BW}}|\mathscr{S}_{\mathsf{BB}'}|. \tag{2.9}$$

This means that it is always convenient to separate completely the phase A from the boundary by inserting an infinitely thin layer of phase B. In this case $\overline{F}_0$ has the same form as the energy E_0 in (1.1), and no line tension appears.

Remark 2.5. In the limit case $\mathsf{c} = 0$ Theorem 2.1 gives a formula for the relaxation $\overline{E}_0$ of the energy E_0 in (1.1). When the wetting condition (2.8) is satisfied: $\overline{E}_0 = E_0$, (that is, E_0 is lower semicontinuous on X), otherwise $\overline{E}_0$ is given by (2.9) (at least when $\sigma_{\mathsf{AW}} > \sigma_{\mathsf{BW}} + \sigma_{\mathsf{AB}}$).

Hence the relaxation of E_0 has always the same form as E_0, only the coefficients change. This specific property of E_0 explains why the relaxation step is usually skipped: one deals directly with the relaxed form by assuming *a priori* that the wetting condition (2.8) is fulfilled, while from our point of view this is only a consequence of the relaxation procedure.

2.3. The Γ-Convergence Theorem

As before, Ω is a bounded open subset of $\mathbb{R}^3$ with boundary of class C^1; W and V are non-negative continuous functions on $\mathbb{R}$ with growth at least linear at infinity and vanish respectively in the double-well $I := \{\alpha,\beta\}$ with $\alpha < \beta$, and $I' := \{\alpha',\beta'\}$ with $\alpha' < \beta'$. The symbol ε denotes a parameter decreasing to 0, while λ_ε is a parameter which goes to infinity as $\varepsilon \to 0$ and satisfies

$$\lim_{\varepsilon\to 0} \varepsilon \log \lambda_\varepsilon = K \quad \text{with } 0 < K < \infty. \tag{2.10}$$

The function H is a primitive of $2\sqrt{W}$, and we set

$$\boldsymbol{\sigma} := \big|H(\beta) - H(\alpha)\big| = 2\int_\alpha^\beta \sqrt{W} \qquad \text{and} \qquad \mathbf{c} := (\beta' - \alpha')^2\,\frac{K}{\pi}. \tag{2.11}$$

For every $\varepsilon > 0$ and $u \in H^1(\Omega)$ we define the functional

$$F_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 + \frac{1}{\varepsilon}\int_\Omega W(u) + \lambda_\varepsilon \int_{\partial\Omega} V(Tu), \tag{2.12}$$

where Tu is the trace of u on $\partial\Omega$.

First we want to briefly account for the choice of the double-well potential in the boundary energy $\int_{\partial\Omega} V(Tu)$ and of the scaling (2.10). The case $\lambda_\varepsilon = 0$ (that is, when no boundary energy is taken into account in F_ε) was already considered in [Mo1] (cf. Theorem 4.2 below). The term $\varepsilon^{-1}\int W(u)$ forces u_ε to take values close to α and β, while the term $\varepsilon\int|Du|^2$ penalizes the oscillations of u_ε. When ε tends to 0, the functions u_ε converge (up to a subsequence) to a function $u\in BV(\Omega)$ which takes only the values α and β. Moreover each u_ε has a transition from the value α to the value β in a thin layer close to the surface Su which separates the bulk phases $\{u=\alpha\}$ and $\{u=\beta\}$. Since the energy $F_\varepsilon(u_\varepsilon)$ tends to concentrate in this layer, the limit energy is distributed on Su with surface density $\boldsymbol{\sigma}$ (surface tension).

In [Mo2] this analysis has been extended to the case $\lambda_\varepsilon=1$ (V being any positive continuous function). In this case the traces Tu_ε of the minimizers u_ε converge to a function v on $\partial\Omega$. This function is constant on the trace of each bulk phase, namely $\{Tu=\alpha\}$ and $\{Tu=\beta\}$, but differs from Tu. The transition of u_ε from Tu to v occurs in a thin boundary layer, and since part of the total energy $F_\varepsilon(u_\varepsilon)$ concentrates in this layer, an additional surface density appears in the limit $\varepsilon\to 0$.

In this paper we investigate the case when λ_ε tends to infinity. If we assume that V is a double-well potential, the boundary part of F_ε forces the traces Tu_ε to take values close to α' and β', while the oscillations of the traces Tu_ε are penalized by the bulk integral $\varepsilon\int|Du|^2$. Then we expect that the traces Tu_ε converge to a function v which takes only the values α' and β' and that a concentration of energy occurs along line Sv which separates the boundary phases $\{v=\alpha'\}$ and $\{v=\beta'\}$.

The interest of this asymptotic model lies in the possible connection between this line concentration of energy and the line tension phenomenon. In order to establish such a connection, we first have to ensure that the transition of Tu_ε from α' and β' does take place in a thin layer. This brings to the fore scaling (2.10), which also provides a uniform control on the oscillations of Tu_ε. In fact we can prove that under (2.10) the traces Tu_ε converge (up to a subsequence) to a function v in $BV(\partial\Omega,I')$, and then the boundary phases $\{v=\alpha'\}$ and $\{v=\beta'\}$ are divided by the rectifiable curve Sv.

At this stage, we investigate the relation between v and Tu. In particular we wonder whether the boundary phases agree with the traces of the volume phases. In general the answer is negative, and indeed this situation is quite similar to the one described in the previous subsection: the asymptotic behavior of the functionals F_ε is described by a functional Φ which depends on the two variables u and v. Since the total energy $F_\varepsilon(u_\varepsilon)$ is partly concentrated in a thin layer close to Su (where u_ε has a transition from α to β), partly in a thin layer close to the boundary (where u_ε has a transition from Tu to v), and partly in the vicinity of Sv (where Tu_ε has a transition from α' to β'), we expect that the limit energy is the sum of a surface energy on concentrated on Su, a boundary energy on $\partial\Omega$ (with density depending on the gap between Tu and v), and a line energy concentrated along Sv.

Precisely we have the following theorem (see section 4 for the proof), which is the main result of this paper.

Theorem 2.6. *For every $u\in BV(\Omega,I)$ and $v\in BV(\partial\Omega,I')$ we set*

$$\Phi(u,v):=\boldsymbol{\sigma}\,\mathscr{H}^2(Su)+\int_{\partial\Omega}\big|H(Tu)-H(v)\big|\;+\mathbf{c}\,\mathscr{H}^1(Sv).\qquad(2.13)$$

Then the following three statements hold.

(i) Compactness: *let* $(u_\varepsilon) \subset H^1(\Omega)$ *be a sequence such that* $\varepsilon \to 0$ *and* $F_\varepsilon(u_\varepsilon)$ *is bounded. Then the sequence* $(u_\varepsilon, Tu_\varepsilon)$ *is pre-compact in* $L^1(\Omega) \times L^1(\partial\Omega)$ *and every cluster point belongs to* $BV(\Omega, I) \times BV(\partial\Omega, I')$*.*

(ii) Lower bound inequality: *for every* (u, v) *in* $BV(\Omega, I) \times BV(\partial\Omega, I')$ *and every sequence* $(u_\varepsilon) \subset H^1(\Omega)$ *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$ *and* $Tu_\varepsilon \to v$ *in* $L^1(\partial\Omega)$*, there holds*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v). \tag{2.14}$$

(iii) Upper bound inequality: *for every* (u, v) *in* $BV(\Omega, I) \times BV(\partial\Omega, I')$ *there exists an approximating sequence* $(u_\varepsilon) \subset H^1(\Omega)$ *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$*,* $Tu_\varepsilon \to v$ *in* $L^1(\partial\Omega)$ *and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \Phi(u, v). \tag{2.15}$$

This theorem can be easily rewritten in term of Γ-convergence (for the definition and the main properties of Γ-convergence we refer the reader to [DM, Chapters 6-9], see also [Al]). To this end we extend each F_ε to $+\infty$ on $L^1(\Omega) \setminus H^1(\Omega)$, and from Theorem 2.6 we immediately deduce the following corollary.

Corollary 2.7. *The* Γ*-limit on* $L^1(\Omega)$ *of the functionals* F_ε *is given by*

$$F(u) := \begin{cases} \inf\big\{\Phi(u, v) : \, v \in BV(\partial\Omega, I')\big\} & \text{if } u \in BV(\Omega, I), \\ \\ +\infty & \text{elsewhere in } L^1(\Omega). \end{cases} \tag{2.16}$$

Note that the functional $F(u)$ is nonlocal with respect to u, in the sense that it cannot be expressed by integration of a local density depending on u and Du.

Statement (iii) of Theorem 2.6 can be refined by choosing the approximating sequence (u_ε) so that $\int_\Omega u_\varepsilon = \int_\Omega u$ for every ε (we will not prove this refinement, in fact one has to slightly modify the construction of the approximating sequence (u_ε) in Lemma 4.15). In this way we can accommodate a prescribed mass constraint: if we take m such that $\alpha|\Omega| < m < \beta|\Omega|$, then the functionals F_ε Γ-converge to F also on the subspace of all $u \in L^1(\Omega)$ such that $\int_\Omega u = m$. By a well-known property of Γ-convergence and statement (i) of Theorem 2.6, we immediately deduce the following result:

Corollary 2.8. *For every* $\varepsilon > 0$ *let* u_ε *be a solution of the problem*

$$\min\big\{F_\varepsilon(u) : \textstyle\int_\Omega u = m\big\}. \tag{2.17}$$

Then the sequence (u_ε) *is pre-compact in* $L^1(\Omega)$*, and every cluster point belongs to* $BV(\Omega, I)$ *and solves*

$$\min\big\{F(u) : \textstyle\int_\Omega u = m\big\}. \tag{2.18}$$

2.4. Comparison of the Results

In this subsection we make a brief comparison of the results obtained in subsections 2.2 and 2.3. The energies Φ_0 and Φ that we have derived in the study of the relaxation of F_0 and of the Γ-limit of F_ε can be written in the following general geometric form:

$$\begin{aligned}\Phi_{\mathrm{gen}}(\mathsf{A},\mathsf{A}') := \sigma_{\mathsf{AB}}\,|\mathscr{S}_{\mathsf{AB}}| + \sigma_{\mathsf{AA}'}\,|\mathscr{S}_{\mathsf{AA}'}| + \sigma_{\mathsf{AB}'}\,|\mathscr{S}_{\mathsf{AB}'}| + \\ + \sigma_{\mathsf{BA}'}\,|\mathscr{S}_{\mathsf{BA}'}| + \sigma_{\mathsf{BB}'}\,|\mathscr{S}_{\mathsf{BB}'}| + \mathsf{c}\,|\mathscr{L}_{\mathsf{A}'\mathsf{B}'}|.\end{aligned} \tag{2.19}$$

where (A,A') belongs to the space of admissible configurations $X \times X'$. More precisely, the functional Φ_0 defined by (2.4) agrees with Φ_{gen} if we set

$$\begin{aligned}\sigma_{\mathsf{AA}'} &:= \sigma_{\mathsf{AW}}, \quad & \sigma_{\mathsf{AB}'} &:= \sigma_{\mathsf{AB}} + \sigma_{\mathsf{BW}}, \\ \sigma_{\mathsf{BB}'} &:= \sigma_{\mathsf{BW}}, \quad & \sigma_{\mathsf{BA}'} &:= \sigma_{\mathsf{AB}} + \sigma_{\mathsf{AW}}.\end{aligned} \tag{2.20}$$

On the other hand, if for every $u \in BV(\Omega, I)$ and $v \in BV(\partial\Omega, I')$ we consider the bulk phase $\mathsf{A} := \{u(x) = \alpha\} \subset \Omega$ and the boundary phase $\mathsf{A}' := \{v(x) = \alpha'\} \subset \partial\Omega$, then the functional Φ defined in (2.13) satisfies the identity $\Phi(u, v) = \Phi_{\mathrm{gen}}(\mathsf{A},\mathsf{A}')$ provided that we set

$$\begin{aligned}\sigma_{\mathsf{AB}} &:= \boldsymbol{\sigma}, & \mathsf{c} &:= \mathbf{c}, \\ \sigma_{\mathsf{AA}'} &:= \big|H(\alpha) - H(\alpha')\big|, \quad & \sigma_{\mathsf{AB}'} &:= \big|H(\alpha) - H(\beta')\big|, \\ \sigma_{\mathsf{BA}'} &:= \big|H(\beta) - H(\alpha')\big|, \quad & \sigma_{\mathsf{BB}'} &:= \big|H(\beta) - H(\beta')\big|,\end{aligned} \tag{2.21}$$

where H, $\boldsymbol{\sigma}$ and $\mathbf{c}$ are given in (2.11).

One can easily check that the coefficients of the functional Φ_{gen} can be written in the form (2.20) (for a suitable choice of σ_{AB}, σ_{AW} and σ_{BW}) if and only if they satisfy the relations

$$\sigma_{\mathsf{AB}'} = \sigma_{\mathsf{AB}} + \sigma_{\mathsf{BB}'}, \quad \sigma_{\mathsf{BA}'} = \sigma_{\mathsf{AB}} + \sigma_{\mathsf{AA}'}. \tag{2.22}$$

On the other hand, taking into account that the function H is strictly increasing it is easy to show that that the coefficients in (2.21) fulfills the relations in (2.22) if and only if the relative positions of the wells $\alpha, \beta, \alpha', \beta'$ are the following:

$$\alpha' \le \alpha < \beta \le \beta'. \tag{2.23}$$

Therefore when (2.23) is assumed we derive in a rigorous way that the model of capillarity with line tension (associated with F_0) is recovered from the Cahn-Hilliard model (associated with F_ε) in the limit $\varepsilon \to 0$. This carries out the main issue of our initial program.

Now we briefly account for some general features of the energies Φ_0, Φ and Φ_{gen}. Clearly the functional Φ_{gen} is coercive on $X \times X'$ because the energy densities σ_{AB} and c are strictly positive. The semicontinuity is discussed in the following statement (proved in section 3).

Theorem 2.9. *The functional $\Phi_{\rm gen}$ is lower semicontinuous on $X \times X'$ if and only if the coefficients in* (2.19) *verifies the following* generalized wetting conditions *(cf.* (1.2)*):*

$$\left|\sigma_{\mathsf{AA}'} - \sigma_{\mathsf{BA}'}\right| \le \sigma_{\mathsf{AB}}, \quad \left|\sigma_{\mathsf{AB}'} - \sigma_{\mathsf{BB}'}\right| \le \sigma_{\mathsf{AB}}. \tag{2.24}$$

Remark 2.10. Obviously (2.24) is satisfied when the coefficients in $\Phi_{\rm gen}$ are given either by (2.20) or by (2.21); hence we recover the lower semicontinuity of Φ_0 and Φ (cf. Theorems 2.1 and 2.6).

We may compare the models associated with the energies Φ_0, Φ and $\Phi_{\rm gen}$ by discussing the number N of independent parameters which drive the geometry of the equilibrium configurations (i.e., the number of their degrees of freedom). Notice that the equilibrium configurations of $\Phi_{\rm gen}$ (subject to some volume constraint) do not change if we multiply all the coefficients in (2.19) by a constant factor, or if we add the same constant to the boundary coefficients $\sigma_{\mathsf{AA}'}$, $\sigma_{\mathsf{AB}'}$, $\sigma_{\mathsf{BA}'}$ and $\sigma_{\mathsf{BB}'}$. Hence for $\Phi_{\rm gen}$ we have $N = 4$.

For Φ_0 we have to consider the two additional conditions in (2.22), and then $N = 2$.

For Φ the number N depends on the relative positions of $\alpha, \beta, \alpha', \beta'$: in the case $\alpha' < \alpha < \beta < \beta'$, $N = 2$ because we can reduce to the case Φ_0. We let the reader check that $N = 3$ in the remaining five cases.

These considerations about the value of N suggest the possibility of an experimental validation either of the line tension model F_0 (i.e., Φ_0), or of the model derived from the Cahn-Hilliard model F_ε (i.e. Φ), or of the more general $\Phi_{\rm gen}$. In fact all these models seem to be physically acceptable. We conclude this section with some comments on their physical background.

Capillary energy with line tension, like F_0, is frequently considered in physics (see [RW] or [WW]), and we have proved that the well-posed problems naturally associated with F_0 can only be defined through Φ_0. In other words the interaction between the fluids and the wall can be efficiently described only by considering two boundary phases which are independent of the bulk phases. Notice that the idea of phase transition between surface phases on the wall of the container has already been suggested (see for instance [DG]).

As noticed before, Φ_0 is obtained from $\Phi_{\rm gen}$ imposing the restrictions (2.22). In the relaxation procedure which leads to Φ_0, the interface $\mathscr{S}_{\mathsf{AB}'}$ is viewed as the part of the wall where an infinitesimal layer of the phase B is interposed between the phase A and the wall of the container (a similar argument applies to $\mathscr{S}_{\mathsf{BA}'}$), and the relations (2.22) are a consequence of the fact that the energy density of such a layer is simply the sum of σ_{AB} (due to the transition from A to B) and σ_{BW} (due to the transition from B to the wall). On a physical level, such a superposition principle has no reason to hold: consider for instance a layer whose thickness has the same order as the range of the interaction forces which generate the surface tension. Then it is quite natural to consider generalized energies of the form $\Phi_{\rm gen}$.

The functional Φ which corresponds to the asymptotic limit of the Cahn-Hilliard model, appears as an intermediate case between Φ_0 and $\Phi_{\rm gen}$ (and indeed for Φ we have $N = 2$ or $N = 3$). The Cahn-Hilliard model, despite its relative simplicity, is known to describe efficiently many interfacial phenomena. In this paper we show that it can be used to describe line tension phenomena as well. One may question the physical ground of the boundary energy we postulated, and in particular on the double-well potential V and the scaling (2.10) for λ_ε. Indeed these assumptions are totally different from those of Cahn & Hilliard [CH] or Modica [Mo2] (where λ_ε does not depend on ε and V is a monotone function). To our knowledge, the boundary energy cannot be reached by direct experiments, but only through its effects on the macroscopic equilibrium. We justify our assumptions *a posteriori* by the relevance of the model associated with the limit energy Φ.

3. Proof of the Relaxation Result

This section is devoted to the proof of Theorem 2.1 and Theorem 2.9.

We follow here the notation introduced in subsection 2.2; in particular, given sets A and B in Ω (resp. in $\partial\Omega$) the identity $A = B$ must be intended up to negligible subsets, that is, in the sense of the space X (resp. X'). We also recall that ∂A denotes the essential boundary of A, and not the topological one. All statements and proofs in this section can be adapted without essential modifications to arbitrary dimension.

Lemma 3.1. *Let be given $B \subset \partial\Omega$. Then for every $\delta > 0$ there exists E with finite perimeter in Ω such that*

(i) *B is the trace of E on $\partial\Omega$, that is, $B = \partial E \cap \partial\Omega$;*

(ii) *$|E| \le \delta$ and $|\partial E \cap \Omega| \le |B| + \delta$.*

Proof. This statement is an immediate corollary of a well-known result of Gagliardo (see for instance [Gi, Theorem 2.16]). □

Lemma 3.2. *Let be given $M \subset \partial\Omega$. Then the functional $\mathsf{A} \mapsto |\partial\mathsf{A} \cap \Omega| - |\partial\mathsf{A} \triangle M|$ is lower semicontinuous on X.*

Proof. We apply Lemma 3.1 with Ω and B replaced by $\mathbb{R}^3 \setminus \overline{\Omega}$ and $\partial\Omega \setminus M$ respectively, and we find a set $E \subset \mathbb{R}^3 \setminus \overline{\Omega}$ with finite perimeter in $\mathbb{R}^3 \setminus \overline{\Omega}$ so that $\partial E \cap \partial\Omega = \partial\Omega \setminus M$.

Thus $\partial\Omega \setminus \partial E = M$ and, since $\mathsf{A} \cap E = \emptyset$ for every $\mathsf{A} \in X$, we have that $\partial(\mathsf{A} \cup E)$ is the disjoint union of $\partial\mathsf{A} \cap \Omega$, $\partial E \setminus \overline{\Omega}$, and $\partial\Omega \cap \partial(\mathsf{A} \cup E) = \partial\Omega \setminus (\partial\mathsf{A} \triangle M)$ (see Figure 2).

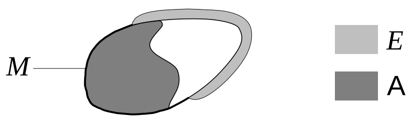

Fig. 2. The sets M and E.

Hence
$$|\partial\mathsf{A} \cap \Omega| - |\partial\mathsf{A} \triangle M| = |\partial(\mathsf{A} \cup E)| - |\partial E \setminus \overline{\Omega}| - |\partial\Omega|.$$

Since E is fixed, the thesis follows from the lower semicontinuity of the perimeter $|\partial(\mathsf{A}\cup E)|$ with respect to A. $\square$

Proof of Theorem 2.9. We assume first that the generalized wetting condition (2.24) does not hold, and in particular that $\sigma_{\mathsf{AA}'} > \sigma_{\mathsf{AB}}+\sigma_{\mathsf{BA}'}$ (the other three cases can be treated in the same way). We argue now as for the lack of semicontinuity of F_0 (see subsection 2.2).

Fix a configuration $(\mathsf{A},\mathsf{A}')\in X\times X'$ such that $|\mathscr{S}_{\mathsf{AA}'}|>0$. For every $\delta>0$ we apply Lemma 3.1 to find a set $E_\delta\subset\Omega$ such that $\partial E_\delta\cap\partial\Omega=\mathsf{A}'\cap\partial\mathsf{A}$, $|E_\delta|\le\delta$, and $|\partial E_\delta\cap\Omega|\le|\mathsf{A}'\cap\partial\mathsf{A}|+\delta$, and then we set $\mathsf{A}_\delta:=\mathsf{A}\setminus E_\delta$.

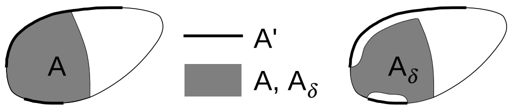

Fig. 3. The sets A, A', and A_δ.

Hence A_δ converge to A in X as $\delta\to0$. Moreover for the configuration $(\mathsf{A}_\delta,\mathsf{A})$ there holds $\mathscr{S}_{\mathsf{A}_\delta\mathsf{A}'}=\emptyset$, $\mathscr{S}_{\mathsf{B}_\delta\mathsf{A}'}=\mathsf{A}'=\mathscr{S}_{\mathsf{BA}'}\cup\mathscr{S}_{\mathsf{AA}'}$, $|\mathscr{S}_{\mathsf{A}_\delta\mathsf{B}_\delta}|\le|\mathscr{S}_{\mathsf{AB}}|+|\mathscr{S}_{\mathsf{AA}'}|+\delta$, while $\mathscr{S}_{\mathsf{A}_\delta\mathsf{B}'}=\mathscr{S}_{\mathsf{AB}'}$ and $\mathscr{S}_{\mathsf{B}_\delta\mathsf{B}'}=\mathscr{S}_{\mathsf{BB}'}$. Then

$$\Phi_{\rm gen}(\mathsf{A}_\delta,\mathsf{A}')\le\Phi_{\rm gen}(\mathsf{A},\mathsf{A}')-(\sigma_{\mathsf{AB}}+\sigma_{\mathsf{BA}'}-\sigma_{\mathsf{AA}'})\,|\mathscr{S}_{\mathsf{AA}'}|+\delta,$$

and since both $(\sigma_{\mathsf{AB}}+\sigma_{\mathsf{BA}'}-\sigma_{\mathsf{AA}'})$ and $|\mathscr{S}_{\mathsf{AA}'}|$ are positive we obtain

$$\liminf_{\delta\to0}\Phi_{\rm gen}(\mathsf{A}_\delta,\mathsf{A}')<\Phi_{\rm gen}(\mathsf{A},\mathsf{A}'),$$

which proves that $\Phi_{\rm gen}$ is not lower semicontinuous at (A,A').

We prove now the opposite implication. Let us assume that (2.24) holds and let be given $\mathsf{A}_n\to\mathsf{A}$ in X and $\mathsf{A}'_n\to\mathsf{A}'$ in X'. We may assume that $\sup_n\Phi_{\rm gen}(\mathsf{A}_n,\mathsf{A}'_n)$ is finite, so that $|\partial\mathsf{A}\cap\Omega|$ and $|\mathscr{L}_{\mathsf{A}'\mathsf{B}'}|$ are finite. By applying Lemma 3.2 with $M:=\partial\mathsf{A}\cap\partial\Omega$, we obtain the following lower bound:

$$\liminf_{n\to\infty}\big(|\partial\mathsf{A}_n\cap\Omega|-|\partial\mathsf{A}\cap\Omega|-|(\partial\mathsf{A}_n\triangle\partial\mathsf{A})\cap\partial\Omega|\big)\ge0. \tag{3.1}$$

By the lower semicontinuity of the perimeter, the functional $\mathsf{A}'\mapsto\Phi_{\rm gen}(\mathsf{A},\mathsf{A}')$ is lower semicontinuous on X'. Hence

$$\liminf_{n\to\infty}\Phi_{\rm gen}(\mathsf{A},\mathsf{A}'_n)\ge\Phi_{\rm gen}(\mathsf{A},\mathsf{A}'). \tag{3.2}$$

On the other hand, let $\rho_n(x)$ and $\hat\rho_n(x)$ denote respectively the surface energy densities at x of the configurations $(\mathsf{A}_n,\mathsf{A}'_n)$ and $(\mathsf{A},\mathsf{A}'_n)$ (for every n and every $x\in\partial\Omega$); one easily verifies that if $x\notin\partial\mathsf{A}_n\triangle\partial\mathsf{A}$ then $\rho_n(x)=\hat\rho_n(x)$, while if $x\in\partial\mathsf{A}_n\triangle\partial\mathsf{A}$ the inequalities in (2.24) implies $\rho_n(x)\ge\hat\rho_n(x)-\sigma_{\mathsf{AB}}$. Thus we can write

$$
\begin{aligned}
&\Phi_{\rm gen}(\mathsf{A}_n,\mathsf{A}'_n)\\
&\qquad = \sigma_{\mathsf{AA'}}|\partial\mathsf{A}_n\cap\partial\Omega| + \int_{\partial\Omega}\rho_n(x) \ \ + \mathrm{c}|\mathscr{L}_{\mathsf{A}'_n\mathsf{B}'_n}|\\
&\qquad \geq \sigma_{\mathsf{AA'}}|\partial\mathsf{A}_n\cap\partial\Omega| + \int_{\partial\Omega}\hat\rho_n(x) - \sigma_{\mathsf{AB}}|(\partial\mathsf{A}_n\triangle\partial\mathsf{A})\cap\partial\Omega| + \mathrm{c}|\mathscr{L}_{\mathsf{A}'_n\mathsf{B}'_n}|\\
&\qquad \geq \Phi_{\rm gen}(\mathsf{A},\mathsf{A}'_n) + \sigma_{\mathsf{AB}}\big(|\partial\mathsf{A}_n\cap\Omega)| - |\partial\mathsf{A}\cap\Omega)| - |(\partial\mathsf{A}_n\triangle\partial\mathsf{A})\cap\partial\Omega|\big).
\end{aligned}
$$

Now we take the lower limit as $n\to\infty$, and with the help of (3.1) and (3.2) we deduce

$$\liminf_{n\to\infty}\Phi_{\rm gen}(\mathsf{A}_n,\mathsf{A}'_n)\geq\Phi_{\rm gen}(\mathsf{A},\mathsf{A}'). \quad \square$$

Proof of Theorem 2.1. The coefficients in the functionals Φ_0 given in (2.4) fulfill the generalized wetting condition (2.24), and then Φ_0 is lower semicontinuous on $X\times X'$ by Theorem 2.9.

Let be given now $\mathsf{A}_n\to\mathsf{A}$ in X, and assume that $F_0(\mathsf{A}_n)$ is bounded. The sets A_n have uniformly bounded perimeters in $\partial\Omega$ (cf. (1.4)); then the sequence $(\mathscr{S}_{\mathsf{A}_n\mathsf{W}})$ is pre-compact in X', and, possibly passing to a subsequence, we may assume that it converge to some $\mathsf{A}'\in X'$. Now identity (2.5) and the semicontinuity of Φ_0 imply

$$\liminf_{n\to\infty}F_0(\mathsf{A}_n)=\liminf_{n\to\infty}\Phi_0(\mathsf{A}_n,\mathscr{S}_{\mathsf{A}_n\mathsf{W}})\geq\Phi_0(\mathsf{A},\mathsf{A}'). \tag{3.3}$$

Inequality (3.3) shows that the left hand side of (2.6) is larger than the right hand side. To obtain the equality it suffices to find, for every configuration $(\mathsf{A},\mathsf{A}')\in X\times X'$ with finite energy Φ_0, an approximating sequence $\mathsf{A}_n\to\mathsf{A}$ such that

$$\liminf_{n\to\infty}F_0(\mathsf{A}_n)\leq\Phi_0(\mathsf{A},\mathsf{A}'). \tag{3.4}$$

Here we use an argument similar to the first part of the proof of Theorem 2.9: by Lemma 3.1, for every $n>0$ we find a set E_n with finite perimeter in Ω such that

(i) $\partial E_n\cap\partial\Omega=\mathsf{A}'\triangle\partial\mathsf{A}$,

(ii) $|E_n|\leq 1/n$ and $|\partial E_n\cap\Omega|\leq|\mathsf{A}'\triangle\partial\mathsf{A}|+1/n$.

We set $\mathsf{A}_n:=\mathsf{A}\triangle E_n$: by (i) we have $\mathscr{S}_{\mathsf{A}_n\mathsf{W}}=\partial\mathsf{A}_n\cap\partial\Omega=\mathsf{A}'$ and by (ii)

$$\mathsf{A}_n\to\mathsf{A} \text{ in } X \quad , \quad |\partial\mathsf{A}_n\cap\Omega|\leq|\partial\mathsf{A}\cap\Omega|+|\mathsf{A}'\triangle\partial\mathsf{A}|+1/n.$$

Hence

$$
\begin{aligned}
F_0(\mathsf{A}_n) &\leq \sigma_{\mathsf{AB}}\big(|\partial\mathsf{A}\cap\Omega|+|\mathsf{A}'\triangle\partial\mathsf{A}|+1/n\big)+\sigma_{\mathsf{AW}}|\mathsf{A}'|+\sigma_{\mathsf{BW}}|\mathsf{B}'|+\mathrm{c}|\partial\mathsf{A}'|\\
&= \sigma_{\mathsf{AB}}|\partial\mathsf{A}\cap\Omega|+\sigma_{\mathsf{AW}}|\mathsf{A}'\cap\partial\mathsf{A}|+(\sigma_{\mathsf{AW}}+\sigma_{\mathsf{AB}})\,|\mathsf{A}'\setminus\partial\mathsf{A}|+\\
&\qquad +\sigma_{\mathsf{BW}}|\mathsf{B}'\setminus\partial\mathsf{A}|+(\sigma_{\mathsf{BW}}+\sigma_{\mathsf{AB}})\,|\mathsf{B}'\cap\partial\mathsf{A}|+\mathrm{c}|\partial\mathsf{A}'|+\frac{\sigma_{\mathsf{AB}}}{n}\\
&= \Phi_0(\mathsf{A},\mathsf{A}')+\frac{\sigma_{\mathsf{AB}}}{n}.
\end{aligned}
$$

We obtain (3.4) by letting n tend to ∞. $\square$

4. Proof of the Γ-Convergence Result

In this section we prove Theorem 2.6. In order to simplify the proof, we will make two additional assumptions: first we will assume that $\partial\Omega$ is of class C^2. This restriction is used in the proof of statement (iii) of Theorem 2.6, and can be relaxed with some additional work to $\partial\Omega$ of class C^1. However we cannot go below the C^1 regularity. The second assumption concerns the potentials V and W:

$$\begin{array}{l}\text{there exists } m \text{ so that } -m \leq \alpha, \alpha', \beta, \beta' \leq m, \\ W(x) \geq W(m) \text{ and } V(x) \geq V(m) \text{ for } x \geq m, \text{ and} \\ W(x) \geq W(-m) \text{ and } V(x) \geq V(-m) \text{ for } x \leq -m.\end{array} \tag{4.1}$$

For instance, this condition is verified when V and W are increasing on $[m, +\infty)$ and decreasing on $(-\infty. -m]$ for some positive m. Assumption (4.1) will allow us to use the truncation argument given Lemma 4.1. It can be removed but in that case the proof of Proposition 4.7 would require more delicates truncation arguments which we prefer to avoid.

From now on we always use the term "sequence" also to denote families (of functions) labelled by the continuous parameter ε, which tends to 0. On this line, a subsequence of (u_ε) is any sequence (u_{ε_n}) such that $\varepsilon_n \to 0$ as $n \to \infty$, and we say that (u_ε) is pre-compact if every subsequence admits a converging sub-subsequence. To simplify the notation we often omit to relabel subsequences, and we say "a countable sequence (u_ε)" to mean a sequence defined only for countably many $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$ (see for instance the statements (i) of Theorems 4.2 and 4.4).

To begin we introduce the localization of the functionals F_ε: for every domain $A \subset \mathbb{R}^3$, every set $A' \subset \partial A$ and every $u \in H^1(A)$ we set

$$F_\varepsilon(u, A, A') := \varepsilon \int_A |Du|^2 + \frac{1}{\varepsilon} \int_A W(u) + \lambda_\varepsilon \int_{A'} V(Tu) \tag{4.2}$$

(according to our convention the measure in the last integral is $\mathscr{H}^2$). Notice that $F_\varepsilon(u) = F_\varepsilon(u, \Omega, \partial\Omega)$ for every $u \in H^1(\Omega)$.

Lemma 4.1. *Let be given a domain $A \subset \mathbb{R}^3$ and a set $A' \subset \partial A$, and a sequence $(u_\varepsilon) \subset H^1(A)$ with uniformly bounded energies $F_\varepsilon(u_\varepsilon, A, A')$. If we take the truncated functions $\bar{u}_\varepsilon(x) := (u_\varepsilon(x) \wedge m) \vee -m$, then $F_\varepsilon(\bar{u}_\varepsilon, A, A') \leq F_\varepsilon(u_\varepsilon, A, A')$, and both $\|\bar{u}_\varepsilon - u_\varepsilon\|_{L^1(A)}$ and $\|T\bar{u}_\varepsilon - Tu_\varepsilon\|_{L^1(A')}$ vanish as $\varepsilon \to 0$.*

Proof. The inequality $F_\varepsilon(\bar{u}_\varepsilon, A, A') \leq F_\varepsilon(u_\varepsilon, A, A')$ follows immediately from (4.1). The rest of the statement follows from the fact that both W and V have growth at least linear at infinity and the integrals $\int W(u_\varepsilon)$ and $\int V(Tu_\varepsilon)$ vanish as $\varepsilon \to 0$. This is a standard argument, and we omit it (see for instance [AB, Lemma 1.11]). $\square$

In order to prove Theorem 2.6 we need some Γ-convergence results which we group in the following subsection.

4.1. Preliminary Convergence Results

We begin with the basic Γ-convergence result for functionals of Cahn-Hilliard type: for every domain $A \subset \mathbb{R}^3$ and every real function $u \in H^1(A)$ we set

$$G^1_\varepsilon(u, A) := \varepsilon \int_A |Du|^2 + \frac{1}{\varepsilon} \int_A W(u), \tag{4.3}$$

where W is the double-well potential given in subsection 2.3. Notice that $G^1_\varepsilon(u, A) = F_\varepsilon(u, A, \emptyset)$.

Theorem 4.2. ([MM, Mo]). *For every domain $A \subset \mathbb{R}^3$ the following three statements hold:*

(i) *every countable sequence $(u_\varepsilon) \subset H^1(A)$ with uniformly bounded energies $G^1_\varepsilon(u_\varepsilon, A)$ is pre-compact in $L^1(A)$ and every cluster point belongs to $BV(A, I)$;*

(ii) *for every $u \in BV(A, I)$ and every sequence $(u_\varepsilon) \subset H^1(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ there holds*
$$\liminf_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, A) \geq \boldsymbol{\sigma} \mathscr{H}^2(Su);$$

(iii) *for every $u \in BV(A, I)$ there exists a sequence $(u_\varepsilon) \subset H^1(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and*
$$\limsup_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, A) \leq \boldsymbol{\sigma} \mathscr{H}^2(Su);$$
moreover when Su is a closed Lipschitz surface in A, the functions u_ε may be required to be (C/ε)-Lipschitz continuous and to converge to u uniformly on every set with positive distance from Su (here C is the supremum of $\sqrt{W}$ in $[\alpha, \beta]$).

Proof. This version of the Modica-Mortola theorem can be found in [Mo1] (see also [Al]). However the second part of statement (iii) is not explicitly stated there, and therefore we briefly sketch its proof.

Let $\varphi : \mathbb{R} \to [\alpha, \beta]$ be an optimal profile for the 1-dimensional functional $\int(\dot{v}^2 + W(v))$, that is, a global solution of the ordinary differential $\dot{\varphi} = \sqrt{W(\varphi)}$ with $\varphi(0)$ arbitrarily taken in $]\alpha, \beta[$. Then φ is increasing, converges to β at $+\infty$ and to α at $-\infty$, and satisfies

$$\int_{\mathbb{R}} (\dot{\varphi}^2 + W(\varphi)) = \int_{\mathbb{R}} 2\sqrt{W(\varphi)}\, \dot{\varphi} = H(\beta) - H(\alpha) = \boldsymbol{\sigma}. \tag{4.4}$$

Let now be given $u \in BV(\Omega, I)$ such that Su is a Lipschitz surface, and denote by d the oriented distance from Su given by $d(x) := \operatorname{dist}(x, Su)$ when $x \in \{u = \beta\}$, and by $d(x) := -\operatorname{dist}(x, Su)$ when $x \in \{u = \alpha\}$. We set $u_\varepsilon(x) := \varphi\big(d(x)/\varepsilon\big)$ for every $\varepsilon > 0$ and $x \in \Omega$. One readily checks that each u_ε is (C/ε)-Lipschitz continuous (because φ is C-Lipschitz continuous and d is 1-Lipschitz continuous) and converge to u uniformly on every set with positive distance from Su. Taking into account that $|Dd| = 1$ a.e. in Ω, by the coarea formula one gets

$$G^1_\varepsilon(u_\varepsilon, A) = \int_A \frac{1}{\varepsilon}\big(\dot\varphi^2(d/\varepsilon) + W(d/\varepsilon)\big) = \int_{\mathbb{R}} \big(\dot\varphi^2(t) + W(t)\big)\,\mathscr{H}^2(\Sigma_{\varepsilon t})\,dt, \tag{4.5}$$

where $\Sigma_s := \{x : d(x) = s\}$ is the s-level set of d. Since Su is Lipschitz regular, $\mathscr{H}^2(\Sigma_s)$ converges to $\mathscr{H}^2(Su)$ as $s \to 0$, and if we use (4.4) and apply the dominated convergence theorem in (4.5), we obtain that $G^1_\varepsilon(u_\varepsilon, A)$ converges to $\boldsymbol{\sigma}\mathscr{H}^2(Su)$ as $\varepsilon \to 0$. $\square$

Theorem 4.2 captures completely the asymptotic behavior of the energies F_ε in the interior of Ω, and justifies the term $\boldsymbol{\sigma}\mathscr{H}^2(Su)$ in the limit energy Φ (see (2.13)). The second term in Φ, namely $\int_{\partial\Omega}\big|H(Tu) - H(v)\big|$, will be derived from the following proposition.

Proposition 4.3. *Assume that $A \subset \mathbb{R}^3$ is a domain with boundary piecewise of class C^1, and A' is a subset of ∂A with Lipschitz boundary, and let be given $u \in L^1(A)$, $v \in L^1(A')$. Then*

(i) *for every sequence $(u_\varepsilon) \subset H^1(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and $Tu_\varepsilon \to v$ in $L^1(A')$ there holds*

$$\liminf_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, A) \ge \int_{A'} \big|H(Tu) - H(v)\big|;$$

(ii) *if v is constant on A' and u is constant on A with $u \equiv \alpha$ or $u \equiv \beta$, there exists a sequence (u_ε) such that $Tu_\varepsilon = v$ on A', u_ε converges uniformly to u on every set with positive distance from A' and*

$$\limsup_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, A) \le \int_{A'} \big|H(Tu) - H(v)\big|;$$

moreover each u_ε can be taken (C/ε)-Lipschitz continuous, where C is the supremum of $\sqrt{W}$ over any interval which contains the values of u and v.

Proof. Statement (i) is the key lemma in the proof of the main result of [Mo2], statement (ii) is essentially contained in that paper, but not stated in this form. The proof is a modification of the argument of the proof of Theorem 4.2. We consider the case $u \equiv \beta$ and $v \equiv \gamma$, with $\alpha < \gamma < \beta$ (the other cases can be treated in a similar way).

Let $\varphi : [0, +\infty[\to [\gamma, \beta]$ be a solution of the ordinary differential $\dot\varphi = \sqrt{W(\varphi)}$ with $\varphi(0) = \gamma$; then φ is increasing, converges to β at $+\infty$, and satisfies (cf. (4.4))

$$\int_0^\infty (\dot\varphi^2 + W(\varphi)) = \int_0^\infty 2\sqrt{W(\varphi)}\,\dot\varphi = H(\beta) - H(\gamma).$$

Denote by $d(x)$ the distance of x from A' and set $u_\varepsilon(x) := \varphi\big(d(x)/\varepsilon\big)$ for every $\varepsilon > 0$ and $x \in \Omega$. One readily checks that u_ε converges to u uniformly on every set with positive distance from A', u_ε is (C/ε)-Lipschitz continuous, and $G^1_\varepsilon(u_\varepsilon, A)$ converge to $\big(H(\beta)-H(\gamma)\big)\mathscr{H}^2(A')$. $\square$

The last term in Φ, namely $\mathbf{c}\,\mathscr{H}^1(Sv)$, requires a more delicate treatment. The next steps are crucial in the proof of the statements (i) and (ii) of Theorem 2.6. We begin with a singular perturbation theorem for one-dimensional functionals: for every interval $E\subset\mathbb{R}$ and every function $v\in L^1(E)$ we set

$$G^2_\varepsilon(v,E) := \frac{\varepsilon}{2\pi}\int_{E^2}\Big|\frac{v(x')-v(x)}{x'-x}\Big|^2 dx'dx + \lambda_\varepsilon\int_E V(v). \tag{4.6}$$

Here we have replaced the usual Dirichlet integral by a nonlocal energy which is directly related to the square of the norm of the space $H^{\frac12}(E)$. We will use $G^2_\varepsilon(v,E)$ to write the value of $F_\varepsilon(u, B\cap\Omega, B\cap\partial\Omega)$ in term of the trace v of u on $B\cap\partial\Omega$ in the particular case where $B\cap\partial\Omega$ is a flat disk (see Proposition 4.7).

Theorem 4.4.(cf. [ABS1]). *Let V be given as in subsection 2.3. Then the following statements hold:*

(i) *every countable sequence $(v_\varepsilon)\subset L^1(E)$ with uniformly bounded energies $G^2_\varepsilon(v_\varepsilon,E)$ is pre-compact in $L^1(E)$ and every cluster points belongs to the space $BV(E,I')$.*

(ii) *For every $v\in BV(E,I')$ and every sequence (v_ε) such that $v_\varepsilon\to v$ in $L^1(E)$ there holds*
$$\liminf_{\varepsilon\to0} G^2_\varepsilon(v_\varepsilon,E)\ge \mathbf{c}\,\#(Sv) \tag{4.7}$$
where $\#(Sv)$ denotes as usual the number of points in Sv.

In order to prove Theorem 4.4 we need the following estimate:

Lemma 4.5. *Let be given δ such that $0<\delta<\frac{\beta'-\alpha'}{2}$. For every interval $J\subset E$, $\varepsilon>0$ and $v\in L^1(E)$, let $A=A(J,\varepsilon,v)$ and $B=B(J,\varepsilon,v)$ be the sets of all points $x\in J$ such that $v(x)\le\alpha'+\delta$ and $v(x)\ge\beta'-\delta$ respectively, and set*

$$\begin{gathered} a=a(J,\varepsilon,v):=\frac{|A\cap J|}{|J|},\quad b=b(J,\varepsilon,v):=\frac{|B\cap J|}{|J|},\\ \rho:=\inf\big\{V(t):\ \alpha'+\delta\le t\le\beta'-\delta\big\}. \end{gathered}\tag{4.8}$$

Then

$$G^2_\varepsilon(u_\varepsilon,J)\ge\frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2\big[\log(ab)+\log(\lambda_\varepsilon)\big].\quad \textit{for } \varepsilon\le\frac{\pi\rho|J|}{(\beta'-\alpha'-2\delta)^2} \tag{4.9}$$

Proof. The proof relies on the following key inequality, which is obtained by applying Proposition 6.1 with $\Psi(s):=1/s^2$ and $[t,y]:=J$

$$\int_{A\times B}\frac{dx'dx}{|x'-x|^2}\ge\log\Big[1+\frac{ab}{1-a-b}\Big]. \tag{4.10}$$

By (4.6) and (4.8) we get

$$G^2_\varepsilon(v,J) \geq \frac{\varepsilon}{2\pi} \int\limits_{(A\times B)\cup(B\times A)} \Big|\frac{v(x')-v(x)}{x'-x}\Big|^2 dx'dx \;+ \lambda_\varepsilon \int\limits_{J\setminus(A\cup B)} V(v)$$

$$\geq \frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2 \int\limits_{A\times B} \frac{dx'dx}{|x'-x|^2} \;+ \lambda_\varepsilon\rho\,\big|J\setminus(A\cup B)\big|$$

and by (4.10) this exceeds

$$\frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2 \log\Big[1+\frac{ab}{1-a-b}\Big] + \lambda_\varepsilon\rho(1-a-b)\,|J|$$

$$\geq \frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2 \Big[\log(ab)-\log(1-a-b)+\frac{\pi\lambda_\varepsilon\rho|J|}{\varepsilon(\beta'-\alpha'-2\delta)^2}(1-a-b)\Big].$$

Now we apply the inequality $-\log t + Mt \geq \log M$ with $M := \frac{\pi\lambda_\varepsilon\sigma|J|}{\varepsilon(\beta_2-\alpha_2-2\delta)^2}$ and $t := 1-a-b$, and recalling the assumptions on ε we get that our last expression exceeds

$$\frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2 \Big[\log(ab)+\log\Big(\frac{\pi\lambda_\varepsilon\rho|J|}{\varepsilon(\beta'-\alpha'-2\delta)^2}\Big)\Big]$$

$$\geq \frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2\big[\log(ab)+\log\lambda_\varepsilon\big]. \quad \square$$

Proof of Theorem 4.4. The proof reduces to the following statement: given a countable sequence (v_ε) such that $G^2_\varepsilon(v_\varepsilon,E)$ is bounded, possibly passing to a subsequence we have that v_ε converge in $L^1(E)$ to some $v\in BV(E,I')$, and inequality (4.7) holds.

By a standard truncation argument we can assume from the beginning that $\alpha'\leq v_\varepsilon\leq\beta'$ for every $\varepsilon>0$. Possibly passing to a subsequence we can assume that the sequence (v_ε) converges weakly* in $L^\infty(E)$ to some function v and generates a Young measure $x\mapsto\nu_x$ (for a detailed exposition of the theory of Young measures, we refer to [Va1, Va2]).

Since $\lambda_\varepsilon\to\infty$ as $\varepsilon\to 0$ and $G^2_\varepsilon(v_\varepsilon,E)$ is bounded in ε, we deduce that the integral $\int_E V(v_\varepsilon)$ vanishes as $\varepsilon\to 0$, and then

$$\int_E\Big(\int_{\mathbb{R}} V(t)\,d\nu_x(t)\Big)\,dx = 0.$$

As $W(t)=0$ if and only if $t=\alpha'$ or $t=\beta'$, the probability measure ν_x is supported on $I'=\{\alpha',\beta'\}$ for a.a. x; in other words there exists a function $\eta: E\to[0,1]$ such that

$$\nu_x = \eta(x)\cdot\delta_{\alpha'} + (1-\eta(x))\cdot\delta_{\beta'} \quad \text{for a.e. } x\in E. \tag{4.11}$$

We claim that η belongs to $BV(E,\{0,1\})$. Take indeed an interval $J\subset E$ and δ such that $0<\delta<\frac{\beta'-\alpha'}{2}$, and define $a_\varepsilon := a(J,\varepsilon,v_\varepsilon)$ and $b_\varepsilon := b(J,\varepsilon,v_\varepsilon)$ as in (4.8). By Lemma 4.5 we obtain that for ε small enough

$$G^2_\varepsilon(v_\varepsilon,J) \geq \frac{\varepsilon}{\pi}(\beta'-\alpha'-2\delta)^2\big[\log(a_\varepsilon b_\varepsilon)+\log\lambda_\varepsilon\big]. \tag{4.12}$$

Furthermore one readily checks that when $\varepsilon \to 0$

$$a_\varepsilon \to a(J) := \tfrac{1}{|J|} \int_J \eta \quad \text{and} \quad b_\varepsilon \to b(J) := \tfrac{1}{|J|} \int_J (1-\eta).$$

If $a(J) \cdot b(J) > 0$, when we pass to the limit in (4.12) we get

$$\liminf_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, J) \geq \frac{K}{\pi} (\beta' - \alpha')^2 = \mathbf{c} \tag{4.13}$$

(recall that $\varepsilon \log \lambda_\varepsilon \to K$ and δ can be taken arbitrarily small).

Consider now the set S of all $x \in E$ such that the approximate limit of η at x does not exists or belongs to $]0,1[$. For every finite integer $m \leq \#(S)$ we can find pairwise disjoint open intervals J_i, $i = 1, \dots, m$, such that $J_i \cap S \neq \emptyset$. Thus $a(J_i) \cdot b(J_i) > 0$ and (4.13) becomes

$$\liminf_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, J_i) \geq \mathbf{c},$$

and since $G^2_\varepsilon(v_\varepsilon, \cdot)$ is super-additive on disjoint sets,

$$\liminf_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, E) \geq \sum_{i=1}^m \liminf_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, J_i) \geq m\, \mathbf{c}. \tag{4.14}$$

Hence S is finite, and since η has approximate limit equal to 0 or 1 outside of S, we deduce that η belongs to $BV(E, \{0,1\})$ and $S\eta = S$. The claim is proved.

According to (4.11) we deduce that ν_x is a Dirac mass for almost every x; hence v_ε converge strongly to v and

$$v(x) := \alpha' \eta(x) + \beta'(1 - \eta(x)) \quad \text{for a.e. } x \in E.$$

Then v belongs to $BV(E, I')$, $Sv = S\eta = S$ and by taking $m = \#(S)$ in (4.14) we get

$$\liminf_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, E) \geq \mathbf{c}\, \#(Sv). \quad \square$$

Remark 4.6. In [ABS1], we proved that the lower bound given in (4.9) is in fact optimal: for every $v \in BV(E, I')$ we can find a sequence $(v_\varepsilon) \subset H^1(E)$ such that $v_\varepsilon \mapsto v$ in $L^1(E)$ and

$$\lim_{\varepsilon \to 0} G^2_\varepsilon(v_\varepsilon, E) = \mathbf{c}\, \#(Sv). \tag{4.15}$$

Therefore the functionals $G^2_\varepsilon(\cdot, E)$ Γ-converge in $L^1(E)$ to the functional which is equal to $\mathbf{c}\,\#(Sv)$ for every $v \in BV(E, I')$ and to $+\infty$ elsewhere.

Using Theorem 4.4 and a suitable slicing argument we can obtain the optimal lower bound for the energies $F_\varepsilon(u, B \cap \Omega, B \cap \partial\Omega)$ when B is a ball centered on $\partial\Omega$ and $B \cap \partial\Omega$ is a flat disk (Proposition 4.7). Later on we will show that this flatness assumption can be dropped when B is sufficiently small.

Proposition 4.7. *For every $r > 0$, let D_r be the open half-ball of all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $|x| < r$ and $x_3 > 0$, and E_r the disk of all x such* *that* $|x| < r$ *and* $x_3 = 0$. *Let* $(u_\varepsilon) \subset H^1(D_r)$ *be a countable sequence with uniformly bounded energies* $F_\varepsilon(u_\varepsilon, D_r, E_r)$. *Then the traces* Tu_ε *are pre-compact in* $L^1(E_r)$ *and every cluster point belongs to* $BV(E_r, I')$*; moreover if* $Tu_\varepsilon \to v$ *in* $L^1(E_r)$*, then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D_r, E_r) \geq \mathbf{c}\Big|\int_{E_r\cap Sv} \nu_v\Big|. \tag{4.16}$$

Proof. By Lemma 4.1 we can assume that $|u_\varepsilon| \leq m$ where m is the constant in (4.1). To simplify the notation we write D and E.

The idea is to reduce to statement (i) of Theorem 4.4 via a suitable slicing argument. We fix now an arbitrary unit vector e in the plane $P := \{x_3 = 0\}$, and we denote by M the orthogonal complement of E in P and by π the projection of $\mathbb{R}^3$ onto M. The segment $\pi(E)$ is called E_e; for every $y \in E_e$, E^y denotes the segment $\pi^{-1}(y) \cap E$ and D^y the half-disk $\pi^{-1}(y) \cap D$ (see Figure 4 below).

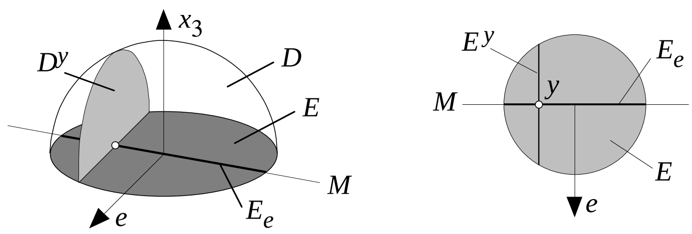

Fig. 4. The sets D, E, E_e, E^y and D^y.

For every $y \in E_e$ and every function u on D, u^y denotes the restriction of u on D^y, and for every function v on E, v^y denotes the restriction of v on E^y. If $u \in H^1(D)$, then for a.e. $y \in E_e$ the function u^y belongs to $H^1(D^y)$, the gradient of u^y agrees a.e. in D^y with the projection of Du on the plane spanned by the vector e and the axis x_3, and the trace of u^y on E^y agrees a.e. in E^y with $(Tu)^y$ (cf. Proposition 6.8). Taking into account these facts and Fubini's theorem, for every $\varepsilon > 0$ we get

$$\begin{aligned} F_\varepsilon(u, D, E) &\geq \varepsilon \int_D |Du|^2 + \lambda_\varepsilon \int_E V(Tu) \\ &\geq \int_{E_e} \Big[\varepsilon \int_{D^y} |Du^y|^2 + \lambda_\varepsilon \int_{E^y} V(Tu^y)\Big]\, dy \end{aligned}$$

We apply now the trace inequality (6.6) to each function u^y on the half-disk D^y, and then

$$\begin{aligned} F_\varepsilon(u, D, E) &\geq \int_{E_e} \Big[\frac{\varepsilon}{2\pi} \int_{(E^y)^2} \Big|\frac{Tu^y(x') - Tu^y(x)}{x' - x}\Big|\, dx'dx + \lambda_\varepsilon \int_{E^y} V(Tu^y)\Big]\, dy \\ &= \int_{E_e} G^2_\varepsilon(Tu^y, E^y)\, dy. \end{aligned} \tag{4.17}$$

Let us prove that the sequence (Tu_ε) is pre-compact in $L^1(E)$. To this end it suffices to show that the family $\mathcal{F} := \{Tu_\varepsilon\}$ satisfies the assumptions of Theorem

6.6 for every of the unit vector e. Thus we fix $\delta > 0$ and we choose a constant C such that

$$F_\varepsilon(u_\varepsilon, D, E) \le C, \tag{4.18}$$

for every ε we take $v_\varepsilon : E \to [-m, m]$ defined by

$$v_\varepsilon^y := \begin{cases} Tu_\varepsilon^y & \text{for all } y \in E_e \text{ s.t. } G_\varepsilon^2(Tu_\varepsilon^y, E^y) \le 2mrC/\delta, \\ \alpha' & \text{otherwise.} \end{cases} \tag{4.19}$$

By (4.17), (4.18) and (4.19) we have $v_\varepsilon^y = Tu_\varepsilon^y$ for all $y \in E_e$ apart a subset of measure smaller than $\delta/(2mr)$. Hence $v_\varepsilon = Tu_\varepsilon$ in E minus a set of measure smaller that δ/m and, since $|Tu_\varepsilon| \le m$, we deduce that $\|v_\varepsilon - Tu_\varepsilon\|_{L^1(E)} \le \delta$. Therefore the family $\mathcal{F}_\delta := \{v_\varepsilon\}$ is δ-dense in $\mathcal{F}$ in the sense of subsection 6.3; by (4.19), $G_\varepsilon^2(v_\varepsilon^y, E^y) \le 2mrC/\delta$ for every $y \in E_e$ and every ε, and hence statement (i) of Theorem 4.4 implies that the sequence (v_ε^y) is pre-compact in $L^1(E^y)$. Thus $\mathcal{F}$ satisfies condition (6.9) in Theorem 6.6 for every e, and then the sequence (Tu_ε) is pre-compact in $L^1(E)$.

It remains to prove that if $Tu_\varepsilon \to v$ in $L^1(E_r)$, then v belongs to $BV(E_r, I')$ and inequality (4.16) holds. replacing u by u_ε in (4.17) and passing to the limit as $\varepsilon \to 0$, by Fatou's lemma we deduce that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \ge \int_{E_e} \liminf_{\varepsilon\to 0} G_\varepsilon^2(Tu_\varepsilon^y, E^y)\, dy,$$

and then $\liminf G_\varepsilon^2(Tu_\varepsilon^y, E^y)$ is finite for a.e. $y \in E_e$.

Since $Tu_\varepsilon \to v$ in $L^1(E_r)$, possibly passing to a subsequence we have that $Tu_\varepsilon^y \to v^y$ in $L^1(E_r)$ for a.e. $y \in E_e$ (cf. Remark 6.7). Then statements (i) and (ii) of Theorem 4.4 yield $v^y \in BV(E^y, I')$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \ge \int_{E_e} \mathbf{c}\, \#(Sv^y)\, dy. \tag{4.20}$$

The right hand side of (4.20) is finite, and then Proposition 6.9 implies that v belongs to $BV(E, I')$, and that Sv^y agrees with $Sv \cap E^y$ for a.e. $y \in E_e$. By (6.13), we may rewrite (4.20) as

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, D, E) \ge \mathbf{c} \int_{D\cap Sv} \langle \nu_v, e\rangle. \tag{4.21}$$

Finally (4.16) follows from (4.21) by choosing a suitable unit vector e. □

4.2. Reduction to the Flat Case

The contribution of the wall to the limit energy Φ will be obtained by estimating the asymptotic behavior of $F_\varepsilon(u, B \cap \Omega, B \cap \partial\Omega)$ when B is a small ball centered on $\partial\Omega$. This estimate will be derived by Proposition 4.7, provided we can evaluate the error we make when we perturb $B \cap \Omega$ to get an half-ball. We expect of course that this error goes to zero with the radius of B and that it is controlled by the flatness of the boundary $\partial\Omega$, but making this argument precise requires some computations. We first describe the behavior of F_ε under change of variable.

Definition 4.8. Given two domains $A_1, A_2 \subset \mathbb{R}^3$ and a bi-Lipschitz homeomorphism $\Psi : \overline{A}_1 \to \overline{A}_2$, the *isometry defect* $\delta(\Psi)$ of Ψ is the smallest constant δ such that

$$\operatorname{dist}\big(D\Psi(x), O(3)\big) \le \delta \quad \text{for a.e. } x \in A_1. \tag{4.22}$$

Here $O(3)$ is the set of linear isometries on $\mathbb{R}^3$, and $D\Psi(x)$ is regarded as a linear mapping of $\mathbb{R}^3$ into $\mathbb{R}^3$; the distance between linear mappings is induced by the norm $\|\cdot\|$, which for every T is defined as the supremum of $|Tv|$ over all v such that $|v| \le 1$.

Given T and L such that L is an isometry, the inequality $\|T - L\| \le \delta$ with $\delta < 1$ implies that T is invertible and $\|T^{-1} - L^{-1}\| \le \delta\big/(1-\delta)$. Hence (4.22) implies that $\operatorname{dist}\big(D\Psi^{-1}(y), O(3)\big) \le \delta\big/(1-\delta)$ for a.e. $y \in A_2$, that is, $\delta(\Psi^{-1}) \le \delta(\Psi)\big/(1-\delta(\Psi))$.

Inequality (4.22) also implies that $\|D\Psi\| \le 1 + \delta$ a.e. in A_1, and then Ψ is $(1+\delta)$-Lipschitz continuous on every convex subset of A_1; similarly, Ψ^{-1} is $(1-\delta)^{-1}$-Lipschitz continuous on every convex subset of A_2.

Proposition 4.9. *Let be given A_1, A_2 and Ψ as above, and assume that Ψ maps a certain set $A'_1 \subset \partial A_1$ onto $A'_2 \subset \partial A_2$. Then for every $u \in H^1(A_2)$ there holds*

$$F_\varepsilon(u, A_2, A'_2) \ge \big(1 - \delta(\Psi)\big)^5 F_\varepsilon(u \circ \Psi, A_1, A'_1). \tag{4.23}$$

Proof. Let $\delta := \delta(\Psi)$ and assume that $\delta < 1$. By (4.22) we get $\|D\Psi\| \le 1 + \delta$ a.e. in A_1, and then

$$\big|D(u \circ \Psi)\big| \le (1+\delta)\,\big|(Du) \circ \Psi\big| \quad \text{a.e. in } A_1. \tag{4.24}$$

Let g and g' denote the inverse of Ψ and the restriction of the inverse of Ψ to the boundary of A_2. The maps g and g' are locally $(1-\delta)^{-1}$-Lipschitz continuous, and then the Jacobian determinants satisfy $|Jg| \le (1-\delta)^{-3}$ a.e. on A_2 and $|Jg'| \le (1-\delta)^{-3}$ a.e. on ∂A_2. Using these estimates and the inequality $(1-\delta)^{-1} \ge 1 + \delta$, we derive (4.23) from (4.2) by (4.24) and the usual change of variable formula. □

Proposition 4.10. *For every $x \in \partial\Omega$ and every positive r smaller than a certain critical value $r_x > 0$, there exists a bi-Lipschitz map $\Psi_r : \overline{D}_r \to \overline{\Omega \cap B_r(x)}$ such that*

(a) *Ψ_r takes D_r onto $\Omega \cap B_r(x)$ and E_r onto $\partial\Omega \cap B_r(x)$;*

(b) *Ψ_r is of class C^1 on D_r and $\|D\Psi_r - I\| \le \delta_r$ everywhere in D_r, where $\delta_r \to 0$ as $r \to 0$.*

(Here I denotes the identity map on $\mathbb{R}^3$). In particular the isometry defect of Ψ_r vanishes as $r \to 0$.

Proof. We assume that $x = 0$ and the tangent plane $T_x(\partial\Omega)$ agrees with the plane $\{x_3 = 0\}$, and we write B_r for $B_r(x)$. For every positive $\gamma < 1$ and for every r sufficiently small, we construct a map Ψ which fulfills (a) and $\|D\Psi - I\| \le O(\gamma)$.

Since $\partial\Omega$ is of class C^1, for r sufficiently small we reduce to the situation described in Figure 5 below:

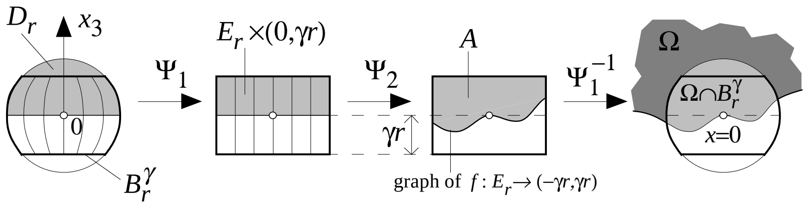

Fig. 5. Construction of $\Psi := \Psi_1^{-1} \circ \Psi_2 \circ \Psi_1$.

Here B_r^γ is the set of all $x \in B_r$ such that $-\gamma r < x_3 < \gamma r$; the map Ψ_1 which takes $D_r \cap B_r^\gamma$ into the cylinder $E_r \times (0, \gamma r)$ is given by

$$\Psi_1(x_1, x_2, x_3) := \Big(\frac{x_1}{\sqrt{1 - (x_3/r)^2}}, \, \frac{x_2}{\sqrt{1 - (x_3/r)^2}}, \, x_3 \Big).$$

For r small enough, Ψ_1 takes the set $\Omega \cap B_r^\gamma$ into a set of the form $A := \big\{ x \in \mathbb{R}^3$ s.t. $(x_1, x_2) \in E_r$ and $f(x_1, x_2) < x_3 < \gamma r \big\}$, where f is a suitable real function of class C^1 on E_r and satisfies $f(0) = 0$, $Df(0) = 0$, $|Df| \le \gamma^2$ on E_r. The map Ψ_2 which takes the cylinder $E_r \times (0, \gamma r)$ into A is given by

$$\Psi_2(x_1, x_2, x_3) := \big(x_1, \, x_2, \, x_3 + \big(1 - \big(\tfrac{x_3}{\gamma r}\big)^2\big) f(x_1, x_2) \big).$$

If I denote the identity map on $\mathbb{R}^3$, then $\|D\Psi_1 - I\| = O(\gamma^2)$ and $\|D\Psi_2 - I\| = O(\gamma)$; therefore the map $\Psi := \Psi_1^{-1} \circ \Psi_2 \circ \Psi_1$ (extended to the identity in $D_r \setminus B_r^\gamma$) satisfies (a), is of class C^1 on D_r and $\|D\Psi - I\| = O(\gamma)$. $\square$

4.3. Proof of Theorem 2.6, Part I

Proof of statement (i). Let be given a countable sequence $(u_\varepsilon) \subset H^1(\Omega)$ such that $F_\varepsilon(u_\varepsilon)$ is bounded in ε. Since $F_\varepsilon(u_\varepsilon) \ge G_\varepsilon^1(u_\varepsilon, \Omega)$ (see (2.12) and (4.3)), the sequence (u_ε) is pre-compact in $L^1(\Omega)$ by statement (i) of Theorem 4.2.

We have to prove that the sequence of the traces (Tu_ε) is pre-compact in $L^1(\partial\Omega)$. In view of Proposition 4.10 we can cover $\partial\Omega$ with finitely many balls B^i centered on $\partial\Omega$ so that $\Omega \cap B^i$ is the image of an half-ball under a map Ψ^i with isometry defect smaller than 1. Hence it suffices to show that the sequence (Tu_ε) is pre-compact in $L^1(\partial\Omega \cap B^i)$ for every i.

For every fixed i, let $\bar{u}_\varepsilon := u_\varepsilon \circ \Psi^i$. Since the isometry defect of Ψ^i is smaller than 1, Proposition 4.9 implies that $F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r)$ is bounded. Hence the pre-compactness of the traces Tu_ε in $L^1(\partial\Omega \cap B^i)$ is implied by the pre-compactness of the traces $T\bar{u}_\varepsilon$ in $L^1(E_r)$, which in turn follows from Proposition 4.7. $\square$

Proof of statement (ii). Let be given a sequence $(u_\varepsilon) \subset H^1(\Omega)$ such that $u_\varepsilon \to u \in BV(\Omega, I)$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v \in BV(\partial\Omega, I')$ in $L^1(\partial\Omega)$. We have to show that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v). \tag{4.25}$$

Clearly we can assume that the liminf at the left hand side of (4.25) is finite.

For every $\varepsilon > 0$ let μ_ε be the energy distribution associated with the configuration u_ε, that is, the positive measure which for every Borel set $B \subset \mathbb{R}^3$ is given by

$$\mu_\varepsilon(B) := \varepsilon \int_{\Omega\cap B} |Du_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\Omega\cap B} W(u_\varepsilon) + \lambda_\varepsilon \int_{\partial\Omega\cap B} V(Tu_\varepsilon). \tag{4.26}$$

Then the total mass $\|\mu_\varepsilon\|$ of the measure μ_ε is equal to $F_\varepsilon(u_\varepsilon)$, and possibly passing to a subsequence we can assume that $\|\mu_\varepsilon\|$ is bounded and that μ_ε converges in the sense of measures to some finite measure μ on $\mathbb{R}^3$.

We also associate to each of the three terms in (2.13) which give $\Phi(u,v)$ the energy distributions μ^1, μ^2 and μ^3 defined by

$$\mu^1(B) := \boldsymbol{\sigma}\mathscr{H}^2(Su\cap B), \quad \mu^2(B) := \int_{\partial\Omega\cap B} \big|H(Tu) - H(v)\big|, \quad \mu^3(B) := \mathbf{c}\,\mathscr{H}^1(Sv\cap B).$$

Thus $\Phi(u,v)$ is equal to $\|\mu^1\| + \|\mu^2\| + \|\mu^3\|$, and since the measures μ^i are mutually singular and $\liminf F_\varepsilon(u_\varepsilon) = \liminf \|\mu_\varepsilon\| \geq \|\mu\|$, inequality (4.25) follows from

$$\mu \geq \mu^i \quad \text{for } i = 1,2,3. \tag{4.27}$$

We prove that $\mu \geq \mu^1$ by showing that $\mu(B) \geq \mu^1(B)$ for all sets $B \subset \mathbb{R}^3$ such that $B \cap \Omega$ is a Lipschitz domain and $\mu(\partial B) = 0$ (one readily checks that this class is large enough to imply the inequality $\mu(B) \geq \mu^1(B)$ for all Borel sets B). Indeed for such a B there holds

$$\mu(B) = \lim_{\varepsilon\to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, B\cap\Omega) \geq \boldsymbol{\sigma}\mathscr{H}^2(Su\cap B) = \mu^1(B),$$

where the first equality follows from the assumption $\mu(\partial B) = 0$, the first inequality follows from (4.26) and (4.3), and the second one by statement (ii) of Theorem 4.2 with $A := B \cap \Omega$.

We prove that $\mu \geq \mu^2$ in the same way: taken B as before we get

$$\mu(B) \geq \liminf_{\varepsilon\to 0} G^1_\varepsilon(u_\varepsilon, B\cap\Omega) \geq \int_{\partial\Omega\cap B} \big|H(Tu) - H(v)\big| = \mu^2(B)$$

(apply statement (ii) of Proposition 4.3 with $A := B \cap \Omega$ and $A' := B \cap \partial\Omega$).

The inequality $\mu \geq \mu^3$ requires a different argument. Since μ^3 is the restriction of $\mathscr{H}^1$ to the rectifiable set Sv multiplied by the factor $\mathbf{c}$, the following density estimate will suffice:

$$\lim_{r\to 0} \frac{\mu(B_r(x))}{2r} \geq \mathbf{c} \quad \text{for } \mathscr{H}^1\text{-a.e. } x \in Sv. \tag{4.28}$$

The limit at the left hand side of (4.28) is the one-dimensional density of the measure μ at x. Since Sv is one-rectifiable, this density exists and agrees with the Radon-Nikodym derivative of the measure μ with respect to μ^3 for $\mathscr{H}^1$-a.e. $x \in Sv$.

In fact we prove (4.28) for every point x in Sv such that the limit at left hand side of (4.28) exists, the set Sv has 1-dimensional density equal to one, and the unit normal ν_v is approximately continuous at x (notice that these three conditions are verified for $\mathscr{H}^1$-a.e. $x \in Sv$).

We fix such a point x. For r sufficiently small we choose a map Ψ_r as in Proposition 4.10; we assume moreover that $\mu(\partial B_r(x)) = 0$ (this condition is verified by all r but countably many).

We set $\bar{u}_\varepsilon := u_\varepsilon \circ \Psi_r$ and $\bar{v} := v \circ \Psi_r$. Hence $T\bar{u}_\varepsilon \to \bar{v}$ in $L^1(E_r)$, $v \in BV(E_r, I')$ and

$$\begin{aligned}
\mu(B_r(x)) = \lim_{\varepsilon\to 0} \mu_\varepsilon(B_r(x)) &= \lim_{\varepsilon\to 0} F_\varepsilon\big(u_\varepsilon, \Omega\cap B_r(x), \partial\Omega \cap B_r(x)\big) \\
&\geq \big(1-\delta(\Psi_r)\big)^5 \liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r) \\
&\geq \big(1-\delta(\Psi_r)\big)^5 \,\mathbf{c}\,\Big|\int_{S\bar{v}\cap E_r} \nu_{\bar{v}}\Big|, \qquad (4.29)
\end{aligned}$$

where the first inequality follows from (4.23) and the second from (4.16).

Notice that $Sv \cap B_r(x) = \Psi_r(S\bar{v} \cap E_r)$, and $\nu_v\big(\Psi_r(y)\big) = D\Psi_r(y)\cdot \nu_{\bar{v}}(y)$ for a.e. $y \in S\bar{v}$; taking into account that $\|D\Psi_r - I\| \leq \delta_r$ and $\delta(\Psi_r) \leq \delta_r$ where δ_r vanishes as $r \to 0$ (cf. Proposition 4.10), and the choice of the point x, one can easily prove that

$$\Big|\int_{S\bar{v}\cap E_r} \nu_{\bar{v}}\Big| = 2r + o(r). \qquad (4.30)$$

Inequality (4.28) follows from (4.29) and (4.30). □

4.4. Proof of Theorem 2.6, Part II

We need the following extension result.

Lemma 4.11. *Let a domain $A \subset \mathbb{R}^3$ be given, and take $\varepsilon \in]0,1]$, a set $A' \subset \partial A$, and a Lipschitz function $v : A' \to [-m, m]$, where m is given in (4.1). Then v admits an extension $u : \overline{A} \to [-m, m]$ such that* $\mathrm{Lip}(u) \leq 1/\varepsilon + \mathrm{Lip}(v)$ *and*

$$G^1_\varepsilon(u, A) \leq \big((\varepsilon \mathrm{Lip}(v) + 1)^2 + C\big)\,\big(|\partial A| + o(1)\big)\,\rho, \qquad (4.31)$$

where C is the supremum of W on the interval $[-m, m]$, the error $o(1)$ is a function of ε which depends only on the choice of A (and not on v), and ρ is the infimum between $\|v-\alpha\|_\infty$ and $\|v-\beta\|_\infty$.

Proof. Since we can extend v to the rest of ∂A without increasing its Lipschitz constant, we assume from the beginning that $A' = \partial A$. We also assume that $\rho := \|v-\alpha\|_\infty$ (the other case is similar).

Let U_t be the set of all $x \in A$ such that $0 < \operatorname{dist}(x, \partial A) < t$. We set $u := v$ on ∂A and $u = \alpha$ on $A \setminus U_{\varepsilon\rho}$. Then u is $(\operatorname{Lip}(v) + 1/\varepsilon)$-Lipschitz continuous on $\overline{A} \setminus U_{\rho\varepsilon}$, and we extend it to the rest of $\overline{A}$ without increasing its Lipschitz constant. Then

$$G^1_\varepsilon(u, A) = \int_{U_{\varepsilon\rho}} \varepsilon |Du|^2 + \tfrac{1}{\varepsilon} W(u) \leq \big[(\varepsilon \operatorname{Lip}(v) + 1)^2 + C\big] \frac{|U_{\varepsilon\rho}|}{\varepsilon}. \tag{4.32}$$

Finally it is enough to notice that $|U_t| = t\,|\partial A| + o(t)$ because ∂A is Lipschitz. □

Statement (iii) of Theorem 2.6 is a direct corollary of the following approximation result.

Lemma 4.12. *Let be given $u \in BV(\Omega, I)$ and $v \in BV(\partial\Omega, I')$ so that Su and Sv are closed manifolds of class C^2 without boundary respectively in Ω and $\partial\Omega$. Then for every $\eta > 0$ and every $\varepsilon > 0$ we can find $u_\varepsilon \in H^1(\Omega)$ such that*

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^1(\Omega)} \leq \eta, \quad \limsup_{\varepsilon\to 0} \|Tu_\varepsilon - v\|_{L^1(\partial\Omega)} \leq \eta, \tag{4.33}$$

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \boldsymbol{\sigma}\mathscr{H}^2(Su) + \int_{\partial\Omega} \big|H(Tu) - H(v)\big| + \mathbf{c}\,\mathscr{H}^1(Sv) + \eta. \tag{4.34}$$

Proof. Possibly modifying u and v in a negligible subset we can assume that they are constant in each connected component of $\Omega \setminus Su$ and $\partial\Omega \setminus Sv$ respectively.

Let us fix some notation. All the functions we consider in this proof will take values in $[-m, m]$, where m is given in (4.1). We fix a constant $C > 2m$ which is larger than the constant C in Lemma 4.11, and of the suprema of $\sqrt{W}$, W and V on the interval $[-m, m]$. In particular C is larger than the constants in statement (iii) of Theorem 4.2 and statement (ii) of Proposition 4.3. For every $x \in \Omega$ we set $d(x) := \operatorname{dist}(x, \partial\Omega)$, while $d' : \partial\Omega \to \mathbb{R}$ is the oriented distance from Sv defined by

$$d'(x) := \begin{cases} \operatorname{dist}(x, Sv) & \text{if } x \in \{v = \beta'\}, \\[2ex] -\operatorname{dist}(x, Sv) & \text{if } x \in \{v = \alpha'\}. \end{cases}$$

Since Sv is a boundary in $\partial\Omega$, the intersection of a tubular neighborhood of Sv and Ω is diffeomorphic to the product of Sv and an half-disk. Such diffeomorphism Ψ can be constructed as follows: given $x \in \overline{\Omega}$, let x' be the projection of x on $\partial\Omega$, let x'' be the projection of x' on Sv, and set $\Psi(x) := \big(x'', d'(x'), d(x)\big) \in Sv \times \mathbb{R} \times [0, +\infty[$. Using the tubular neighborhood theorem one can show that the map Ψ is well-defined and is a diffeomorphism of class C^2 on $\overline{\Omega} \cap U$ for some neighborhood U of Sv (here we use the fact that $\partial\Omega$ and Sv are of class C^2). Moreover Ψ takes $\Omega \cap U$ into $Sv \times \mathbb{R} \times]0, +\infty[$ and $\partial\Omega \cap U$ into $Sv \times \mathbb{R} \times \{0\}$, and for every $x \in \partial\Omega$, $\Psi(x) = x$ and $D\Psi(x)$ is an isometry.

Let A_r be the set of all $x \in \Omega$ such that $\operatorname{dist}(x, Sv) < r$. Since $D\Psi$ is continuous, we deduce that the isometry defect δ_r of the restriction of Ψ to the A_r satisfies

$$\lim_{r\to 0} \delta_r = 0. \tag{4.35}$$

We denote by $\mathscr{S}_r$ the set of all $x \in \Omega$ such that $d(x) = r$, and we fix now $r > 0$ so that

(a) $\mathscr{S}_r$ and $\mathscr{S}_{2r}$ are Lipschitz surfaces,

(b) $Su \cap \mathscr{S}_r$ is a Lipschitz curve (not necessarily connected),

(c) the set A_r is included in U.

Notice that (a) and (c) are verified by every r sufficiently small, while (b) is verified by a.e. r sufficiently small (apply Sard's theorem to the function d on the surface Su). We construct now a partition of Ω:

$$\begin{aligned}
B_1 &:= \big\{x \in \Omega : \operatorname{dist}\big(x, Sv \cup (Su \cap \mathscr{S}_r)\big) < 3r\big\},\\
A_1 &:= \big\{x \in \Omega \setminus \overline{B}_1 : d(x) < r\big\},\\
B_2 &:= \big\{x \in \Omega \setminus \overline{B}_1 : r < d(x) < 2r\big\},\\
A_2 &:= \big\{x \in \Omega \setminus \overline{B}_1 : 2r < d(x)\big\},
\end{aligned}$$

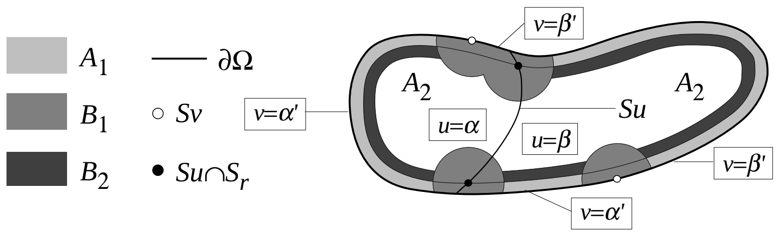

Fig. 6. The partition of Ω.

For every $\varepsilon < r$ we construct the Lipschitz function u_ε in four steps: first we define it on A_2 and A_1, and then on B_2 and B_1.

On the set A_2 we take u_ε as in the second part of statement (iii) of Theorem 4.2 (with A replaced by A_2) and we extend it to ∂A_2 by continuity. Hence u_ε is (C/ε)-Lipschitz continuous on $\overline{A}_2$, it converges to u pointwise on A_2 and uniformly on $\partial A_2 \cap \partial B_2$, and

$$F_\varepsilon(u_\varepsilon, A_2, \emptyset) = G^1_\varepsilon(u_\varepsilon, A_2) \le \boldsymbol{\sigma}\mathscr{H}^2(Su \cap A_2) + o(1) \le \boldsymbol{\sigma}\mathscr{H}^2(Su) + o(1). \quad (4.36)$$

The function u is constant (equal to α or β) on every connected component A of A_1, while v is constant (equal to α' or β') on $\partial A \cap \partial\Omega$ (cf. Figure 6); thus we take u_ε as in statement (ii) of Proposition 4.3 (with A' replaced by $\partial A \cap \partial\Omega$) and we extend it to ∂A_1 by continuity. Since the distance between two different connected components of A_1 is larger than r and $C/\varepsilon > 2m/r$, then u_ε is (C/ε)-Lipschitz continuous on $\overline{A}_1$ and agrees with v on $\partial A_1 \cap \partial\Omega$. Moreover it converges to u pointwise on A_1 and uniformly on $\partial A_1 \cap \partial B_2$, and satisfies

$$F_\varepsilon(u_\varepsilon, A_1, \partial A_1 \cap \partial\Omega) = G^1_\varepsilon(u_\varepsilon, A_1) \le \int_{\partial\Omega} \big|H(Tu_\varepsilon) - H(v)\big| + o(1). \quad (4.37)$$

Since the distance between A_1 and A_2 is equal to r and $C/\varepsilon \geq 2m/r$, it follows that u_ε is (C/ε)-Lipschitz continuous also on $\overline{A}_1 \cup \overline{A}_2$. Then we can apply Lemma 4.11 to each connected component B of B_2 to extend the function u_ε, which is defined only on $(\partial A_1 \cup \partial A_2) \cap \partial B$, to the rest of B; since u is constant (equal to α or β) on each connected component of B_1, if we denote by ρ_ε the infimum of $|u_\varepsilon - u|$ on $(\partial A_1 \cup \partial A_2) \cap \partial B_1$, then $\rho_\varepsilon = o(1)$ and (4.31) yields

$$F_\varepsilon(u_\varepsilon, B_2, \emptyset) = G^1_\varepsilon(u_\varepsilon, B_2) \leq \big((1+C)^2 + C\big)\big(|\partial B_2| + o(1)\big)\,\rho_\varepsilon = o(1). \tag{4.38}$$

Moreover u_ε is $(2C/\varepsilon)$-Lipschitz continuous on $\overline{B}_2$.

It remains to construct u_ε on B_1. This last step is slightly more elaborated than the previous ones. First of all we define a function $w_\varepsilon : \mathbb{R} \times [0, +\infty[\to [-m, m]$ as follows: in polar coordinates $\theta \in [0, \pi]$, $\rho \in [0, +\infty]$,

$$w_\varepsilon(\theta, \rho) := \begin{cases} (\rho\lambda_\varepsilon/\varepsilon)\,(\theta\alpha' + (1-\theta)\beta') + (1 - \rho\lambda_\varepsilon/\varepsilon)\,\frac{\alpha'+\beta'}{2} & \text{if } 0 \leq \rho < \varepsilon/\lambda_\varepsilon, \\[2ex] \theta\alpha' + (1-\theta)\beta' & \text{if } \varepsilon/\lambda_\varepsilon \leq \rho. \end{cases}$$

For every $t > 0$ let D_t be the half-disk of all $y \in \mathbb{R}\times]0, +\infty[$ such that $|y| < t$, and let E_t be the segment of all $y \in \mathbb{R} \times \{0\}$ such that $|y| < t$. A direct computation gives

$$\int_{D_1} |Dw_\varepsilon|^2 = \frac{(\beta' - \alpha')^2}{\pi} \log(\lambda_\varepsilon/\varepsilon) + O(1). \tag{4.39}$$

We set $\bar{w}_\varepsilon(x, y) := w_\varepsilon(y)$ for every $x \in Sv$ and $y \in \mathbb{R} \times [0, +\infty[$, and using (4.39) we obtain

$$\begin{aligned} F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{2\varepsilon}, Sv \times E_{2\varepsilon}) &= \\ &= \mathscr{H}^1(Sv) \cdot \Big[\varepsilon \int_{D_{2\varepsilon}} |Dw_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{D_{2\varepsilon}} W(w_\varepsilon) + \lambda_\varepsilon \int_{E_{\varepsilon/\lambda_\varepsilon}} V(w_\varepsilon)\Big] \\ &\leq \mathscr{H}^1(Sv) \cdot \Big[\frac{(\beta' - \alpha')^2}{\pi} \varepsilon \log(\lambda_\varepsilon/\varepsilon) + 2\pi\varepsilon C + 2\varepsilon C\Big] \\ &= \mathbf{c}\, \mathscr{H}^1(Sv) + o(1). \end{aligned} \tag{4.40}$$

We define u_ε on the set $\overline{A}_\varepsilon$ by $u_\varepsilon := \bar{w}_\varepsilon \circ \Psi$, where Ψ and A_ε are given at the beginning of this proof. Since the isometry defect of Ψ on A_ε tends to 0 as $\varepsilon \to 0$ (cf. (4.35)), for ε small enough the function Ψ is 2-Lipschitz continuous (see Definition 4.8 and subsequent remarks), and then Ψ takes A_ε into $Sv \times D_{2\varepsilon}$ and $\partial A_\varepsilon \cap \partial\Omega$ into $Sv \times E_{2\varepsilon}$. Then (4.40) and Proposition 4.9 yield

$$\begin{aligned} (1 - \delta_\varepsilon)^5 F_\varepsilon(u_\varepsilon, A_\varepsilon, \partial A_\varepsilon \cap \partial\Omega) &\leq F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{2\varepsilon}, Sv \times E_{2\varepsilon}) \\ &\leq \mathbf{c}\, \mathscr{H}^1(Sv) + o(1). \end{aligned} \tag{4.41}$$

We extend u_ε by setting $u_\varepsilon := v$ in the rest of $\partial B_1 \cap \partial\Omega$; then $u_\varepsilon = v$ on $\partial\Omega \setminus \partial A_\varepsilon$. Now u_ε is defined on the whole boundary of $B_1 \setminus \overline{A}_\varepsilon$ and is $(2C/\varepsilon)$-Lipschitz continuous. Hence we can use Lemma 4.11 to extend u_ε to $B_1 \setminus \overline{A}_\varepsilon$, and inequality (4.32) yields

$$\begin{aligned} F_\varepsilon\big(u_\varepsilon, B_1 \setminus \overline{A}_\varepsilon, \partial(B_1 \setminus \overline{A}_\varepsilon) \cap \partial\Omega\big) &= G^1_\varepsilon\big(u_\varepsilon, B_1 \setminus \overline{A}_\varepsilon\big) \\ &\leq \big((1+2C)^2 + C\big)\frac{|U_{\varepsilon\rho}|}{\varepsilon}, \end{aligned} \tag{4.42}$$

where ρ is the infimum of $\|u_\varepsilon - \alpha\|$ and $\|u_\varepsilon - \beta\|$, and $U_{\varepsilon\rho}$ is the set of all $x \in B_1 \setminus \overline{A}_\varepsilon$ such that $\operatorname{dist}(x, \partial(B_1 \setminus \overline{A}_\varepsilon)) \leq \rho\varepsilon$. Since $\rho \leq 2m$ and $|U_{\varepsilon\rho}| = \varepsilon\rho\,|\partial B_1| + o(\varepsilon\rho)$, (4.42) becomes

$$F_\varepsilon\big(u_\varepsilon, B_1 \setminus \overline{A}_\varepsilon, \partial(B_1 \setminus \overline{A}_\varepsilon) \cap \partial\Omega\big) \leq C'|\partial B_1| + o(1), \tag{4.43}$$

where $C' := (1+2C)^2 + C$.

The function u_ε is now defined on the whole of $\overline{\Omega}$ and is Lipschitz continuous. Putting together inequalities (4.36), (4.37), (4.38), (4.41) and (4.43) we finally obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \boldsymbol{\sigma}\mathscr{H}^2(Su) + \int_{\partial\Omega}\big|H(Tu) - H(v)\big| \; + \mathbf{c}\,\mathscr{H}^1(Sv) + C'|\partial B_1|. \tag{4.44}$$

Moreover $u_\varepsilon \to u$ pointwise on A_1 and A_2, and $u_\varepsilon = v$ on $\partial\Omega \setminus \partial A_\varepsilon$, and then

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^1(\Omega)} \leq 2m\big(|B_1| + |B_2|\big) \quad \text{and} \quad \lim_{\varepsilon\to 0} \|Tu_\varepsilon - v\|_{L^1(\partial\Omega)} = 0. \tag{4.45}$$

Notice that $|\partial B_1|$, $|B_1|$ and $|B_2|$ have order r, r^2 and r respectively, then taking r small enough we deduce (4.34) and (4.33) from (4.44) and (4.45). □

Proof of statement (iii) of Theorem 2.6. We first remark that every pair $(u,v) \in BV(\Omega, I) \times BV(\partial\Omega, I')$ can be approximated in $L^1(\Omega) \times L^1(\partial\Omega)$ by a sequence (u_n, v_n) which fulfill the regularity assumptions of Lemma 4.12 and satisfies $\mathscr{H}^2(Su_n) \to \mathscr{H}^2(Su)$ and $\mathscr{H}^1(Sv_n) \to \mathscr{H}^1(Sv)$ (see for instance [Gi, Theorem 1.24]). Therefore $\Phi(u_n, v_n) \to \Phi(u, v)$. To conclude the proof of statement (iii) of Theorem 2.6, we just need to apply Lemma 4.12 to each pair (u_n, v_n) and then apply a suitable diagonal argument. □

5. Application to Capillary Equilibrium with Line Tension

In this section we describe some mechanical features of the equilibrium configurations associated with the relaxation $\overline{F}_0$ of the energy F_0 (see subsection 2.2) or with the limit energy F obtained in subsection 2.3. We follow the notation of subsections 2.2 and 2.4.

We recall that $\overline{F}_0$ and F are given in term of Φ_0 and Φ (see (2.6), (2.16)), which in turn can be viewed as special cases of the more general energy Φ_{gen} given in (2.19) (following subsection 2.4, here we view F as a function of $\mathsf{A} \in X$ instead of $u \in BV(\Omega, I)$, and Φ as a function of $(\mathsf{A}, \mathsf{A}') \in X \times X'$ instead of $(u, v) \in BV(\Omega, I) \times BV(\partial\Omega, I'))$.

The functional Φ_{gen} depends on the bulk phase A and the boundary phase A' (which determine respectively the other bulk phase B and the other boundary phase B'). A configuration $\mathsf{A} \in X$ is at equilibrium with respect to $\overline{F}_0$ (resp. F) under the volume constraint $|\mathsf{A}| = v$ if and only if there exists $\mathsf{A}' \in X'$ such that $(\mathsf{A}, \mathsf{A}')$ is an equilibrium configuration for Φ_0 (resp. Φ).

In subsection 5.1 we briefly describe the equilibrium conditions for a configuration $(\mathsf{A}, \mathsf{A}')$ with respect to the energy Φ_{gen}; in particular we notice that, at equilibrium, the contact angle ϕ satisfies a different condition than the usual Young's law (prescribed by the capillary energy E_0 in (1.1)). This modification depends heavily whether the contact line $\mathscr{L}_{\mathrm{c}}$ and the dividing line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ coincide or not. In subsection 5.2 we exhibit examples where $\mathscr{L}_{\mathrm{c}}$ and $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ coincide and examples where they do not.

5.1. Equilibrium Conditions for the Energy Φ_{gen}

The general model Φ_{gen} is characterized by the coefficients σ_{AB}, $\sigma_{\mathsf{AA}'}$, $\sigma_{\mathsf{AB}'}$, $\sigma_{\mathsf{BA}'}$, $\sigma_{\mathsf{BB}'}$ and c; we assume that the generalized wetting conditions (2.24) are verified. A configuration $(\mathsf{A}, \mathsf{A}')$ is in equilibrium if it minimizes Φ_{gen} under the volume constraint $|\mathsf{A}| = v$ for some v such that $0 < v < |\Omega|$, that is, if it solves the problem

$$\min_{\substack{\mathsf{A}\in X, |\mathsf{A}|=v \\ \mathsf{A}'\in X'}} \Big\{ \sigma_{\mathsf{AB}}|\mathscr{S}_{\mathsf{AB}}| + \sigma_{\mathsf{AA}'}|\mathscr{S}_{\mathsf{AA}'}| + \sigma_{\mathsf{AB}'}|\mathscr{S}_{\mathsf{AB}'}| + \\ + \sigma_{\mathsf{BA}'}|\mathscr{S}_{\mathsf{BA}'}| + \sigma_{\mathsf{BB}'}|\mathscr{S}_{\mathsf{BB}'}| + \mathsf{c}|\mathscr{L}_{\mathsf{A}'\mathsf{B}'}| \Big\}. \tag{5.1}$$

We just recall here that since Φ_{gen} is lower semicontinuous on $X \times X'$ (Theorem 2.9), the minimum problem (5.1) admits a solution $(\mathsf{A}, \mathsf{A}')$ where A and A' have finite perimeter (respectively in Ω and $\partial\Omega$). By standard regularity results for sets with minimal perimeter in dimension 3 and 2 (see for instance [Ta, Amb2]), the essential boundary of A in Ω, that is, the interface $\mathscr{S}_{\mathsf{AB}}$, is a closed analytic surface with constant mean curvature, while the essential boundary of A' in $\partial\Omega$, that is $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$, is a closed curve of class $C^{1,1}$.

For the rest of this section we assume that all the objects we consider are sufficiently smooth, and all statements are given without rigorous proofs. Let us recall the geometrical parameters of the problem. Given a configuration $(\mathsf{A}, \mathsf{A}')$, the *contact line* $\mathscr{L}_{\mathrm{c}}$ is the curve determined by the intersection of the interface $\mathscr{S}_{\mathsf{AB}} = \partial\mathsf{A}$ with the boundary of Ω, the *contact angle* θ is defined at every point of $\mathscr{L}_{\mathrm{c}}$ as the angle between the outward normal ν_Ω to $\partial\Omega$ and the outward normal to $\mathscr{S}_{\mathsf{AB}}$ (viewed as a part of the boundary of A); the *dividing line* $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ is the boundary of A', and at every point $x \in \mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ we denote by $\mathcal{K}_g(x)$ the scalar product of the outward co-normal versor of $\partial A'$ (denoted $\nu_{A'}$) by the mean curvature vector of $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$; in other words the real number $\mathcal{K}_g(x)$ represents the signed geodesic curvature of $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$, oriented by the tangent vector t so that the Darboux system $(t, \nu_{A'}, \nu_\Omega)$ is direct.

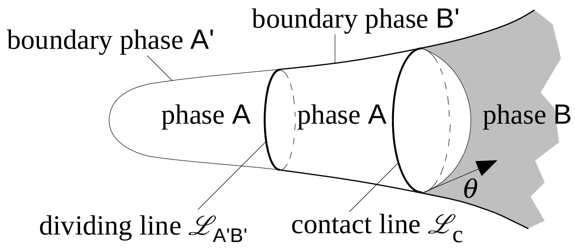

Fig. 7. An example of equilibrium configuration

We define now the angles $\theta_1, \theta_2 \in [0, \pi]$, the dimensionless parameter τ and the characteristic length ℓ as follows:

$$\begin{aligned} \cos(\theta_1) &:= \frac{\sigma_{\mathsf{AA'}} - \sigma_{\mathsf{BA'}}}{\sigma_{\mathsf{AB}}}, \qquad & \cos(\theta_2) &:= \frac{\sigma_{\mathsf{AB'}} - \sigma_{\mathsf{BB'}}}{\sigma_{\mathsf{AB}}}, \\ \tau &:= \frac{\sigma_{\mathsf{BB'}} + \sigma_{\mathsf{AB'}} - \sigma_{\mathsf{AA'}} - \sigma_{\mathsf{BA'}}}{\sigma_{\mathsf{AB}}}, & \ell &:= \frac{\mathrm{c}}{\sigma_{\mathsf{AB}}}. \end{aligned} \tag{5.2}$$

In the following we assume that $\theta_1 \geq \theta_2$, the other case being similar.

Let $(\mathsf{A}, \mathsf{A}')$ be an equilibrium configuration for Φ_{gen}, that is, a solution of (5.1). Then the mean curvature of the interface $\mathscr{S}_{\mathsf{AB}}$ is constant, moreover we can derive some equilibrium conditions for $\mathcal{K}_g$ and θ. More precisely, the contact angle θ verifies

$$\begin{aligned} &\theta = \begin{cases} \theta_1 & \text{on } (\mathscr{L}_{\mathrm{c}} \setminus \mathscr{L}_{\mathsf{A'B'}}) \cap \mathsf{A}', \\ \theta_2 & \text{on } (\mathscr{L}_{\mathrm{c}} \setminus \mathscr{L}_{\mathsf{A'B'}}) \cap \mathsf{B}', \end{cases} \\ &\theta \in [\theta_2, \theta_1] \quad \text{on } \mathscr{L}_{\mathrm{c}} \cap \mathscr{L}_{\mathsf{A'B'}}, \end{aligned} \tag{5.3}$$

while the geodesic curvature $\mathcal{K}_g$ verifies

$$-2\ell\, \mathcal{K}_g = \begin{cases} \tau - \cos\theta_1 + \cos\theta_2 & \text{on } (\mathscr{L}_{\mathsf{A'B'}} \setminus \mathscr{L}_{\mathrm{c}}) \cap \mathscr{S}_{\mathsf{AW}}, \\ \tau + \cos\theta_1 - \cos\theta_2 & \text{on } (\mathscr{L}_{\mathsf{A'B'}} \setminus \mathscr{L}_{\mathrm{c}}) \cap \mathscr{S}_{\mathsf{BW}}, \\ \tau - \cos\theta_1 - \cos\theta_2 + 2\cos\theta & \text{on } \mathscr{L}_{\mathsf{A'B'}} \cap \mathscr{L}_{\mathrm{c}} \end{cases} \tag{5.4}$$

(we do not precise here in which weak sense the curvature must be intended; clearly (5.3) and (5.4) will hold in the classical sense up to few exceptional points).

Both equilibrium conditions (5.3) and (5.4) can be easily interpreted in term of forces. Notice that the first two lines in (5.3) are a restatement of Young's law (cf. (1.3) and (5.2)), while the first two lines in (5.4) are the usual constant mean curvature condition for the minimizers of functionals of the type $a|\partial E| + b|E| \quad E \subset \partial\Omega$ with $E \subset \partial\Omega$, that is a Young's law on the manifold $\partial\Omega$. In the intersection of $\mathscr{L}_{\mathrm{c}}$ and $\mathscr{L}_{\mathsf{A'B'}}$, the balance between forces due to surface tension, line tension and boundary adhesion leads to the relation between θ and $\mathcal{K}_g$ stated in the last line of (5.4).

We remark that the dividing line $\mathscr{L}_{\mathsf{A'B'}}$ may be empty, namely when A' or B' is empty; in this case condition (5.4) disappears and eventually (5.3) reduces to the usual Young's law $\theta = \theta_2$ or $\theta = \theta_1$ on $\mathscr{L}_{\mathrm{c}}$, respectively.

Both (5.3) and (5.4) depends only on the four independent parameters θ_1, θ_2, τ and ℓ which determine the equilibrium configurations of $\Phi_{\rm gen}$. This is in accordance with subsection 2.4, where we claimed that the model associated with $\Phi_{\rm gen}$ has indeed four degrees of freedom.

5.2. An Example: A Bubble Growing in a Cylinder

In this subsection we give some explicit examples of equilibrium configurations. We restrict our attention to the particular case

$$\sigma_{\mathsf{AA'}} = \sigma_{\mathsf{BB'}} = 0, \quad \sigma_{\mathsf{AB'}} = \sigma_{\mathsf{BA'}} = \sigma_{\mathsf{AB}} = \sigma. \tag{5.5}$$

Then $\theta_1 = \pi$, $\theta_2 = 0$, $\tau = 0$, and the only free parameter left is $\ell := \mathrm{c}/\sigma$ (cf. (5.2)). The expression of $\Phi_{\rm gen}$ becomes

$$\Phi_{\rm gen}(\mathsf{A},\mathsf{A'}) = \sigma\big(|\mathscr{S}_{\mathsf{AB}}| + |\mathscr{S}_{\mathsf{AB'}}| + |\mathscr{S}_{\mathsf{BA'}}| + \ell|\mathscr{L}_{\mathsf{A'B'}}|\big). \tag{5.6}$$

Notice that (5.5) implies (2.22), and therefore $\Phi_{\rm gen}$ is a particular case of Φ_0 (corresponding to the relaxation of F_0 when $\sigma_{\mathsf{AW}} = \sigma_{\mathsf{BW}} = 0$) or of Φ (when the wells of V and W satisfy $\alpha = \alpha' < \beta = \beta'$).

We consider now the (limit) case where the container Ω is an infinite cylinder of radius r and the volume of the phase A is a finite number v, and we study the behavior of the equilibrium configurations as v increases from 0 to $+\infty$. Under the additional assumption that $r \leq \ell/2$, we obtain in fact a complete description of the equilibrium configurations for every value of v.

Proposition 5.1. *Assume that $r \leq \ell/2$ and let $v_1 := 4\pi r^3/3$ and $v_2 := \pi r^2(r/3 + 2\ell)$ (hence $v_1 \leq v_2$). Then the equilibrium configurations $(\mathsf{A},\mathsf{A'})$ are given as follows:*

(i) *when $v \leq v_1$, A is any sphere with radius $\rho := \big(\frac{3v}{4\pi}\big)^{1/3}$, $\mathsf{A'}$ is empty, and the total energy is given by*

$$E := 4\pi\rho^2\sigma = (36\pi)^{1/3}\,\sigma\, v^{2/3}; \tag{5.7}$$

(ii) *when $v_1 < v < v_2$, A is the union of two half-spheres of radius r and a cylinder of radius r and height d (see Figure 8 below) where $d := \frac{v-v_1}{\pi r^2}$, $\mathsf{A'}$ is empty, and the total energy is*

$$E := \sigma(4\pi r^2 + 2\pi r d) = \sigma\Big(\frac{4\pi r^2}{3} + \frac{2v}{r}\Big); \tag{5.8}$$

(iii) *when $v \geq v_2$, A is a cylinder of radius r and height $\pi^{-1}r^{-2}v$ and $\mathsf{A'}$ agrees with the interface $\mathscr{S}_{\mathsf{AW}}$ (see Figure 9 below); the total energy is*

$$E := \sigma(2\pi r^2 + 4\pi r\ell). \tag{5.9}$$

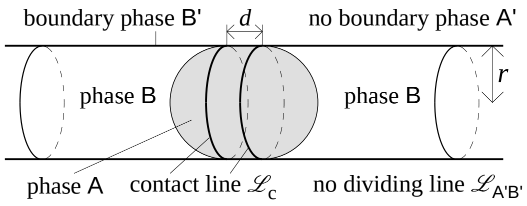

Fig. 8. Equilibrium configuration for $v_1 < v < v_2$.

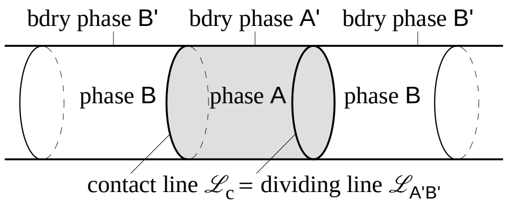

Fig. 9. Equilibrium configuration for $v_2 \leq v$.

The result of Proposition 5.1 can be interpreted as follows: for v smaller than the critical volume v_2, the minimal energy is achieved when A' is empty; this means that the dividing line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ is empty no line tension appears. When v is smaller than v_1, A is a spherical bubble which touches the wall of the container in at most one point (and the contact line is empty); when v reaches the value v_1 the sphere A becomes tangent to the cylinder (on a circle) and then it grows as shown in Figure 8. In the intermediate range $v_1 < v < v_2$, the contact line $\mathscr{L}_{\mathrm{c}}$ consists of two circles (delimiting the part of the wall corresponding to $\mathscr{S}_{\mathsf{AW}}$) and the contact angle θ is everywhere equal to $\theta_2 = 0$. When v passes the critical value v_2 we have a sudden change: the boundary phase A' appears and agrees with the interface $\mathscr{S}_{\mathsf{AW}}$; the contact line $\mathscr{L}_{\mathrm{c}}$ and the dividing line $\mathscr{L}_{\mathsf{A}'\mathsf{B}'}$ coincide and have vanishing geodesic curvature $\mathcal{K}_g(x)$. Then (5.4) shows that the contact angle θ is equal to $\frac{\pi}{2}$.

In other words, if we consider the quasistatic evolution of the system when the volume v of phase A increases continuously from 0 to $+\infty$, the bubble will experience a discontinuity in θ (from $\theta = 0$ to $\theta = \frac{\pi}{2}$) when v reaches the critical value v_2. This example shows that for a good understanding of this model of capillarity with line tension it is crucial to admit boundary phases which may not agree with the interfaces between the bulk phases and the wall.

Remark 5.2. In the previous example we have assumed condition (5.5) only to provide explicit computations. Another interesting situation is obtained when the container Ω is an a half-space and the coefficients of Φ_{gen} satisfy, instead of (5.5),

$$\sigma_{\mathsf{AA}'} = \sigma_{\mathsf{BB}'} = 0 \quad , \quad 0 < \sigma_{\mathsf{AB}'} = \sigma_{\mathsf{BA}'} = \sigma' < \sigma_{\mathsf{AB}} = \sigma .$$

In this case the angle θ_2 lies in interval $(0, \frac{\pi}{2})$, $\theta_1 = \pi - \theta_2$, and $\tau = 0$ (see (5.2)).

Under these assumptions we expect the following picture (which has been partially confirmed by numerical computations): when the volume v of the phase A

is small, the optimal configuration is obtained when A' is empty and the interface $\mathscr{S}_{\mathsf{AB}}$ is a spherical surface which meets the wall $\partial\Omega$ with constant contact angle $\theta = \theta_2$; in this regime A grows homotetically with v, the dividing line is empty and there is no line tension.

When v is larger than a certain critical value v_0, the optimal configuration is obtained when A' agrees with $\mathscr{S}_{\mathsf{AW}}$ and $\mathscr{S}_{\mathsf{AB}}$ is a spherical surface which meets $\partial\Omega$ with constant contact angle $\theta \in (\theta_2, \frac{\pi}{2})$. When v passes v_0 the contact angle θ increase discontinuously from θ_2 to a certain $\theta_0 \in (\theta_2, \frac{\pi}{2})$; also the radius of the disk $\mathscr{S}_{\mathsf{AB}}$ admits a discontinuity at $v = v_0$. In the regime $v > v_0$ the dividing line agrees with the contact line, the radius of the disk $\mathscr{S}_{\mathsf{AB}}$ increases with v, while the relative contribution of line tension to the total energy decreases, and the contact angle increases to $\pi/2$ as $v \to \infty$.

Sketch of the proof of Proposition 5.1. Since $\mathscr{L}_{\mathsf{A'B'}} = \partial\mathsf{A}'$, $\mathscr{S}_{\mathsf{AB}} = \partial\mathsf{A} \cap \Omega$, $\mathscr{S}_{\mathsf{AB'}} = \partial\mathsf{A} \setminus \mathsf{A}'$, $\mathscr{S}_{\mathsf{BA'}} = \mathsf{A}' \setminus \partial\mathsf{A}$, and $\ell = \mathsf{c}/\sigma$, we can rewrite the functional Φ_{gen} in (5.6) as

$$\Phi_{\text{gen}}(\mathsf{A}, \mathsf{A}') = \sigma\big(|\partial\mathsf{A} \triangle \mathsf{A}'| + \ell|\partial\mathsf{A}'|\big). \tag{5.10}$$

We consider now a minimizer $(\mathsf{A}, \mathsf{A}')$ of Φ_{gen} under the constraint $|\mathsf{A}| = v$.

Assertion 1. *If A' is empty, then A is a sphere as long as $v \leq v_1 = 4\pi r^3/3$, and otherwise is given as in Figure 8; the corresponding energies are given by* (5.7) *and* (5.8) *respectively.*

Proof. If A' is empty then A minimizes $\sigma|\partial\mathsf{A}|$ under the volume constraint $|\mathsf{A}| = v$ (cf. (5.10)). Then A must be a sphere as long as a sphere of volume v is contained in Ω, that is, for $v \leq 4\pi r^3/3$. For larger v, we can easily prove that A is axially symmetric (by a standard application of Steiner symmetrization), $\partial\mathsf{A}$ has constant mean curvature in Ω and meets $\partial\Omega$ with constant contact angle $\theta = 0$ (cf. (5.3)). The only possibility is the one in Figure 8.

Assertion 2. *if A' is not empty, then $|\partial\mathsf{A}'| \geq 4\pi r$.*

Proof. We assume first that $\partial\mathsf{A}'$ consists of one connected component γ only. Since the closed curve γ is a boundary within $\partial\Omega$, it is homotopically trivial. Now the Gaussian curvature of $\partial\Omega$ vanishes and by the theorem of Gauss-Bonnet, the integral over γ of the modulus of the geodesic curvature $\mathcal{K}_g(x)$ is exactly 2π. But we know from (5.4) that $|\mathcal{K}_g(x)| \leq 1/\ell$ (recall that $\tau = 0$). Then γ has length at least $2\pi\ell$, and the thesis follows by the assumption $r \leq \ell/2$.

Clearly this argument runs also if we assume that $\partial\mathsf{A}'$ contains at least one homotopically trivial connected component. In all other cases, $\partial\mathsf{A}'$ contains at least two closed curves which wind around the cylinder and therefore we have again $|\partial\mathsf{A}'| \geq 4\pi r$.

Assertion 3. *if A' is not empty then $v > \pi^2 r^3$.*

Proof. By Step 2 we know that $|\partial\mathsf{A}'| \geq 4\pi r$, and then the energy of the configuration $(\mathsf{A}, \mathsf{A}')$ is at least $4\pi r\ell\sigma$, which is strictly larger than (5.7) and (5.8) if $v \leq \pi^2 r^3$; by Step 1 we deduce that $(\mathsf{A}, \mathsf{A}')$ cannot be a minimizer for $v \leq \pi^2 r^3$.

Assertion 4. *if* A' *is not empty then* A *is given as in Figure 9, and the energy is given in* (5.9).

Proof. By Step 2 we know that $|\partial\mathsf{A}'| \geq 4\pi r$ and by Step 3 that $|\mathsf{A}| > \pi^2 r^3$. Then Proposition 6.10 yields $|\partial\mathsf{A}\cap\Omega| \geq 2\pi r^2$ and therefore the total energy is larger than $\sigma(2\pi r^2+4\pi r\ell)$. On the other hand this lower bound is achieved by the configuration described in Figure 9 only.

To conclude the proof, it is enough to notice that the configuration in Figure 9 is preferable to the one in Figure 8 only when v is larger than $v_2 = \pi r^2(r/3 + 2\ell)$ (just compare the values of the energy in (5.8) and (5.9)). $\square$

6. Appendix

We give here some technical lemmas we used in the previous sections.

6.1. A Rearrangement Result

Let ψ be a positive decreasing function on $[0, +\infty[$. For every couple of Borel sets A and B in $\mathbb{R}$ we set

$$\Psi(A, B) := \int_{A\times B} \psi(|x' - x|)\, dx' dx. \tag{6.1}$$

Now, for every $t, y \in \mathbb{R}$, we denote by $L_t(A)$ the interval $[t, t + |A|]$ and by $R_y(B)$ the interval $[y - |B|, y]$. The following result can be found in [ABS1] (see also [Br]); for the convenience of the reader we give also the proof.

Proposition 6.1. *If* A *and* B *are disjoint sets included in the interval* $[t, y]$, *then*

$$\Psi(A, B) \geq \Psi\big(L_t(A), R_y(B)\big). \tag{6.2}$$

In other words, if we fix an interval I and restrict our attention to the class of all $A, B \subset I$ with prescribed measures a and b (with $a + b \leq |I|$), then the infimum of Ψ is achieved when A and B are intervals and are taken as much distant as possible.

Proof. We write $A \prec B$ if $\sup A \leq \inf B$. We remark that if $t \prec A \prec B \prec y$ then

$$\Psi(A, B) \geq \Psi(L_t(A), B) \quad \text{and} \quad \Psi(A, B) \geq \Psi(A, R_y(B)). \tag{6.3}$$

Indeed, by setting $h(x) := t + \int_t^x 1_A(s)\, ds$, we have that $h(x) \leq x$ for all $x \geq t$, and since ψ is decreasing we get

$$\begin{aligned}
\Psi(A, B) &= \int_B \Big[\int_t^{+\infty} \psi(x' - x)\, 1_A(x)\, dx\Big]\, dx' \\
&\geq \int_B \Big[\int_t^{+\infty} \psi(x' - h(x))\, h'(x)\, dx\Big]\, dx' \\
&= \int_B \Big[\int_t^{t+a} \psi(x' - u)\, du\Big]\, dx' = \Psi(L_t(A), B).
\end{aligned}$$

This proves the first inequality in (6.3). The second one may be proved in the same way.

Next we observe that it suffices to prove inequality (6.2) when A and B are finite unions of closed intervals, the general case will follow by a standard approximation argument. Let $A = A_1 \cup A_2 \cup \ldots \cup A_{n_A}$, $B = B_1 \cup B_2 \cup \ldots \cup B_{n_B}$, where A_i and B_j are pairwise disjoint closed intervals in $[t, y]$.

The proof is achieved by induction on the total number of intervals $n = n_A + n_B$. When $n = 1$, either A or B is empty and the proposition is trivial. Now, we assume the proposition true for n and we prove it for $n + 1$.

Let be given A and B such that $n_A + n_B = n + 1$. With no loss in generality we may assume that A is non-empty and $A_1 \prec A_i$ for all $i > 1$ and $A_1 \prec B$; we set $c := |A_1|$ and $A' := A_2 \cup A_3 \cup \ldots \cup A_{n_A}$. Then we may write $\Psi(A, B)$ as $\Psi(A_1, B) + \Psi(A', B)$, and since $t \prec A_1 \prec B \prec y$, inequalities (6.3) yield

$$\Psi(A_1, B) \geq \Psi\big(L_t(A_1), B\big) \geq \Psi\big(L_t(A_1), R_y(B)\big).$$

Moreover, A' and B are disjoint subsets of $[t + c, y]$ and $n_{A'} + n_B = n$; therefore the inductive hypothesis yields

$$\Psi(A', B) \geq \Psi\big(L_{t+c}(A'), R_y(B)\big).$$

Hence

$$\Psi(A, B) \geq \Psi\big(L_t(A_1), R_y(B)\big) + \Psi\big(L_{t+c}(A'), R_y(B)\big)$$

and since $L_t(A_1) \cup L_{t+c}(A') = L_t(A)$, we deduce (6.2). $\square$

6.2. Optimal Constants for Some Trace Inequalities

The following three statements are concerned with the optimal constant for some trace inequalities involving the L^2 norm of the gradient of a function defined on a two-dimensional domain and the $H^{1/2}$ norm of its trace on a line. For the time being $u = u(x, y)$ is a real function on $\mathbb{R}^2$, $v = v(x)$ is the trace of u on the line $\mathbb{R} \times \{0\}$, $\hat{u} = \hat{u}(\xi, \nu)$ is the Fourier Transform of u and $\hat{v} = \hat{v}(\xi)$ is the Fourier Transform of v.

Lemma 6.2. *Let u be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ with derivative in L^2. Then u belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and the trace of u on the line $\mathbb{R} \times \{0\}$ is a well-defined function $v \in L^2_{\mathrm{loc}}(\mathbb{R})$. Moreover*

$$\int_{\mathbb{R}^2} \Big| \frac{v(x') - v(x)}{x' - x} \Big|^2 dx'dx \leq \pi \int_{\mathbb{R}^2} |Du|^2 \, dx \, dy. \tag{6.4}$$

Proof. First we prove inequality (6.4) when u is a smooth function with compact support by a standard Fourier Transform argument (cf. [Ne, Chapter 2, Section

5]):

$$\begin{aligned}
\int_{\mathbb{R}^2} \Big|\frac{v(x')-v(x)}{x'-x}\Big|^2 dx'dx &= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |v(x+h)-v(x)|^2 dx\Big]\frac{dh}{h^2} \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \big|\hat v(\xi)(e^{2\pi i h\xi}-1)\big|^2 d\xi\Big]\frac{dh}{h^2} \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \frac{2-2\cos(2\pi h\xi)}{h^2}\, dh\Big]|\hat v(\xi)|^2 d\xi \\
&= 4\pi^2 \int_{\mathbb{R}} |\hat v(\xi)|^2 |\xi|\, d\xi
\end{aligned}$$

(here the second equality follows from Plancherel Theorem and the identity $\widehat{\tau_h v}(\xi) = e^{2\pi i h\xi}\,\hat v(\xi)$, while the last equality follows from the identity $\int_{\mathbb{R}}(2 - 2\cos(2\pi h\xi))\, h^{-2} dh = 4\pi^2|\xi|$).

Now we notice that $\hat v(\xi) = \int_{\mathbb{R}} \hat u(\xi,\nu)\, d\nu$, and then

$$\begin{aligned}
\int_{\mathbb{R}^2} \Big|\frac{v(x')-v(x)}{x'-x}\Big|^2 dx'dx &= 4\pi^2 \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \hat u(\xi,\nu)|\xi|^{1/2}\, d\nu\Big]^2 d\xi \\
&= 4\pi^2 \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \Big(\hat u(\xi,\nu)(\xi^2+\nu^2)^{1/2}\Big)\Big(\frac{|\xi|}{\xi^2+\nu^2}\Big)^{1/2} d\nu\Big]^2 d\xi \\
&\le 4\pi^2 \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\hat u|^2(\xi^2+\nu^2)\, d\nu\Big]\Big[\int_{\mathbb{R}} \frac{|\xi|}{\xi^2+\nu^2} d\nu\Big]\, d\xi \\
&= 4\pi^3 \int_{\mathbb{R}^2} |\hat u|^2(\xi^2+\nu^2)\, d\xi\, d\nu \\
&= \pi \int_{\mathbb{R}^2} |Du|^2\, dx\, dy
\end{aligned}$$

(the inequality follows from Schwartz-Hölder inequality, while the last equality follows from Plancherel theorem and the identity $\widehat{Du}(\xi,\nu) = 2\pi i\, \hat u(\xi,\nu)\cdot(\xi,\nu)$).

Now we want to extend inequality (6.4) to all functions in the Beppo-Levi space $X := \big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \,:\, Du \in L^2\big\}$. We recall that X is Fréchet space whose topology is generated by the L^1_{loc} topology and the semi-norm $\|Du\|_2$. We will use a density argument.

Notice that the right hand side of inequality (6.4) is continuous on X (by the definition of the topology of X), while the left hand side is lower semicontinuous in L^1_{loc}, and then also in X, by the Fatou's Lemma. Hence it is enough to prove that the space $\mathcal{D}(\mathbb{R}^2)$ of all smooth functions with compact support is dense in X. Since $\mathcal{D}(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$ and $X \cap L^\infty$ is dense in X (any $u \in X$ may be approximated by the truncated functions $u_n := (u \wedge n) \vee (-n)$), it remains to show that H^1 is dense in $X \cap L^\infty$ (with respect to the topology of X).

For every bounded function u in X and every integer $n > 1$ we set $u_n(x) := g_n(x)u(x)$ where

$$g_n(x) = \begin{cases} 1 & \text{if } |x| \le n^{1/e}, \\ \log(\log n) - \log(\log|x|) & \text{if } n^{1/e} \le |x| \le n, \\ 0 & \text{if } n \le |x|. \end{cases}$$

Each u_n belongs to H^1 and u_n tends to u in $L^1_{\rm loc}$. Moreover $Du_n = g_n Du + u\, Dg_n$, and $u\, Dg_n \to 0$ in L^2 because u is bounded and $Dg_n \to 0$ in L^2 (this can be checked by a direct computation); hence $Du_n \to Du$ in L^2, and thus we have proved that u_n tends to u in X. $\square$

Corollary 6.3. *Let A be the half-plane $\{(x,y) : \, y>0\}$ and let u be a function in $L^1_{\rm loc}(A)$ such that $Du \in L^2$. Then the trace v of u on $\mathbb{R}\times\{0\}$ is well-defined and*

$$\int_{\mathbb{R}^2} \Big| \frac{v(x')-v(x)}{x'-x} \Big|^2 dx' dx \le 2\pi \int_A |Du|^2 \, dx\, dy. \tag{6.5}$$

Proof. Extend the function u to the whole $\mathbb{R}^2$ by reflection and then apply Lemma 6.2. $\square$

Corollary 6.4. *Let D be the half-disk $\big\{(x,y) : \; x^2+y^2<r, \; y>0\big\}$ where $r>0$, and let u be a function in $H^1(\Omega)$. Then the trace of u on the segment $E\times\{0\}$ (with $E =]-r,r[$) belongs to $H^{1/2}(E)$ and*

$$\int_{E^2} \Big| \frac{v(x')-v(x)}{x'-x} \Big|^2 dx\, dx' \le 2\pi \int_D |Du|^2 \, dx\, dy. \tag{6.6}$$

Proof. We extend the function u to the whole half-plane $A := \{(x,y) : \; y>0\}$ by setting

$$\tilde u(z) = \begin{cases} u(z) & \text{if } |z|<r, \\ u(r^2/\bar z) & \text{if } |z|>r \end{cases}$$

(we identify the points (x,y) with the complex numbers $z = x+iy$). Since $z \mapsto r^2/\bar z$ is a conformal mapping, we have $\int_{A\setminus D} |D\tilde u|^2 = \int_D |D\tilde u|^2 = \int_D |Du|^2$. Thus $D\tilde u$ belongs to $L^2(A)$ and by Corollary 6.3 we get

$$\begin{aligned} 4\pi \int_D |Du|^2 &= 2\pi \int_A |D\tilde u|^2 \\ &\ge \int_{\mathbb{R}^2} \Big| \frac{\tilde v(x')-\tilde v(x)}{x'-x} \Big|^2 dx' dx \\ &\ge \int_{E^2} \Big| \frac{\tilde v(x')-\tilde v(x)}{x'-x} \Big|^2 dx' dx + \int_{(\mathbb{R}\setminus E)^2} \Big| \frac{\tilde v(x')-\tilde v(x)}{x'-x} \Big|^2 dx' dx \\ &= \int_{E^2} \Big| \frac{v(x')-v(x)}{x'-x} \Big|^2 dx' dx + \int_{E^2} \Big| \frac{v(x')-v(x)}{r^2/x'-r^2/x} \Big|^2 \frac{r^4}{(x'x)^2} dx' dx \\ &= 2\int_{E^2} \Big| \frac{v(x')-v(x)}{x'-x} \Big|^2 dx' dx. \quad \square \end{aligned}$$

Remark 6.5. The constants in the trace inequalities (6.4), (6.5) and (6.6) are optimal. The proof of this claim clearly reduces to prove that the constant 2π in (6.6) is sharp. To this end we consider for every $\lambda > 1/r$ the functions $u_\lambda : D \to [0,1]$ given in polar coordinates $\theta \in]0,\pi[$, $\rho \in]0,r[$ by

$$u_\lambda(\rho,\theta) := \begin{cases} \frac{\lambda}{\pi}\theta\rho & \text{for } 0 < \rho < 1/\lambda, \\ \frac{1}{\pi}\theta & \text{for } 1/\lambda < \rho < r. \end{cases} \tag{6.7}$$

The trace of u_λ on E is the function $v_\lambda(x) = 0$ for $x > 0$ and $v_\lambda(x) = 1$ for $x < 0$. By a straightforward computation one gets

$$\int_{E^2} \left| \frac{v_\lambda(x') - v_\lambda(x)}{x' - x} \right|^2 dx\, dx' = 2\log\lambda + O(1)$$

and

$$\int_D |Du_\lambda|^2\, dx\, dy = \tfrac{1}{\pi}\log\lambda + O(1).$$

The conclusion follows by letting $\lambda \to +\infty$.

6.3. Some Slicing Results

We establish now a connection between the compactness of a family of functions in $L^1(\mathbb{R}^h)$ and the compactness of the traces of these functions on lines. We fix $L > 0$ and we assume throughout this subsection that every function takes values in the interval $[-L, L]$.

Let us fix some notation: A is a bounded open subset of $\mathbb{R}^N$, e is a unit vector in $\mathbb{R}^N$ and u a function on A; we denote by M the orthogonal complement of e, by A_e the projection of A onto M; for every $y \in M$, $A_e^y := \{t \in \mathbb{R} : y + te \in A\}$ and u_e^y is the trace of u on A_e^y, that is, $u_e^y(t) := u(y + te)$. Accordingly, for every family $\mathcal{F}$ of functions on A we set $\mathcal{F}_e^y := \{u_e^y : u \in \mathcal{F}\}$, so that $\mathcal{F}_e^y$ is a family of functions on A_e^y.

The simplest statement which connects the pre-compactness of $\mathcal{F}$ in $L^1(A)$ with the pre-compactness of $\mathcal{F}_e^y$ in $L^1(A_e^y)$ is the following: *assume that there exist N linearly independent unit vectors e such that:*

$$\mathcal{F}_e^y \text{ is pre-compact in } L^1(A_e^y) \text{ for } \mathscr{H}^{N-1} \text{ a.e. } y \in A_e. \tag{6.8}$$

Then $\mathcal{F}$ is pre-compact in $L^1(A)$.

Unfortunately this statement does not fit our purposes. A sufficiently general result is obtained by allowing the possibility of replacing $\mathcal{F}$ in (6.8) with a perturbation of $\mathcal{F}$. More precisely, for every $\delta > 0$ we say that a family $\mathcal{F}'$ is *δ-dense* in $\mathcal{F}$ if $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}'$ with respect to the $L^1(A)$ topology, and then we have the following:

Theorem 6.6. *Let $\mathcal{F}$ be a family of functions $v : A \to [-L, L]$ and assume that there exists N linearly independent unit vectors e which satisfy the following property:*

$$\text{for every } \delta > 0 \text{ there exists a family } \mathcal{F}_\delta \ \ \delta\text{-dense in } \mathcal{F} \text{ such that } (\mathcal{F}_\delta)^y_e \text{ is pre-compact in } L^1(A^y_e) \text{ for } \mathscr{H}^{N-1} \text{ a.e. } y \in A_e. \tag{6.9}$$

Then $\mathcal{F}$ is pre-compact in $L^1(A)$.

Proof. With no loss in generality, we may assume that $L = 1$ and $|A^y_e| \le 1$ for all y. Every function defined on A is extended to 0 on $\mathbb{R}^N \setminus A$, and accordingly every function defined on A^y_e is extended to 0 on $\mathbb{R} \setminus A^y_e$. Fix for the moment a unit vector e which satisfies (6.9). For all $y \in A_e$ and all $s > 0$ we set

$$\omega^y_\delta(s) := \sup\Big\{ \int_{\mathbb{R}} \big|v^y_e(t+h) - v^y_e(t)\big|\, dt \ : \ v \in \mathcal{F}_\delta,\ h \in [-s, s]\Big\}. \tag{6.10}$$

Since $|v^y_e| \le 1$ and $|A^y_e| \le 1$, then $\omega^y_\delta(s) \le 2$ for all $s > 0$, and since $(\mathcal{F}_\delta)^y_e$ is pre-compact in $L^1(A^y_e)$, the Fréchet-Kolmogorov Theorem yields that $\omega^y_\delta(s) \searrow 0$ as $s \searrow 0$. Take now $u \in \mathcal{F}$ and $\delta > 0$, and choose $v \in \mathcal{F}_\delta$ such that $\|u - v\|_1 \le \delta$ (in $L^1(A)$). By (6.10) we obtain, for every h

$$\begin{aligned} \int_{\mathbb{R}^N} \big|u(x+he) - u(x)\big|\, dx &\le 2\delta + \int_{\mathbb{R}^N} \big|v(x+he) - v(x)\big|\, dx \\ &= 2\delta + \int_{A_e} \Big(\int_{\mathbb{R}} \big|v^y_e(t+h) - v^y_e(t)\big|\, dt\Big)\, dy \\ &\le 2\delta + \int_{A_e} \omega^y_\delta(|h|)\, dy. \end{aligned} \tag{6.11}$$

For every $\delta > 0$ and $s > 0$ we set $\omega_\delta(s) := \int_{A_e} \omega^y_\delta(s)\, dy$. Then $\omega_\delta \le 2|A_e|$ because $\omega^y_\delta \le 2$, and $\omega_\delta(s) \searrow 0$ as $s \searrow 0$ because the same holds true for all ω^y_δ (apply the dominated convergence theorem). Now, for all $s > 0$ we set $\omega(s) := \inf_{\delta>0} \big(2\delta + \omega_\delta(s)\big)$: the function ω is bounded and $\omega(s) \searrow 0$ as $s \searrow 0$, and (6.11) yields

$$\int_{\mathbb{R}^h} \big|u(x+he) - u(x)\big|\, dx \le \omega(|h|) \quad \forall h \in \mathbb{R},\ v \in \mathcal{F}. \tag{6.12}$$

Finally we take linearly independent unit vectors e_i with $i = 1, \dots, N$ which satisfy (6.9) and we take ω_i such that (6.12) holds (with e and ω replaced by e_i and ω_i resp.). Since the vectors e_i span $\mathbb{R}^N$, the Fréchet-Kolmogorov Theorem implies that $\mathcal{F}$ is pre-compact in $L^1(A)$. □

We conclude this subsection by recalling some results about the slicing of Sobolev functions and finite perimeter sets, which are well-known to experts but not available in this form in standard reference books. For simplicity we consider only one-dimensional slicings, but the following results are valid for slicings with arbitrary dimension.

Remark 6.7. Let A, e, A_e and A^y_e be given as before, and take a Borel function u on A; by Fubini's theorem u belongs to $L^p(A)$ (with $1 \le p < \infty$) if and only if u^y_e belongs to $L^p(A^y_e)$ for a.e. $y \in A_e$ and the function $y \mapsto \|u^y_e\|_p$ belongs to $L^p(A_e)$.

Similarly, given a sequence $(u_n) \subset L^p(A)$ which converges to u in $L^p(A)$, possibly passing to a subsequence we have that $(u_n)_e^y$ converges to u_e^y in $L^p(A_e^y)$ for a.e. $y \in A_e$. Conversely, if $(u_n)_e^y \to u_e^y$ in $L^p(A_e^y)$ for a.e. $y \in A_e$ and the functions $|u_n|^p$ are equi-integrable, then $u_n \to u$ in $L^p(A)$.

Proposition 6.8. (cf. [EG, Section 4.9]). *Let be given $u \in L^p(A)$. If e is an arbitrary unit vector and u belongs to $W^{1,p}(A)$, then $u_e^y \in W^{1,p}(A_e^y)$ for a.e. $y \in A_e$, and the derivative $Du_e^y(t)$ agrees with the partial derivative $D_e u(y + te)$ for a.e. $y \in A_e$ and $t \in A_e^y$. Conversely u belongs to $W^{1,p}(A)$ if there exist N linearly independent unit vectors e such that $u_e^y \in W^{1,p}(A_e^y)$ for a.e. $y \in A_e$ and the function $y \mapsto \|Du_e^y\|_p$ belongs to $L^p(A_e)$.*

Proposition 6.9. (see [Amb1], cf. also [EG, section 5.10]). *Let be given a Borel set $E \subset A$. If e is an arbitrary unit vector and E has finite perimeter in A, then E_e^y has finite perimeter in A_e^y and $\partial(E_e^y \cap A_e^y) = (\partial E \cap A)_e^y$ for a.e. $y \in A_e$, and*

$$\int_{A_e} \#(\partial E_e^y \cap A_e^y)\, dy = \int_{\partial E \cap A} \langle \nu_E ; e \rangle. \tag{6.13}$$

Conversely, E has finite perimeter in A if there exist N linearly independent unit vectors e such that the integral of $\#(\partial E_e^y \cap A_e^y)$ over all $y \in A_e$ is finite.

6.4. An Inequality of Isoperimetric Type

In this last subsection we consider finite perimeter sets A in $\mathbb{R}^3$, as usual ∂A denotes the essential boundary of A. The result we are interested in reads as follows:

Proposition 6.10. *Let be given an open infinite cylinder Ω with radius r in $\mathbb{R}^3$, and a finite perimeter set $A \subset \Omega$ with volume $|A| \geq \pi^2 r^3$. Then $|\partial A \cap \Omega| \geq 2\pi r^2$.*

Proof. Let denote points in $\mathbb{R}^3$ by $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$, and let P be the projection on $\mathbb{R}^2$, that is, $P(x, t) := x$. We assume Ω is of the form $D \times \mathbb{R}$ where D is the open disk with center 0 and radius r in $\mathbb{R}^2$, and that every point of A is a point of density one. For all $t \in \mathbb{R}$ we denote by A_t the set of all x such that $(x, t) \in A$, and by δ the measure of $D \setminus P(A)$. We apply to each set A_t the isoperimetric inequality on the disk D:

$$\min\big\{|A_t|,\, \pi\, r^2 - |A_t|\big\} \leq C\, |\partial A_t \cap D|^2, \tag{6.14}$$

where $C := \pi/8$.

By the definition of δ we obtain $\pi r^2 - |A_t| \geq \delta$, and if we apply the inequality $\min\{a, b\} \geq a^2 b$ (valid for $0 \leq a, b \leq 1$) with $a := |A_t|/\pi r^2$ and $b := \delta/\pi r^2$, we get $\min\big\{|A_t|, \pi r^2 - |A_t|\big\} \geq \delta\big(|A_t|/\pi r^2\big)^2$. Then (6.14) yields $|\partial A_t \cap D| \geq \sqrt{\delta/C}\,\big(|A_t|/\pi r^2\big)$, and integrating this inequality over all t leads to

$$\int_{\mathbb{R}} |\partial A_t \cap D|\, dt \geq \frac{\sqrt{\delta/C}}{\pi r^2}\, |A|. \tag{6.15}$$

We recall now that for every two-dimensional rectifiable set $S \subset \mathbb{R}^3$ there holds

$$\Big[\mathscr{H}^2(S)\Big]^2 \geq \Big[\int_D \#\big(S \cap \{x\} \times \mathbb{R}\big)\, dx\Big]^2 + \Big[\int_{\mathbb{R}} \mathscr{H}^1(S_t)\, dt\Big]^2$$

(this inequality can be easily derived for surfaces of class C^1, and therefore immediately extended to any rectifiable set). Now we apply this inequality to $S = \partial A \cap \Omega$, and since $(\partial A \cap \Omega)_t = \partial A_t \cap D$ for a.e. $t \in \mathbb{R}$ and $\#\big(\partial A \cap \{x\} \times \mathbb{R}\big) \geq 2$ for a.e. x in $P(A)$ (cf. Proposition 6.9), by (6.15) we get

$$\begin{aligned} |\partial A \cap \Omega|^2 &\geq 4(\pi r^2 - \delta)^2 + \frac{\delta |A|^2}{C\pi^2 r^4} \\ &\geq (2\pi r^2)^2 + \Big(\frac{|A|^2}{C\pi^2 r^4} - 8\pi r^2\Big)\,\delta + 4\delta^2. \end{aligned}$$

Finally, inequality $|\partial A \cap \Omega| \geq 2\pi r^2$ follows when $|A| \geq \sqrt{8C\pi^3 r^6} = \pi^2 r^3$. □

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References.

[AB] G. Alberti & G. Bellettini: A nonlocal anisotropic model for phase transitions – Asymptotic behavior of rescaled energies, *European J. Appl. Math.* **9** (1998), 261-284.

[ABS1] G. Alberti, G. Bouchitté & P. Seppecher: Un résultat de perturbations singulières avec la norme $H^{1/2}$, *C. R. Acad. Sci. Paris*, **319-I** (1994), 333-338.

[ABS2] G. Alberti, G. Bouchitté & P. Seppecher: Boundary effects in phase transitions, in *Curvature Flows and Related Topics* (A. Visintin ed.), pp. 1-11, Gakkotosho, Tokyo 1995.

[Al] G. Alberti: Variational models for phase transitions, an approach via Γ-convergence, proceedings of the Summer School on *Differential Equations and Calculus of Variations*, Pisa, September 1996, to appear.

[Amb1] L. Ambrosio: A compactness theorem for a special class of functions of bounded variation, *Boll. Un. Mat. It.* B(7), **3** (1989), 857-881.

[Amb2] L. Ambrosio: *Corso Introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*, Lecture Notes, Scuola Normale Superiore, Pisa 1997.

[Ba] S. Baldo: Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990), 289-314.

[BS] G. Bouchitté & P. Seppecher: Cahn and Hilliard fluid on an oscillating boundary, in *Motion by Mean Curvature and Related Topics* (G. Buttazzo and A. Visintin eds.), pp. 23-42, de Gruyter, Berlin 1994.

[BDS] G. Bouchitté, C. Dubs & P. Seppecher: Transitions de phase avec un potentiel dégénéré à l'infini, application à l'équilibre de petites gouttes, *C. R. Acad. Sci. Paris*, **323-I** (1996), 1103-1108.

[Br] F. Brock: A reverse convolution inequality, *J. Convex Anal.* **1** (1994) 259-264.

[CH] J.W. Cahn & J.E. Hilliard: Free energy of a non-uniform system I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), 258-267.

[DG] P.G. de Gennes: Wetting: statics and dynamics, *Rev. Modern Phys.* **57** (1985), 827-863.

[DM] G. Dal Maso: *An Introduction to Γ-convergence*, Progress in Nonlinear Diff. Equat. and their Appl. **8**, Birkhäuser, Boston 1993.

[EG] L.C. Evans & R.F. Gariepy: *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton 1992.

[Fi] R. Finn: *Equilibrium Capillary Surfaces*, Springer, New York 1986.

[Gi] E. Giusti: *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Math. **80**, Birkhäuser, Boston, Basel 1984.

[Gu] M.E. Gurtin: On a theory of phase transitions with interfacial energy, *Arch. Rational Mech. & Anal.* **87** (1984), 187-212.

[Mo1] L. Modica: The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. & Anal.* **98** (1987), 123-142.

[Mo2] L. Modica: Gradient theory of phase transitions with boundary contact energy, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5** (1987), 453-486.

[MM] L. Modica & S. Mortola: Un esempio di Γ^--convergenza, *Boll. Un. Mat. Ital.* B(5), **14** (1977), 285-299.

[Ne] J. Nečas: *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et Cie, Paris 1967.

[RW] J.S. Rowlinson & B. Widom: *Molecular Theory of Capillarity*, Clarendon Press, Oxford 1984.

[Ta] I. Tamanini: *Regularity Results for Almost Minimal Oriented Hypersurfaces in* $\mathbf{R}^n$, Quaderni del Dipartimento di Matematica **1**, Università di Lecce, Lecce 1984.

[Va1] M. Valadier: Young measures, in *Methods of Nonconvex Analysis* (A. Cellina ed.), Lectures Notes in Math. **1446**, Springer, Berlin 1990.

[Va2] M. Valadier: A course on Young measures, *Rend. Istit. Mat. Univ. Trieste* **26** (1994) suppl., 349-394.

[WW] B. Widom & H. Widom: Model for line tension in three-phase equilibrium, *Physica A*, **173** (1991), 72-110.

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