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Gap Phenomenon for Some Autonomous Functionals

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We give an example of an autonomous functional $F(u) = \int_\Omega f(u, Du)dx$ (Ω open subset of $\mathbb{R}^2$, $u : \Omega \to \mathbb{R}^2$ in the Sobolev space $W^{1,1}$) which is sequentially weakly lower semicontinuous in $W^{1,p}$ for every $p \geq 1$ but does not agree with the relaxation of the same functional restricted to smooth functions when $p < 2$. A Lavrentiev phenomenon occurs for a related boundary problem.

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1. - Introduction

When dealing with variational problems involving integral functionals, a very surprising and interesting phenomenon is the so-called Lavrentiev phenomenon; let T be a space of weakly differentiable functions, X a dense subset of smooth functions and F a weakly lower semicontinuous integral functional on T: then it may happen that the infimum of F on T is strictly lower than the infimum on X (the first example of such behaviour was given in [18], many examples and references may be found in [6], [10] and [9], section 3.4.3).
For example Lavrentiev phenomenon may occur with $T = \big\{u \in W^{1,2}(0,1) :\ u(0) = 0,\ u(1) = 1\big\}$, $X = \{u \in T : u \text{ is Lipschitz}\}$ and $F(u) = \int_0^1 f(x, u, u')dx$ with f continuous, convex in the third variable and satisfying $f(x, u, s) \geq |s|^2$ (cf. [2]). Notice that in this particular case F is coercive on T (endowed with the weak topology) and then it attains a "natural" minimum on T.
Following [6], we consider Lavrentiev phenomenon in a more abstract framework: let T be a topological space, X a dense subset of T and F a lower semicontinuous function on T and set for all $u \in T$

$$F(X,T)(u) := \inf(\liminf_{n\to\infty} F(u_n)) \ :\quad u_n \in X \text{ and } u_n \to u. \tag{1.1}$$

Then we have that $F(X,T) \geq F$ but it may happen that equality does not hold: in this case we say that F *has a gap from* X *to* T (see Definition 2.1). Notice that when there is no gap, then there is no Lavrentiev phenomenon (indeed the infima of F on T and X are the same). In particular we have that $F(X,T)(u) = F(u)$ if and only if there exists a sequence $(u_n) \subset X$ such that $u_n \to u$ and $F(u_n) \to F(u)$; hence there is no gap when F is continuous with respect to any topology τ finer than the topology of T and such that X is still τ-dense in T (for example, this happens when T is a Sobolev space $W^{1,p}$ endowed with the weak topology, X includes C^∞ functions, and F is strongly $W^{1,p}$ continuous).

A lot of result are available for integral functional on Sobolev spaces, giving examples of gap and Lavrentiev phenomenon or showing that under some assumption they never occur (see references in [6], [7] and [10]). An interesting case is the one of autonomous integral functionals, i.e., $T = W^{1,p}(\Omega, \mathbb{R}^n)$, X is the subset of all smooth functions (Lipschitz or C^1) and $F(u) = \int_\Omega f(u, Du)dx$.
When Ω is one-dimensional and the integrand f is continuous, it may be proved that gap never occurs (see [1] for a direct proof, even if this can be obtained as a corollary of the regularity of minima of variational integrals proved in [8]). In this paper we show that a gap may occur when $T = W^{1,p}(\Omega, \mathbb{R}^2)$ with $p < 2$ and Ω two-dimensional. In particular, if we take

$$F(u) = \int_\Omega \left(|u|^2 - 1\right)^2 |Du|^2 dx \tag{1.2}$$

and $u_0(x) = (x - x_0)/|x - x_0|$ where $x_0 \in \Omega$, then $F(u_0) = 0$ but $\liminf F(u_n) \geq 2\pi/3$ whenever u_n are smooth functions which weakly converge to u_0 in $W^{1,1}$ (see Theorem 2.3). In Theorem 2.12 we give an example of Lavrentiev phenomenon for a related boundary value problem.
The heuristic idea of the proof is the following: recalling that $|Du|^2 \geq 2|\det Du|$, by area formula we have that for all smooth functions u

$$F(u) \geq 2 \int_\Omega \left(|u|^2 - 1\right)^2 |\det Du| \, dx \geq 2 \int_{u(\Omega)} \left(|y|^2 - 1\right)^2 dy \,. \tag{1.3}$$

Now $F(u_0) = 0$ because the image of u_0 is included in the unit circle in $\mathbb{R}^2$ but since the functions u_n are smooth and converge to u_0, their images have to "cover" the unit ball B of $\mathbb{R}^2$ and then (1.3) yields

$$\liminf_{n \to \infty} F(u_n) \geq 2 \int_B \left(|y|^2 - 1\right)^2 dy = \frac{2\pi}{3} \,.$$

Notice that if we replace weak convergence in $W^{1,1}$ by a weaker one (for example, L^p convergence), this argument does not work; indeed we can approximate u_0 by smooth functions u_n with images included in the unit circle, and then $F(u_n) = 0$ for all n (see Remark 2.8).

Another important remark is that the occurrence of a gap in this example is tightly connected with the structure of the singularity of u_0. In particular we cannot extend this example to Sobolev spaces $W^{1,p}$ with $p \geq 2$ or to the scalar case $W^{1,p}(\Omega, \mathbb{R})$, even taking functionals with different growth. We think it should be very interesting to understand what happens in these cases.
Finally, we want to point out the analogy of our example with the problem of relaxation of Dirichlet energy $F(u) = \int |Du|^2 dx$ for mappings which takes values into the sphere (see [3], [4], [15] and [16]), and indeed the function f in our integrand plays a role which is very close to the geometrical constraints of this problem.

2. - Statements and proofs of the results

In the following (unless differently stated) Ω is a bounded open subset of $\mathbb{R}^2$, u is a function of Ω into $\mathbb{R}^2$ and we denote a point of the domain Ω as $x = (x_1, x_2)$ and a point of the codomain $\mathbb{R}^2$ as $y = (y_1, y_2)$. B is the unit ball in $\mathbb{R}^2$ and $S = \partial B$ is the unit circle in $\mathbb{R}^2$.
If $A = (A_{ij})$ is a matrix in $\mathbb{R}^{2\times 2}$, then the norm of A is given as usual by $|A| := (\sum_{i,j} A_{ij}^2)^{1/2}$. Notice that $|A| \geq 2|\det A|$. I denotes both the 2×2 identity matrix and the identity map on $\mathbb{R}^2$.
If F is a function on the set T and X is a subset of T, then $F \llcorner X$ denotes the restriction of F to X.
If $u : \Omega \to \mathbb{R}^2$ is a continuous function, A is an open set relatively compact in Ω and y belongs to $\mathbb{R}^2 \setminus u(\partial A)$, then $\deg(u, A, y)$ is the degree of u restricted to A over the point y. For the basic results about degree we refer essentially to [11], chapters 1-3.
$W^{1,p}(\Omega, \mathbb{R}^2)$ (with $1 \leq p \leq \infty$), $Lip(\Omega, \mathbb{R}^2)$ and $C^k(\Omega, \mathbb{R}^2)$ are the usual spaces of Sobolev, Lipschitz and C^k functions from Ω into $\mathbb{R}^2$. They are usually endowed with their strong topologies; we write $W^{1,p}_w$ to indicate the Sobolev space $W^{1,p}$ endowed with its weak topology. For every function $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ we denote by $Ju(x)$ the determinant of the matrix $Du(x)$.
If T and V are topological spaces, we write $T \hookrightarrow V$ when T is included in V with continuous inclusion (i.e., when the topology of T is finer than the topology induced on T by V) and sequentially dense image.

Definition 2.1. Let T be a Hausdorff topological space and let F be a sequentially lower semicontinuous function on T. We say that a sequence $\{u_n\} \subset T$ *approximates* u *in energy* if $u_n \to u$ and $F(u_n) \to F(u)$.
Let X be a sequentially dense subset of T (i.e., every point of T is limit of some sequences of points of X). Then we define the following functions on T:
(i) $F_1(X,T)(u) := \inf(\liminf_n F(u_n))$ with $u_n \in X$ and $u_n \to u$,
(ii) $F_2(X,T)$ is the sequential relaxation of $F \llcorner X$ on T (namely, the maximal sequentially lower semicontinuous function on T which is less than or equal to F on X).

Then $F_1(X,T) \geq F_2(X,T) \geq F$. When T is first countable (i.e., every point has a countable neighborhood basis) then $F_1(X,T)$ and $F_2(X,T)$ are the same function,

that is the relaxation of $F \llcorner X$ on T. If we drop the first countability assumption (for example when T is a Sobolev space endowed with the weak topology instead of the norm topology) we have to be more careful; in particular $F_1(X,T)$ may be not sequentially l.s.c. and it may not agree with $F_2(X,T)$ (and also $F_2(X,T)$ may be not l.s.c.). Hence we have two slightly different problems: the approximation problem, that is determine whether $F_1(X,T)(u) = F(u)$ (in other words, whether u can be approximated in energy by functions in X) and the relaxation problem, that is determine whether $F_2(X,T)(u) = F(u)$.
Equality $F_1(X,T) = F$ may not hold in general: in this case we say that F *has a gap from* X *to* T.
Notice that this equality hold when F is sequentially continuous with respect to some topology τ finer than the topology of T and such that X is τ dense in T.

Definition 2.2. For all $u \in W^{1,1}(\Omega, \mathbb{R}^2)$, set

$$F(u) := \int_\Omega f(u)|Du|^2 dx \tag{2.1}$$

where $f : \mathbb{R}^2 \to [0,\infty[$ is a continuous map such that $f(y) = 0$ whenever $|y| = 1$, and set

$$C := 2\int_B f(y)\,dy\ . \tag{2.2}$$

Take $x_0 \in \Omega$ and let u_0 be the function given by

$$u_0(x) := (x - x_0)/|x - x_0|\ . \tag{2.3}$$

Then u_0 belongs to $W^{1,p}(\Omega,\mathbb{R}^2)$ for all $p \in [1,2[)$, and the following facts hold:

Theorem 2.3. *Let F and u_0 be given as in* (2.1) *and* (2.2)*, and assume that $C > 0$. Then*
(i) *F is sequentially lower semicontinuous in the L^1 topology;*
(ii) *F is (strongly) continuous in $W^{1,p}$ for all $p \geq 2$;*
(iii) *$F_1(X,W^{1,p}) = F_2(X,W^{1,p}) = F \llcorner W^{1,p}$ whenever $p \geq 2$, and then F has no gap from X to $W^{1,p}$;*
(iv) *$F_1(X,T) = F_1(W^{1,2},T)$ and $F_2(X,T) = F_2(W^{1,2},T)$ whenever X is a dense subset of $W^{1,2}$ and $W^{1,2} \hookrightarrow T$;*
(v) *$F_1(X,W^{1,p}_w) \geq F_2(X,W^{1,p}_w) \neq F$ whenever $p \in [1,2[$ and X is a dense subset of $W^{1,2}$, and then F has a gap from X to $W^{1,p}_w$.*
In particular, $F(u_0) = 0$ but $F_1(X,W^{1,p}_w)(u_0) = F_2(X,W^{1,p}_w)(u_0) = C$.

Remark 2.4. Statement (iii) means that every $u \in W^{1,p}$ with $p \geq 2$ may be approximated in energy in the $W^{1,p}$ topology by smooth functions (Lipschitz or even C^∞). Statement (v) shows that this is false when $p < 2$ and in particular we can prove that no sequences in $W^{1,2}$ approximates u_0 in energy. More precisely, we have that $F_1(X,W^{1,p}_w)(u_0) = F_2(X,W^{1,p}_w)(u_0) = C$.
The difficult part of this theorem is to prove inequalities $F_1(X,W^{1,p}_w)(u_0) \geq C$ and $F_2(X,W^{1,p}_w)(u_0) \geq C$: the proof of the second inequality relies on a result on currents

we develop in section 3. Of course the first inequality may be deduced directly from the other (recalling that there always holds $F_1(X,T) \geq F_2(X,T)$), but since this is the key lemma of this paper, we prefer to give also another independent proof which relies on simple properties of degree.

Proof of Theorem 2.3
(i). Using Corollary 4.1 in [14], we can prove the lower semicontinuity with respect to the L^1 topology of all functionals of the form

$$u \mapsto \int_\Omega g(u)h(Du)\,dx, \quad u \in W^{1,1}(\Omega,\mathbb{R}^2),$$

where g is bounded and continuous and h is a convex function with linear growth. Since F is (trivially) the limit of an increasing sequence of such functionals, also F is lower semicontinuous.
(ii) is well-known (cf. [9], chapter 3, and [5], chapter 2).
(iii) and (iv) follows from (ii).
(v). We may assume with no loss in generality that $x_0 = 0$.
Taking into account (iv), inequalities $C \geq F_1(X,W^{1,p}_w) \geq F_2(X,W^{1,p}_w)$ are proved if there exists a sequence of Lipschitz functions (v_n) which converges to u_0 in $W^{1,p}$ for all $p \in [1,2[$ and verifies $\lim_n F(v_n) = C$.
Take indeed $\rho_n \downarrow 0$ so that $\rho_n B \subset \Omega$ for all n and set

$$v_n := \begin{cases} x/\rho_n & \text{if } x \in \rho_n B, \\ x/|x| & \text{otherwise.} \end{cases} \tag{2.4}$$

The sequence (v_n) converges to u_0 in $W^{1,p}$ for all $p \in [1,2[$ and recalling that $f(y) = 0$ when $|y| = 1$,

$$\lim_n F(v_n) = \lim_n \int_{\rho_n B} f(x/\rho_n)\frac{2}{\rho_n^2}dx = 2\int_B f(y)\,dy = C\ .$$

Taking into account (iv), it is enough to prove inequalities $F_1(X,W^{1,p}_w)(u_0) \geq C$ and $F_2(X,W^{1,p}_w)(u_0) \geq C$ when X is C^1. This follows from Lemmas 2.5 and 2.6. □

Lemma 2.5. *Let F, C and u_0 be as in* (2.1), (2.2) *and* (2.3) *respectively, and let u_n be a sequence of functions in $C^1(\Omega,\mathbb{R}^2)$ which converges to u_0 in the weak topology of $W^{1,1}$. Then*

$$\liminf_{n\to\infty} F(u_n) \geq 2\int_B f(y)dy = C\ . \tag{2.5}$$

In other words, $F_1(C^1,W^{1,1}_w)(u_0) \geq C$.

Proof. First of all, notice that it is enough to prove that for every sequence (u_n) satisfying the hypothesis of Lemma 2.5 there exists a subsequence which satisfies (2.5).

Now, let (u_n) be fixed and take $r > 0$ so that $rB \subset \Omega$. By Lemma 3.9 below, we may find $\rho \in]0, r[$ and a subsequence (u_k) such that $u_k \llcorner \rho S$ converge to $u_0 \llcorner \rho S$ uniformly. We claim that this subsequence satisfies inequality (2.5).
For every integer k, set $g_k(y) := \mathrm{card}\big(u_k^{-1}(y)\big)$ for all $y \in \mathbb{R}^2$.
Since $|A|^2 \geq 2|\det A|$ for every matrix $A \in \mathbb{R}^{2\times 2}$, recalling area formula we obtain

$$\int_\Omega f(u_k)|Du_k|^2 dx \geq 2\int_\Omega f(u_k)|Ju_k|\,dx = 2\int_{\mathbb{R}^2} f(y)g_k(y)\,dy\ .$$

Thus conclusion follows from Fatou's Lemma once we show that $\liminf g_k(y) \geq 1$ for almost all y in B. To this end, define for all k

$$v_k(x) := \frac{|x|}{\rho}\,u_k\Big(\frac{\rho x}{|x|}\Big) \qquad \forall x \in \Omega\ .$$

Clearly each v_k is Lipschitz (and C^1 in $\Omega \setminus \{0\}$) and $v_k \llcorner \rho S = u_k \llcorner \rho S$. Hence $\deg(v_k, \rho B, y) = \deg(u_k, \rho B, y)$ for all $y \in \mathbb{R}^2 \setminus v_k(\rho S)$ (cf. [11], Theorem 3.1). Moreover, by the choice of ρ, v_k converges uniformly to I/ρ on Ω. Hence for all $y \in B$ there holds definitively $y \notin v_k(\rho S)$ and $\deg(v_k, \rho B, y) = \deg(I/\rho, \rho B, y)$ (cf. [11], Theorem 3.1) and then

$$g_k(y) \geq \deg(u_k, \rho B, y) = \deg(v_k, \rho B, y) = \deg(I/\rho, \rho B, y) = 1\ , \tag{2.6}$$

which ends the proof. □

Lemma 2.6. *Take F, C and u_0 as in* (2.1), (2.2) *and* (2.3) *respectively. Then*

$$F_2(C^1, W^{1,1}_w)(u_0) \geq 2\int_B f(y)dy = C\ .$$

Proof. Let $\varepsilon > 0$ be fixed. Since f is continuous we may find a smooth map ϕ on $\Omega \times \mathbb{R}^n$ with compact support so that $0 \leq \phi(x,y) \leq f(y)$ for all x, y and

$$\int \phi(0,y)\,dy \geq \int_B f(y)\,dy - \varepsilon = C/2 - \varepsilon\ . \tag{2.7}$$

Recalling that $|A|^2 \geq 2(\det A)$ for every matrix A, for all $v \in C^1(\Omega, \mathbb{R}^2)$ we get

$$F(v) = \int_\Omega f(v)|Dv|^2 dx \geq 2\int_\Omega \phi(x,v)Jv\,dx = 2\langle Tv; \omega\rangle \tag{2.8}$$

where $\omega = \phi\, dy^1 \wedge dy^2$ is a 2-form in $\Omega \times \mathbb{R}^2$ and Tv is the 2-dimensional current in $\Omega \times \mathbb{R}^2$ associated to the graph of v as in definition 3.1.
Since the map $v \mapsto \langle Tv; \omega\rangle$ is sequentially weakly continuous on $W^{1,1}$ (Theorem 3.2), by (2.8) and definition 2.1 we get

$$F_2(C^1, W^{1,1}_w)(u_0) \geq \langle Tu_0; \omega\rangle$$

and we claim that $\langle Tu_0; \omega\rangle = 2\int_B \phi(0,y)dy$: taking into account (2.7) and recalling that ε is arbitrarily taken, this would end the proof. To prove our claim, we take indeed v_n as in formula (2.4). The functions v_n converge to u_0 in $W^{1,1}$ and then

$$\begin{aligned}\langle Tu_0; \omega\rangle &= \lim_n \langle Tv_n; \omega\rangle = \lim_n \int_\Omega \phi(x, v_n)Jv_n dx\\ &= \lim_n \int_{\rho_n B} \phi(x, x/\rho_n)\rho_n^{-2} dx = \lim_n \int_B \phi(\rho_n y, y)dy\\ &= \int_B \phi(0,y)dy \quad \square\end{aligned}$$

Remark 2.7. Lemma 2.6 may be proved without GMT framework using a recent result in [12] about functions with given jacobian determinant. Take a smooth function $\phi : \Omega \to [0,1]$ with compact support such that $\phi(0) = 1$ and let $g : \mathbb{R}^2 \to [0, \infty]$ be the continuous function which agrees with f in B and is 0 elsewhere.
Following [12], we may find a Lipschitz function $G : \mathbb{R}^2 \to \overline{B}$ so that $\det DG = g$ everywhere and then, for every $u \in C^1$, $g(u)\,Ju = J(G(u))$, and

$$F(u) = \int_\Omega f(u)|Du|^2 dx \geq \int_\Omega g(u)\,Ju\,\phi\,dx = \int_\Omega J(G(u))\,\phi\,dx\ .$$

One can show that the map $u \to \langle\, \mathrm{Det}D(G(u)); \phi\rangle$ is sequentially weakly continuous on $W^{1,1}$ (cf. [17] and [9], section 4.2). Hence

$$F_2(C^1, W^{1,1}_w)(u_0) \geq \langle\, \mathrm{Det}D(G(u_0)); \phi\rangle\ .$$

Moreover we may compute explicitly the right term of this inequality taking suitable v_n converging to u_0 (e.g., as in formula (2.4)), and we get

$$F_2(C^1, W^{1,1}_w)(u_0) \geq 2\int_B g(y)dy = C\ .$$

Remark 2.8. It is important to notice the following fact: Lemmas 2.5 and 2.6 do not hold if we replace the $W^{1,1}$ weak topology with the BV topology or any other weaker topology (we recall that a sequence in $W^{1,1}$ converges in the BV topology if it converges in the L^1 norm and is bounded in the $W^{1,1}$ norm).
In particular we can find a sequence of C^∞ functions u_n which converges to u_0 in the BV topology and $F(u_n) = 0$ for all n (and then $\lim_n F(u_n) = F(u_0) = 0$). As usual, we assume $x_0 = 0$ and then $u_0(x) = x/|x|$.
Take $r > 0$ such that $rB \supset \Omega$ and set $K := [0, 2\pi] \times [0, r]$, and for all integers n, let $\phi_n : K \to [0,1]$ be a C^∞ function s.t.
(i) $\phi_n = 0$ in a neighborhood of ∂K,
(ii) $\phi_n = 1$ out of a set A_n with measure less than $1/n$,
(iii) $\|D\phi_n\|_1$ are uniformly bounded.

Now let u_n be the function which takes each point $x = (\rho\cos t, \rho\sin t)$ with $(t,\rho) \in K$ into

$$u_n(x) := \big(\cos(\phi_n(t,\rho)\,t), \sin(\phi_n(t,\rho)\,t)\big)\ .$$

Recalling (i), it is not difficult to verify that each function u_n is a well-defined C^∞ function from rB into S and then $F(u_n) = 0$.
By (ii), $u_n(x) = x/|x| = u_0(x)$ out of the set A'_n of all $x = (\rho\cos t, \rho\sin t)$ with $(t,\rho) \in A_n$ and since the measures of A_n converge to 0, the measures of A'_n converge to 0, and then u_n converge to u_0 in $L^1(\Omega,\mathbb{R}^2)$. Using (iii), a simple computation shows that the derivatives of u_n are bounded in the L^1 norm, and then we have convergence in the BV topology.

Notice that the proof of Lemma 2.5 does not work in the BV topology because Lemma 3.8 fails, and this happens essentially because the BV convergence in $W^{1,1}(0,1)$ does not imply uniform convergence (but the $W^{1,1}$ weak convergence does). The proof of Lemma 2.6 fails because Theorem 3.2 fails (cf. Remark 3.4).

Remark 2.9. Let T be $W^{1,1}$ endowed with the BV topology (or any weaker topology) and take u_0 as in Theorem 2.3: in the previous remark, we have showed that $F_1(C^\infty,T)(u_0) = F_2(C^\infty,T)(u_0) = F(u_0) = 0$ (and then, taking into account statement (iv) of Theorem 2.3, the same hold if we replace C^∞ with any dense subset of $W^{1,2}$). Thus the following question arises: is there a gap (for F) from C^∞ to T?

Remark 2.10. *(The higher dimension case)***.**
Let $N > 2$ be fixed. Let Ω be a bounded open subset of $\mathbb{R}^N$ and for all $u \in W^{1,1}(\Omega,\mathbb{R}^N)$ set

$$F(u) := \int_\Omega f(u)|Du|^N dx$$

where $f : \mathbb{R}^N \to [0,\infty[$ is a continuous function such that $f(y) = 0$ whenever $|y| = 1$. Let B be the unit ball of $\mathbb{R}^N$ and assume that $C := N^{N/2}\int_B f(y)dy > 0$. Take $x_0 \in \Omega$ and set $u_0(x) := (x - x_0)/|x - x_0|$ for all $x \in \Omega$ (u_0 belongs to $W^{1,p}$ for all $p < N$).

Then the essential statements of Theorem 2.3 may be generalized as follows (without essential modifications in the proofs): F is lower semicontinuous on $W^{1,1}$ with respect to the L^1 topology, and is continuous on $W^{1,N}$ with respect to the norm topology. Then $F_1(X,W^{1,p}) = F_2(X,W^{1,p}) = F$ whenever $p \geq N$ and X is a dense subset of $W^{1,p}$ (and then there is no gap from X to $W^{1,p}$).
If $p \in]N-1,N[$, $F_1(X,W^{1,p}_w)(u_0) = F_2(X,W^{1,p}_w)(u_0) = C$ while $F(u_0) = 0$ (X is a dense subset of $W^{1,N}$) and then there is a gap from X to $W^{1,p}_w$.
If $p \in [1,N-1]$, then $F_1(X,W^{1,p}_w)(u_0) = F_2(X,W^{1,p}_w)(u_0) = F(u_0) = 0$ whenever X is a dense subset of $W^{1,N}$: in particular we may find a sequence of C^∞ functions u_n which weakly converge to u_0 in $W^{1,N-1}$ such that $F(u_n) = 0$ for all n (cf. Remark 2.8); in this case we are not able to say whether there is gap or not (cf. Remark 2.9).

We end this section with an example of Lavrentiev phenomenon for some boundary value problems.

Definition 2.11. Let $\varepsilon > 0$ and $p \in]1,2[$ be given, and for all u in $W^{1,p}(B,\mathbb{R}^2)$, set

$$G(u) := \int_B \big[f(u)|Du|^2 + \varepsilon|Du - I|^p\big]dx \tag{2.9}$$

where $f : \mathbb{R}^2 \to [0,\infty[$ is taken as in Definition 2.2 (i.e., f is a continuous function such that $f(y) = 0$ whenever $|y| = 1$) and assume $C := 2\int_B f(y)dy > 0$. As usual we set $u_0(x) := x/|x|$ for all $x \in B$.
Let $C^1(\overline{B},\mathbb{R}^2)$ denote the class of all functions in $C^1(B,\mathbb{R}^2)$ which admit a continuous extension to $\overline{B}$.
Thus we have the following result.

Theorem 2.12. *G is a weakly lower semicontinuous functional on $W^{1,p}$ with p-growth, there exist*

$$M_1 := \min\Big\{G(u)\ :\ u \in W^{1,p}(B,\mathbb{R}^2),\ u = I \text{ on } \partial B\Big\}\ , \tag{2.10}$$

$$M_2 := \min\Big\{G(u)\ :\ u \in C^1(\overline{B},\mathbb{R}^2),\ u = I \text{ on } \partial B,\Big\}\ , \tag{2.11}$$

and we have $M_1 \leq G(u_0) \leq \dfrac{5\pi\varepsilon}{2-p}$, $M_2 = G(I) = C$. *Hence $M_1 < M_2$ if ε is small enough.*

Proof. G is weakly lower semicontinuous by well-known theorems. Since it has p-growth, it is coercive on the affine space of all functions $u \in W^{1,p}(B,\mathbb{R}^2)$ such that $u = I$ on ∂B, endowed with the weak topology of $W^{1,p}$. Hence G attains a minimum M_1 and

$$M_1 \leq G(u_0) = \varepsilon\int_B |Du_0 - I|^p dx \leq \frac{5\pi\varepsilon}{2-p}\ .$$

On the other side, $G(I) = \int_B f(y)\,2dy = C$, and then it is enough to prove that $G(u) \geq C$ for all $u \in C^1(\overline{B},\mathbb{R}^2)$ such that $u = I$ on ∂B. To this end, notice that for all such u,

$$\deg(u,B,y) = \deg(I,B,y) = \begin{cases} 1 & \text{if } y \in B, \\ 0 & \text{otherwise} \end{cases}$$

(cf. Theorem 3.1 in [11]) and then, using area formula,

$$\begin{aligned} G(u) &= \int_B f(u)|Du|^2 dx \geq 2\int_B f(u)Ju\,dx \\ &= 2\int_{\mathbb{R}^2} f(y)\deg(u,B,y)\,dy \geq 2\int_B f(y)dy = C \quad \square \end{aligned}$$

3. - Appendix

In this appendix we prove two results used above (Theorem 3.2 and Lemma 3.9). The first one is a theorem in current theory; $\mathscr{D}^k(A)$ is the (locally convex) space of all smooth k-forms with compact support in the open set A, and $\mathscr{D}_k(A)$ is the dual of $\mathscr{D}^k(A)$, i.e., the space of all k-dimensional currents on A. $\mathscr{D}_k(A)$ is always endowed with the dual topology. In this notation $\mathscr{D}^0(A)$ is the space of smooth functions on A with compact support and $\mathscr{D}_0(A)$ the space of Schwartz distributions. For the basic results and notation in current theory, we refer to [19].

Definition 3.1. When $u : \Omega \to \mathbb{R}^2$ is a C^1 function, then its graph is a 2-dimensional submanifold of class C^1 of $\Omega \times \mathbb{R}^2$ without boundary and we may orient it so that the orientation induced on Ω agrees with the standard orientation of $\mathbb{R}^2$. Thus we may see it as a current in $\Omega \times \mathbb{R}^2$ (more precisely, a 2-dimensional current without boundary) which we denote by Tu.
As usual, the standard basis $\mathscr{B}$ of the space of 2-covectors in $\mathbb{R}^2 \times \mathbb{R}^2$ is given by $dx^1 \wedge dx^2$, $dx^i \wedge dy^j$ (with $i = 1, 2$ and $j = 1, 2$) and $dy^1 \wedge dy^2$, and then every 2-form $\omega \in \mathscr{D}^2(\Omega \times \mathbb{R}^2)$ may be written as

$$\omega = \sum_{\alpha \in \mathscr{B}} \omega_\alpha \alpha \tag{3.1}$$

with $\omega_\alpha \in \mathscr{D}^0(\Omega \times \mathbb{R}^2)$. Moreover, for every $u \in C^1(\Omega, \mathbb{R}^2)$ and every $\phi \in \mathscr{D}^0(\Omega \times \mathbb{R}^n)$, we have

$$\langle Tu; \phi\, dx^1 \wedge dx^2 \rangle = \int_\Omega \phi(x, u)\, dx \tag{3.2}$$

$$\langle Tu; \phi\, dx^i \wedge dy^j \rangle = \int_\Omega \phi(x, u)(-1)^i \frac{\partial u^j}{\partial x_{\hat{\imath}}} dx \tag{3.3}$$

$$\langle Tu; \phi\, dy^1 \wedge dy^2 \rangle = \int_\Omega \phi(x, u) Ju\, dx \tag{3.4}$$

where $i = 1, 2$, $j = 1, 2$ and $\hat{\imath} = 1$ if $i = 2$, $\hat{\imath} = 2$ if $i = 1$.
Let $T : C^1(\Omega, \mathbb{R}^2) \to \mathscr{D}_2(\Omega \times \mathbb{R}^2)$ be the map which takes each u in Tu: we claim that T admits a continuous extension to the space $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$. More precisely, we have the following theorem.

Theorem 3.2. *Let T be the function given before. Then there exists a unique sequentially weakly continuous map (which we still denote as T) from $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ into $\mathscr{D}_2(\Omega \times \mathbb{R}^2)$ which extends T.*

Remark 3.3. Uniqueness is obvious, because C^1 is dense in $W^{1,1}_{\text{loc}}$. To prove existence, it is enough to show that for every 2-form $\omega \in \mathscr{D}^2(\Omega, \mathbb{R}^2)$ the map which takes each $u \in C^1$ into $\langle Tu; \omega \rangle$ admits a continuous extension to $W^{1,1}_{\text{loc}}$.
In fact, we shall prove that this extension exists for all 2-forms ω with compact support of class C^1, and not only C^∞, and taking into account the decomposition (3.1) and formulas (3.2), ..., (3.4), it is enough to extend to $W^{1,1}_{\text{loc}}$ the map

$$u \mapsto \int_\Omega \phi(x, u)\, Mu\, dx \tag{3.5}$$

whenever $\phi \in C^1_c(\Omega \times \mathbb{R}^2)$ and Mu is a minor of the matrix Du.
It is well-known how extend (3.5) to the space $W^{1,2}_{\text{loc}}$, because the same formula makes sense for all functions in this space and gives a sequentially weakly continuous functional (see [9], section 4.2), and moreover in this case Tu is an integer multiplicity current without boundary by a well-known compactness theorem (see [19], Theorem 27.3).
In the general case, formula (3.5) does not make sense because the minor of order 2 of Du, namely the Jacobian determinant, is not given for $W^{1,1}$ functions. In this case Tu may be not a locally rectifiable current and may have not locally bounded mass (but still it has no boundary); more precisely we show that for all $u \in W^{1,1}_{\text{loc}}$, Tu actually belongs to the dual of all 2-forms of class C^1 (and not only C^∞) with compact support, but by no means to the dual of all continuous 2-forms with compact support.

Remark 3.4. The (sequential) continuity of the extension does not hold if we replace in $W^{1,1}_{\text{loc}}$ the weak topology with the BV topology (cf. Remark 2.8) or any other weaker topology.
With no loss in generality we may assume that 0 belongs to Ω and then set $u_0 = x/|x|$ for all $x \in \Omega$. Now take u_n as in Remark 2.8 and v_n as in formula (2.4): they are both sequences of Lipschitz functions which converge to u_0 in the BV topology, but Tu_n and Tv_n do not converge to the same current.
Indeed, if we take $\omega := \phi\, dy^1 \wedge dy^2$ where $\phi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth functions with compact support such that

(i) $\phi(x, y) = 1$ in a neighborhood of $\{0\} \times \frac{1}{2}B$,
(ii) $\phi(x, y) = 0$ whenever $|y| = 1$,

an explicit computation shows that $\langle Tu_n; \omega \rangle = 0$ for all n and $\langle Tv_n; \omega \rangle \geq \pi/4$ definitively.

Remark 3.5. Of course, the result of Theorem 3.2 can be extended (without any essential modifications in the proof) to arbitrary dimensions: if Ω is an open subset of $\mathbb{R}^N$, and T is the map which takes each function $u \in C^1(\Omega, \mathbb{R}^M)$ in its (oriented) graph, considered as an N-dimensional current of $\Omega \times \mathbb{R}^M$ (N, M positive integers greater than 1), then we may find a unique continuos extension of T to the space $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^M)$ for all $p \geq N \wedge M - 1$. This extension is (sequentially) weakly continuous unless $p = N \wedge M - 1$ and $N \wedge M > 2$; in this last case we have the strong continuity only. If $N \wedge M = 2$ and $p = 1$, we have (sequential) continuity with respect to the $W^{1,1}_{\text{loc}}$ weak topology but not with respect to the BV topology (or any other weaker topology). If $p < N \wedge M - 1$, there is no continuous extension.

Proof of Theorem 3.2

Taking into account Remark 3.3, we shall prove that the map $u \mapsto \int_\Omega \phi(x, u) Mu\, dx$ may be extended to $W^{1,1}_{\text{loc}}$ whenever $\phi \in C^1_c(\Omega \times \mathbb{R}^2)$ and Mu is a minor of Du.
When we consider minors of order 1 or 0, it is enough to recall that the map

$$u \mapsto \int_\Omega \phi(x, u) Mu\, dx \quad u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \tag{3.6}$$

is a (well-defined) continuous map for every $\phi \in C_c(\Omega \times \mathbb{R}^2)$ (in fact, for every bounded continuous ϕ with support included in $K \times \mathbb{R}^2$ for some compact $K \subset \Omega$, see Lemma 3.6).
When we consider the minor of order 2 (i.e., the Jacobian determinant) there is no trivial extension of the function because Ju is not given in general for functions in $W^{1,1}$. Using Lemma 3.8 below, for all $\phi \in C^1_c(\Omega \times \mathbb{R}^2)$ we may find a bounded $R\phi \in C(\Omega \times \mathbb{R}^2, \mathbb{R}^2)$, with support included in $K \times \mathbb{R}^2$ for some compact $K \subset \Omega$, such that

$$\int_\Omega \phi(x,u) Ju\, dx = \int_\Omega R\phi(x,u) \cdot Du\, dx \quad \forall u \in C^1(\Omega, \mathbb{R}^2) \tag{3.7}$$

and the right term of this equality is a function which can be easily extended to the whole space $W^{1,1}_{\mathrm{loc}}$ using Lemma 3.6. □

Lemma 3.6. *The following facts hold.*
(i) *For every bounded continuous function ϕ on $\Omega \times \mathbb{R}^2$, the mapping*

$$u \mapsto \int_\Omega \phi(x,u)\, dx \quad u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$$

is sequentially weakly continuous.
(ii) *For every bounded continuous function ϕ on $\Omega \times \mathbb{R}^2$ with support included in $K \times \mathbb{R}^2$ for some compact $K \subset \Omega$, the mapping*

$$u \mapsto \int_\Omega \phi(x,u) Du\, dx \quad u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$$

is sequentially weakly continuous.

Proof.
(i). Let u_n be a sequence in $W^{1,1}_{\mathrm{loc}}$ which weakly converges to u. Then it strongly converges in L^1_{loc} and hence, eventually taking a subsequence, we may assume that u_n converges to u a.e.. Thus $\phi(x, u_n(x))$ converges to $\phi(x, u(x))$ for a.a. x because ϕ is continuous, and it is enough to apply Lebesgue theorem.
(ii). Let u_n be a sequence in $W^{1,1}_{\mathrm{loc}}$ which weakly converges to u and let A be a relatively compact open subset of Ω which includes K and set $v_n(x) = \phi(x, u_n(x))$, $v(x) = \phi(x, u(x))$ for all x and n. As before, we may assume that v_n converges a.e. to v. Moreover the functions v_n are uniformly bounded and 0 outside A. Since Du_n weakly converges to Du in L^1_{loc}, it is well-known that $v_n Du_n$ weakly converges to $v\, Du$. □

Lemma 3.7. *Let α be a function in $C^1(\Omega \times \mathbb{R}^2)$ with support included in $K \times \mathbb{R}^2$ for some compact $K \subset \Omega$, and let $u = (u^1, u^2)$ be a function in $C^1(\Omega, \mathbb{R}^2)$. Then we have the following identity:*

$$\int_\Omega \Big[\alpha(x,u) + u^2 \frac{\partial\alpha}{\partial y_2}(x,u)\Big] Ju\, dx = \int_\Omega \Big(u^2 \frac{\partial\alpha}{\partial x_2}(x,u), -u^2 \frac{\partial\alpha}{\partial x_1}(x,u)\Big) \cdot Du\, dx \quad . \tag{3.8}$$

Proof. A simple density argument shows that we may assume α and u of class C^∞. Now, writing Ju as a divergence (cf. [17] and [9], section 4.2)

$$Ju = \frac{\partial}{\partial x_1}\Big(-u^2 \frac{\partial u^1}{\partial x_2}\Big) + \frac{\partial}{\partial x_2}\Big(u^2 \frac{\partial u^1}{\partial x_1}\Big) \tag{3.9}$$

and integrating by parts $\alpha(x,u)Ju$ we obtain (with some computations)

$$\begin{aligned}\int_\Omega \alpha(x,u) Ju\, dx &= \int_\Omega \Big[\frac{\partial}{\partial x_1}(\alpha(x,u))\, u^2 \frac{\partial u^1}{\partial x_2} - \frac{\partial}{\partial x_2}(\alpha(x,u))\, u^2 \frac{\partial u^1}{\partial x_1}\Big] dx \\ &= \int_\Omega \Big(u^2 \frac{\partial\alpha}{\partial x_2}(x,u), -u^2 \frac{\partial\alpha}{\partial x_1}(x,u)\Big) \cdot Du\, dx - \\ &\quad - \int_\Omega \Big[u^2 \frac{\partial\alpha}{\partial y_2}(x,u)\Big] Ju\, dx\ . \quad \square\end{aligned}$$

Lemma 3.8. *For every $\phi \in C^1_c(\Omega \times \mathbb{R}^2)$ there exists a bounded $R\phi \in C(\Omega \times \mathbb{R}^2, \mathbb{R}^2)$ with support included in $K \times \mathbb{R}^2$ for some compact $K \subset \Omega$ which satisfies*

$$\int_\Omega \phi(x,u) Ju\, dx = \int_\Omega R\phi(x,u) \cdot Du\, dx \quad \forall u \in C^1(\Omega, \mathbb{R}^2)\ . \tag{3.10}$$

Proof. Let K be the projection on Ω of the support of ϕ and set

$$\alpha(x,y) := \int_0^1 \phi(x, y_1, t y_2) dt \quad \forall (x,y) \in \Omega \times \mathbb{R}^2\ . \tag{3.11}$$

A simple computation shows that α is a function of class C^1 with support included in $K \times \mathbb{R}^2$ which satisfies the equation $\phi = \alpha + y_2 \frac{\partial\alpha}{\partial y_2}$. Hence formula (3.8) yields

$$\int_\Omega \phi(x,u) Ju\, dx = \int_\Omega \Big(u^2 \frac{\partial\alpha}{\partial x_2}(x,u), -u^2 \frac{\partial\alpha}{\partial x_1}(x,u)\Big) \cdot Du\, dx\ , \tag{3.12}$$

and then it is enough to take

$$R\phi := \Big(y_2 \frac{\partial\alpha}{\partial x_2}, -y_2 \frac{\partial\alpha}{\partial x_1}\Big) \quad . \tag{3.13}$$

Formulas (3.11) and (3.13) immediately yield that $R\phi$ is bounded and continuous, with support included in $K \times \mathbb{R}^2$. □

Lemma 3.9. *Let (u_n) be a sequence of functions in $Lip(\Omega, \mathbb{R}^2)$ which weakly converges to u in $W^{1,1}$, and take r such that $rB \subset \Omega$. Then for almost all $\rho \in]0, r[$, $(u_n \llcorner \rho S)$ admits a subsequence which converges to $u \llcorner \rho S$ uniformly.*

Proof. Since the functions u_n weakly converge to u in $W^{1,1}(\Omega,\mathbb{R}^2)$, they converge also in L^1 and the functions Du_n are uniformly integrable in L^1 (by Dunford-Pettis theorem). Hence there exist a finite constant C and a convex function $f:[0,+\infty]\to[0,+\infty]$ such that $\lim\limits_{t\to\infty} f(t)/t=+\infty$ and

$$\int_\Omega f(|Du_n|)dx \le C \quad \forall n \tag{3.14}$$

(cf. [13], chapter II, section 2). For all n and all $\rho\in]0,r[$ set

$$\begin{aligned} u_n^\rho &:= u_n \llcorner \rho S, \quad u^\rho := u \llcorner \rho S \\ g_n(\rho) &:= \int_{\rho S} |u_n-u|\, d\mathscr{H}_1 \\ h_n(\rho) &:= \int_{\rho S} f(|Du_n|)\, d\mathscr{H}_1 \ . \end{aligned}$$

Then $\int_0^r g_n(\rho)\,d\rho \le \|u_n-u\|_1$ and g_n converges to 0 in $L^1(0,r)$. Hence, eventually passing to a subsequence, we may assume that, for a.a. ρ, $g_n(\rho)$ converges to 0, i.e., u_n^ρ converge to u^ρ in $L^1(\rho S,\mathbb{R}^2)$. Moreover (3.14) yields

$$\int_0^r h_n(\rho)\,d\rho \le \int_\Omega f(|Du_n|)dx \le C \quad \forall n$$

and then Fatou's lemma yields

$$\int_0^r \liminf_{n\to\infty} h_n(\rho)\,d\rho \le \liminf_{n\to\infty} \int_0^r h_n(\rho)\,d\rho \le C \ .$$

Thus, for a.a. ρ, $u_n^\rho\to u^\rho$, $\liminf_n h_n(\rho)<\infty$, and we may find a subsequence n_k so that $h_{n_k}(\rho)$ is bounded. Hence $u_{n_k}^\rho$ is a sequence of functions in $C^1(\rho S,\mathbb{R}^2)$ which converges to u^ρ in L^1 with uniformly integrable derivatives (cf [13], chapter II, section 2) and so we have convergence in the weak topology of $W^{1,1}$, which yields uniform convergence. □

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