A non-local anisotropic model for phase transitions: asymptotic behavior of rescaled energies

GIOVANNI ALBERTI and GIOVANNI BELLETTINI

Dipartimento di Matematica Applicata "U. Dini", via Bonanno 25/B, I-56126 Pisa, Italy

In this paper we consider a nonlocal anisotropic model for phase separation in two-phase fluids at equilibrium, and show that when the thickness of the interface tends to zero in a suitable way, the classical surface tension model is recovered. Relevant examples are given by continuum limits of ferromagnetic Ising systems in equilibrium statistical mechanics.

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1. Introduction

We consider a fluid in a container Ω and assume that every configuration of the system is described by a function $u : \Omega \to \mathbb{R}$ which represents the (macroscopic) density of a scalar intrinsic quantity, and the corresponding free energy is given by

$$E(u) := \frac{1}{4} \int_{\Omega\times\Omega} J(x'-x)\big(u(x')-u(x)\big)^2 dx'dx \ + \int_\Omega W\big(u(x)\big)\, dx \ ,$$

where J is a positive interaction potential which vanishes at infinity and W is double-well potential which vanishes at ± 1 only (see paragraph 1.2 for precise assumptions).

If we consider an energy minimizing configuration u, the second term in E forces u to take values close to the "pure" states $+1$ and -1 (phase separation), while the first term represents an interaction energy which penalizes the spatial inhomogeneity of u (surface tension). Examples of this model are given in equilibrium statistical mechanics by continuum limits of Ising spin systems on lattices; in that setting, u represents a macroscopic magnetization density and J is a ferromagnetic Kac potential (cf. [ABCP] and references therein).

When the potential W is large in comparison with J and we minimize E subject to the mass constraint $\int u = c$, the second term in E prevails: the minimizer takes values close to -1 or $+1$, and the transition between the two phases occurs in a thin layer. This situation can be studied by passing to the thermodynamic limit, that is, studying the asymptotic behavior as $\varepsilon \to 0$ of the rescaled energies

$$F_\varepsilon(u,\Omega) := \frac{1}{4\varepsilon} \int_{\Omega\times\Omega} J_\varepsilon(x'-x)\big(u(x')-u(x)\big)^2 dx'dx \ + \frac{1}{\varepsilon}\int_\Omega W\big(u(x)\big)\, dx \ , \tag{1.1}$$

where ε is a positive scaling parameter, and $J_\varepsilon(y) := \varepsilon^{-N} J(y/\varepsilon)$.

We note that this model closely resembles the Cahn–Hilliard model for phase separation (see [CH]), which is described by the energy functional

$$I_\varepsilon(u) := \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega W(u)\ . \tag{1.2}$$

Indeed, F_ε can be obtained from I_ε by replacing the term $\big|\nabla u(x)\big|$ in the first integral in (1.2) with the average of the finite differences $\frac{1}{\varepsilon}\big|u(x+\varepsilon h)-u(x)\big|$ with respect to the measure distribution $J(h)\,dh$. Modica and Mortola proved [MM] (see also [Mo]) that in the limit $\varepsilon \to 0$ the functionals I_ε converge in a suitable sense to a limit energy I which is finite only when $u = \pm 1$ almost everywhere, and in that case is given by the area of the interface Su which separates the phases $\{u = +1\}$ and $\{u = -1\}$ multiplied by a positive surface tension σ.

It follows immediately from the convergence of the energies that the minimizers of I_ε with prescribed total mass converge to minimizers of I, that is, functions $u : \Omega \to \pm 1$ which minimize the area of the interface Su. In this sense the classical model for phase separation (due to van der Waals) can be derived from the Cahn–Hilliard model in the limit $\varepsilon \to 0$. This result was later extended to more general anisotropic functionals in [Bou], [OS], [BF].

A first result in this direction for the functionals F_ε in (1.1) was proved in [ABCP] for a particular choice of W and a radially symmetric J, that is, in the isotropic case. The limit energy F has the same form as I, only the expression for the surface tension σ is different. In this paper we extend this result to the anisotropic case; more precisely we prove (see Theorem 1.4 and following remarks) that when $\varepsilon \to 0$ the functionals F_ε converge to a limit energy F which is finite only when $u = \pm 1$ a.e., and in that case is given by the area of the interface Su weighted by an anisotropic surface tension σ (cf. (1.10)). As before, the convergence of the energies immediately implies that the minimizers of F_ε with prescribed mass converge to minimizers of F.

Beyond the relevance of the specific model we consider, our result suggests the following consideration: the term $\int |\nabla u|^2$ in (1.2) was derived in [CH] as a first order approximation of a more general and complicated quadratic form, but our theorem suggests that the form of the limit energy is largely independent of the choice of this quadratic form; a first result in this direction was given in [ABS], but no general result is available so far.

In this paper we follow a different approach from [ABCP]; for instance, the existence of the optimal profiles for transitions (cf. [AB1]) plays no rôle in the proof. In particular, it is also possible to extend Theorem 1.4 to the multi-phase case (that is, when u takes values in $\mathbb{R}^m$ and W is a function on $\mathbb{R}^m$ which vanishes at finitely many affinely independent points) even though there are no existence results for the optimal profile (see paragraph 1.12). On the other hand, the assumption $J \geq 0$, which in the statistical model is called the *ferromagnetic assumption*, is crucial in all our proofs (cf. paragraph 1.13), and even if Theorem 1.4 holds also for some potentials J with a small negative part (see [AB2]), the problem of understanding what happens in the general case seems still largely open; indeed a quite different asymptotic behavior is expected, since in particular the ground states associated with the unscaled energy E may be not constant.

We finally recall that the evolution model associated with the energy F_ε is described, after a suitable time scaling, by the nonlocal parabolic equation $u_t = \varepsilon^{-2}\big(J_\varepsilon * u - u - f(u)\big)$ where f is the derivative of W and it is assumed $\|J\|_1 = 1$; the analogue for the energy I_ε is the scaled Allen–Cahn equation $u_t = \triangle u - \varepsilon^{-2} f(u)$. The asymptotic behavior of the solutions of this equation has been widely studied and leads to a motion by mean curvature in the sense of viscosity solutions (see, for instance, [DOPT1-3], [KS1-2] for the isotropic nonlocal equation, and [KS3] for the anisotropic case; see [BK], [DS], [ESS], [Ilm] for the Allen–Cahn equation).

This paper is organized as follows. In the rest of this section, we first give some definitions, and then state the convergence result for the functionals F_ε (Theorem 1.4) and briefly discuss some immediate consequences. In paragraph 1.9 we outline the idea of the proof in the one-dimensional case; in paragraph 1.12 we consider the generalization of Theorem 1.4 to the multi-phase case, while in paragraph 1.13 we discuss the assumptions on J and W.

In section 2 we study the decay of the *locality defect* $\Lambda_\varepsilon(u, A, A')$, which is defined by

$$\Lambda_\varepsilon(u, A, A') := \frac{1}{4\varepsilon}\int_{A\times A'} J_\varepsilon(x'-x)\big(u(x')-u(x)\big)^2 dx'dx \tag{1.3}$$

for every $A, A' \subset \mathbb{R}^N$ and every $u : A \cup A' \to \mathbb{R}$. The functionals F_ε are not local, in the sense that the energy $F_\varepsilon(u, A\cup A')$ stored in $A \cup A'$ is strictly larger than the sum of $F_\varepsilon(u, A)$ and $F_\varepsilon(u, A')$ when A and A' are disjoint, and more precisely we have

$$F_\varepsilon(u, A\cup A') = F_\varepsilon(u, A) + F_\varepsilon(u, A') + 2\Lambda_\varepsilon(u, A, A')\ . \tag{1.4}$$

To guarantee that $\Lambda_\varepsilon(u, A, A')$ vanishes as $\varepsilon \to 0$ whenever the distance between A and A' is strictly positive, we must assume a proper decay of J at infinity, namely (1.6) (see however paragraph 1.13).

Sections 3, 4, and 5 are devoted to the proof of Theorem 1.4.

Before passing to precise statements, we fix some notation. In the following Ω is a bounded open subset of $\mathbb{R}^N$, and it is called *regular* when it has a Lipschitz boundary (for $N = 1$, when it is a finite union of distant open intervals). Unless otherwise stated all sets and functions are assumed to be Borel measurable.

Every set in $\mathbb{R}^N$ is usually endowed with the Lebesgue measure $\mathscr{L}_N$, and we simply write $\int_B f(x)\,dx$ for the integrals over B and $|B|$ for $\mathscr{L}_N(B)$, while we never omit explicit mention of the measure when it differs from $\mathscr{L}_N$. As usual $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

1.1. BV functions and sets of finite perimeter

For every open set Ω in $\mathbb{R}^N$, $BV(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ *with bounded variation*, that is, the functions $u \in L^1(\Omega)$ whose distributional derivative Du is represented by a bounded $\mathbb{R}^N$-valued measure on Ω. We denote by $BV(\Omega, \pm 1)$ the class of all $u \in BV(\Omega)$ which take values ± 1 only. For every function u on Ω, Su is the set of all essential singularities, that is, the points of Ω where u has no approximate limit (in the measure theoretic sense, cf. [EG], chapter 5); if $u \in BV(\Omega)$ the set Su is *rectifiable*, and this means that it can be covered up to an $\mathscr{H}^{N-1}$-negligible subset by countably many hypersurfaces of class C^1.

The essential boundary of a set $E \subset \mathbb{R}^N$ is the set $\partial_* E$ of all points in Ω, where E has density neither unity nor zero. A set $E \subset \Omega$ has finite perimeter in Ω if its characteristic function 1_E belongs to $BV(\Omega)$, or equivalently, if $\mathscr{H}^{N-1}(\partial_* E \cap \Omega)$ is finite; in this case

$\partial_* E$ is rectifiable, and we may endow it with a measure theoretic normal ν_E (defined up to $\mathscr{H}^{N-1}$-negligible subsets) so that the measure derivative $D1_E$ is represented as

$$D1_E(B) = \int_{\partial_* E \cap B} \nu_E \, d\mathscr{H}^{N-1} \quad \text{for every } B \subset \Omega.$$

A function $u : \Omega \to \pm 1$ belongs to $BV(\Omega, \pm 1)$ if and only if $\{u = +1\}$ (or $\{u = -1\}$ as well) has finite perimeter in Ω. In this case, Su agrees with the intersection of the essential boundary of $\{u = +1\}$ with Ω, and the previous formula becomes

$$Du(B) := 2 \int_{Su \cap B} \nu_u \, d\mathscr{H}^{N-1} \quad \text{for every } B \subset \Omega, \tag{1.5}$$

where ν_u is a suitable normal field to Su. We claim that Su is the interface between the phases $\{u = +1\}$ and $\{u = -1\}$ in the sense that it contains every point where both sets have density different from zero. For further results and details about BV functions and finite perimeter sets, we refer the reader to [EG], chapter 5.

1.2. Hypotheses on J and W

Unless otherwise stated, the interaction potential J and the double-well potential W which appear in (1.1) satisfy the following assumptions:

(i) $J : \mathbb{R}^N \to [0, +\infty)$ is an even function (i.e., $J(h) = J(-h)$) in $L^1(\mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} J(h) \, |h| \, dh < \infty \ . \tag{1.6}$$

(ii) $W : \mathbb{R} \to [0, +\infty)$ is a continuous function which vanishes at ± 1 only, and has at least linear growth at infinity (cf. the proof of Lemma 1.14).

1.3. The optimal profile problem and the surface tension σ

We first define the auxiliary unscaled functional $\mathscr{F}$ by

$$\mathscr{F}(u, A) := \frac{1}{4} \int_{x \in A,\, h \in \mathbb{R}^N} J(h) \big(u(x+h) - u(x) \big)^2 dx \, dh \ + \int_{x \in A} W\big(u(x)\big) \, dx \tag{1.7}$$

for every set $A \subset \mathbb{R}^N$ and every $u : \mathbb{R}^N \to \mathbb{R}$. Hence $\mathscr{F}(u, A) = F_1(u, A) + \Lambda_1(u, A, \mathbb{R}^N \setminus A)$.

We fix now a unit vector $e \in \mathbb{R}^N$ and we denote by M the orthogonal complement of e. Hence, every $x \in \mathbb{R}^N$ can be written as $x = y + x_e e$ where y is the projection of x on M and $x_e := \langle x, e \rangle$.

We denote by $\mathscr{C}_e$ the class of all $(N-1)$-dimensional cubes centered at 0 which lie on M; for every $C \in \mathscr{C}_e$, T_C is the strip $T_C := \big\{ y + te : \ y \in C, \, t \in \mathbb{R} \big\}$, while Q_C is the N-dimensional cube centered at 0 such that $Q_C \cap M = C$. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called C-periodic if $u(x + re_i) = u(x)$ for every x and every $i = 1, \dots, N-1$, where r is the side length of C and $e_1, \dots, e_{N-1}$ are its axes. We denote by $X(C)$ the class of all functions $u : \mathbb{R}^N \to [-1, 1]$ which are C-periodic and satisfy

$$\lim_{x_e \to +\infty} u(x) = +1 \quad \text{and} \quad \lim_{x_e \to -\infty} u(x) = -1 \ , \tag{1.8}$$

and finally we set

$$\sigma(e) := \inf \big\{ |C|^{-1} \mathscr{F}(u, T_C) : \, C \in \mathscr{C}_e, \, u \in X(C) \big\} \ . \tag{1.9}$$

The minimum problem (1.9) is called the *optimal profile problem* associated with the direction e, and a solution is called an *optimal profile for transition in direction e*. In [AB1] it was proved that the minimum in (1.9) is attained, and there exists at least one minimizer u which depends only on the variable x_e, and more precisely, $u(x) = \gamma(x_e)$ where $\gamma : \mathbb{R} \to [-1, 1]$ is the optimal profile associated with a certain one-dimensional functional F^e. However, we emphasize that the proof of Theorem 1.4 does not depend upon this existence result (see Remark 1.10).

For the rest of this section, Ω is a fixed regular open subset of $\mathbb{R}^N$; the functionals F_ε are defined in (1.1), while the limit functional F is given by

$$F(u) := \int_{Su} \sigma(\nu_u) \, d\mathscr{H}^{N-1} \quad \text{for } u \in BV(\Omega, \pm 1). \tag{1.10}$$

Theorem 1.4. *Under the previous assumptions the following three statements hold:*

(i) Compactness: *let sequences (ε_n) and $(u_n) \subset L^1(\Omega)$ be given such that $\varepsilon_n \to 0$, and $F_{\varepsilon_n}(u_n, \Omega)$ is uniformly bounded; then the sequence (u_n) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$.*

(ii) Lower bound inequality: *for every $u \in BV(\Omega, \pm 1)$ and every sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq F(u) \ ;$$

(iii) Upper bound inequality: *for every $u \in BV(\Omega, \pm 1)$ there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \leq F(u) \ .$$

Remark 1.5. Statements (ii) and (iii) of Theorem 1.4 can be rephrased by saying that the functionals $F_\varepsilon(\cdot, \Omega)$, or in short F_ε, Γ-converge in the space $L^1(\Omega)$ to the functional F given by (1.10) for all functions $u \in BV(\Omega, \pm 1)$ and extended to $+\infty$ in $L^1(\Omega) \setminus BV(\Omega, \pm 1)$.

For the general theory of Γ-convergence, we refer the reader to [DM]; for the applications of Γ-convergence to phase transition problems, we refer to the early paper of Modica [Mo], and to [Al] for a review of some results and the related mathematical issues.

Remark 1.6. As every Γ-limit is lower semicontinuous, we infer from the previous remark that the functional F given in (1.10) is weakly* lower semicontinuous and coercive on $BV(\Omega, \pm 1)$.

The coercivity of F implies that the infimum of $\sigma(e)$ over all unit vectors $e \in \mathbb{R}^N$ is strictly positive, while the semicontinuity implies that the 1-homogeneous extension of the function σ to $\mathbb{R}^N$, namely the function $x \mapsto |x| \, \sigma\big(x / |x|\big)$, is convex (see, for instance, [AmB], Theorem 3.1). Notice that it is not immediate to recover this convexity result directly from the definition of σ in (1.9).

Remark 1.7. Statement (iii) of Theorem 1.4 can be refined by choosing the approximating sequence (u_ε) so that $\int_\Omega u_\varepsilon = \int_\Omega u$ for every ε (we will not prove this refinement of statement (iii); in fact one has to slightly modify the construction of the approximating sequence (u_ε) given in Theorem 5.2). In this way we can fit with a prescribed mass constraint: given c such that $|c| \le |\Omega|$, then the functionals F_ε also Γ-converge to F on the class Y_c of all $u \in L^1(\Omega)$ which satisfy the mass constraint $\int_\Omega u = c$.

A sequence (v_ε) in Y_c is called a *minimizing sequence* if v_ε minimizes $F_\varepsilon(\cdot,\Omega)$ in Y_c for every $\varepsilon > 0$, and is called a *quasi-minimizing sequence* if $F_\varepsilon(v_\varepsilon,\Omega) = \inf\big\{F_\varepsilon(u,\Omega) : u \in Y_c\big\} + o(1)$. Using the semicontinuity result given in [AB1], Theorem 4.7, and the truncation argument given in Lemma 1.14 below, we can prove that a minimizer of $F_\varepsilon(\cdot,\Omega)$ in Y_c exists provided that W is of class C^2 and $\ddot W(t) \ge -d_\varepsilon$ for every $t \in [-1,1]$, where d_ε is defined by

$$d_\varepsilon := \operatorname*{ess\,inf}_{x\in\Omega} \frac{1}{2}\int_\Omega J_\varepsilon(x'-x)\,dx' \ .$$

Notice that d_ε tends to $\frac{1}{4}\|J\|_1$ as $\varepsilon \to 0$.

By a well-known property of Γ-convergence and statement (i) of Theorem 1.4 we infer the following (cf. [DM], chapter 7):

Corollary 1.8. *Let (v_ε) be a minimizing or a quasi-minimizing sequence for F_ε on Y_c. Then (v_ε) is relatively compact in $L^1(\Omega)$, and every cluster point v minimizes F among all functions $u \in BV(\Omega,\pm 1)$ which satisfy $\int_\Omega u = c$. Equivalently, the set $E := \{v = 1\}$ solves the minimum problem*

$$\min\Big\{\int_{\partial_* E}\sigma(\nu_E)\,d\mathscr{H}^{N-1} : \ E \text{ has finite perimeter in } \Omega \text{ and } |E| = \tfrac{1}{2}(c+|\Omega|)\Big\} \ .$$

1.9. Outline of the proof of Theorem 1.4 for N = 1

To explain the idea of the proof of Theorem 1.4 and the connection with the optimal profile problem, we now briefly sketch the proof of statement (ii) and (iii) for the one-dimensional case (the proof of statement (i) being slightly more delicate).

In this case, σ becomes the infimum of $F_1(\cdot,\mathbb{R})$ over the class X of all $u : \mathbb{R} \to [-1,1]$ which converge to $+1$ at $+\infty$ and to -1 at $-\infty$ (cf. (1.9)). We assume for simplicity that Ω is the interval $(-1,1)$, and that $u(x) = -1$ for $x < 0$, $u(x) = +1$ for $x \ge 0$. Then $Su = \{0\}$, and $\sigma\mathscr{H}^0(Su) = \sigma$; a standard localization argument can be used to prove the result in the general case (cf. [Al], section 3a).

We first note that the functionals F_ε satisfy the following rescaling property: given $\varepsilon > 0$ and $u : \mathbb{R} \to \mathbb{R}$ we set $u^\varepsilon(x) := u(\varepsilon x)$, and then a direct computation gives

$$F_\varepsilon(u,\mathbb{R}) = F_1(u^\varepsilon,\mathbb{R}) \ . \tag{1.11}$$

Let us now consider the lower bound inequality. First, we reduce to a sequence (u_ε) which converges to u in $L^1(\Omega)$ and satisfies $|u_\varepsilon| \le 1$; then we extend each u_ε to the rest of $\mathbb{R}$ by setting $u_\varepsilon(x) := -1$ for $x \le -1$, $u_\varepsilon(x) := 1$ for $x \ge 1$. The key point of the proof is to show that

$$F_\varepsilon(u_\varepsilon,\Omega) \simeq F_\varepsilon(u_\varepsilon,\mathbb{R}) \quad \text{as } \varepsilon \to 0. \tag{1.12}$$

By identity (1.4), (1.12) can be written in term of the locality defect Λ_ε (see (1.3)), and more precisely, it reduces to $\Lambda_\varepsilon(u_\varepsilon,\Omega,\mathbb{R}\setminus\Omega) = o(1)$; notice that in general this equality may be false, but using the decay estimates for the locality defect given in section 2 we can prove that it is true if we replace Ω with another interval, which may be chosen arbitrarily close to Ω.

By (1.11) and the definition of σ we get $F_\varepsilon(u_\varepsilon,\mathbb{R}) = F_1(u^\varepsilon_\varepsilon,\mathbb{R}) \ge \sigma$, and then (1.12) yields

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega) \ge \sigma \ .$$

The proof of the upper bound inequality is even more simple: we take an optimal profile γ (i.e., a solution of the minimum problem which defines σ) and we set $u_\varepsilon(x) := \gamma(x/\varepsilon)$ for every $\varepsilon > 0$. Then $u_\varepsilon(x)$ converge to $u(x)$ for every $x \ne 0$, and (1.11) yields

$$F_\varepsilon(u_\varepsilon,\Omega) \le F_\varepsilon(u_\varepsilon,\mathbb{R}) = F_1(\gamma,\mathbb{R}) = \sigma \ .$$

Remark 1.10. It is clear from this brief sketch that the shape of the optimal profile plays no rôle in the proof of Theorem 1.4, nor does the fact that the minimum in (1.9) is attained: in case no optimal profiles were available, it would suffice to replace γ with functions in X which “almost” minimize $F_1(\cdot,\mathbb{R})$. This could be indeed the case when one considers the vectorial version of this problem (see paragraph 1.12)

Nevertheless, the existence of the optimal profile has a deeper meaning than appears above. Indeed if (v_ε) is a sequence of minimizers of F_ε which converges to some $v \in BV(\Omega,\pm 1)$, then we would expect that if we blow-up the functions v_ε at some fixed singular point $\bar x$ of v by taking the functions $\gamma_\varepsilon(x) := v_\varepsilon(\varepsilon(x-\bar x))$, then γ_ε more and more resembles an optimal profile. In other words we expect the optimal profiles to be the asymptotic shapes of the minimizers v_ε about the discontinuity points of v. Yet a precise statement in this direction is beyond the scope of this paper.

1.12. The multi-phase model

To describe a multi-phase system, one may postulate a free energy of the form (1.1) where u is a *vector* density function on a domain of $\mathbb{R}^N$ taking values in $\mathbb{R}^m$, $W : \mathbb{R}^m \to [0,\infty)$ is a continuous function which vanishes at $k+1$ affinely independent wells $\{\alpha_0,\dots,\alpha_k\}$ (and therefore $k \le m$), and J is the usual interaction potential.

Theorem 1.4 holds provided we make the following modifications: $BV(\Omega,\pm 1)$ is replaced by the class $BV\big(\Omega,\{\alpha_i\}\big)$ of all functions $u \in BV(\Omega,\mathbb{R}^m)$ which takes values in $\{\alpha_0,\dots,\alpha_k\}$ only, and the functional F is now defined by

$$F(u) := \sum_{i<j}\int_{S_{ij}} \sigma_{ij}(\nu_{ij})\,d\mathscr{H}^{N-1} \ , \tag{1.13}$$

where S_{ij} is the interface which separates the phases $\{u = \alpha_i\}$ and $\{u = \alpha_j\}$, and precisely $S_{ij} := \partial_*\{u = \alpha_i\} \cap \partial_*\{u = \alpha_j\} \cap \Omega$ (recall that both phases have finite perimeter in Ω), and ν_{ij} is the measure theoretic normal to S_{ij}. For every unit vector e the value $\sigma_{ij}(e)$ is defined by the following version of the optimal profile problem:

$$\sigma_{ij}(e) := \inf\big\{|C|^{-1}\mathscr{F}(u,T_C) : C \in \mathscr{C}_e,\ u \in X^{ij}(C)\big\} \ , \tag{1.14}$$

where we follow the notation of paragraph 1.3, and $X^{ij}(C)$ is the class of all functions $u:\mathbb{R}^N\to\mathbb{R}^m$ which are C-periodic and satisfy the boundary condition

$$\lim_{x_e\to+\infty} u(x)=\alpha_j \quad\text{and}\quad \lim_{x_e\to-\infty} u(x)=\alpha_i \ .$$

This vectorial generalization of Theorem 1.4 can be proved by adapting the proof for the scalar case given below, and using a suitable approximation result for the functions in $BV\big(\Omega,\{\alpha_i\}\big)$ (cf. the approach in [Ba] for the vectorial version of the Modica–Mortola theorem).

Notice that in this case it is not known whether the optimal profile problem (1.14) admits a solution or not (cf. [AB1], section 4b).

1.13. The optimal assumptions on J

The ferromagnetic assumption $J\geq 0$ plays an essential rôle in the proof of statement (i) of Theorem 1.4, and in particular in the first step of the proof of Theorem 3.1. On the other hand, the proofs of statements (ii) and (iii) do not require the positivity of J, and therefore the task of extending Theorem 1.4 to potentials J which are not ferromagnetic essentially reduces to proving the compactness result in statement (i); in fact, this seems possible under certain restrictions on J (see [AB2]).

About the growth assumptions on J, we can replace the hypotheses in paragraph 1.3, namely $J\in L^1(\mathbb{R}^N)$ and (1.6), with the following more general ones (cf. [AB1], section 4c): J is even, non-negative, and satisfies

$$\int_{\mathbb{R}^N} J(h)\,\big(|h|\wedge|h|^2\big)\,dh<+\infty \ . \tag{1.15}$$

We note that the proof of Theorem 1.4 needs no modifications at all if J does not belong to $L^1(\mathbb{R}^N)$ but still satisfies (1.6), while some additional care has to be taken in the fully general case, and more precisely in the proof of statement (iii) (see in particular the third step in the proof of Theorem 5.2, and the decay of the locality defect in Lemma 2.7), while statements (i) and (ii) can always be recovered from the usual version of Theorem 1.4 by approximating J with an increasing sequence of potentials which satisfy the assumptions in paragraph 1.2.

Finally, we notice that if (1.15) does not hold, then the value of $\sigma(e)$ as given by the optimal profile problem (1.9) is always equal to $+\infty$ (cf. [AB1], Theorem 4.6). This probably means that a different scaling should be considered in the definition of the functionals F_ε. For instance, if $N=1$ and $J(h):=1/h^2$ the right scaling is given by

$$\varepsilon\int_{\Omega\times\Omega}\Big|\frac{u(x')-u(x)}{x'-x}\Big|^2dx'dx\ +e^{1/\varepsilon}\int_\Omega W\big(u(x)\big)\,dx \ ,$$

or equivalently by multiplying the functionals F_ε defined in (1.1) by an infinitesimal factor of order $|\log\varepsilon|^{-1}$. In this case, we again obtain a Γ-limit of the form (1.10) (see, for instance, [ABS]). However, no general result is available when J does not satisfy (1.15).

Warning. Throughout the rest of the paper, we will always restrict ourselves to functions which take values in $[-1,1]$. We are allowed to do this by the following truncation lemma:

Lemma 1.14. *For every function $u:\Omega\to\mathbb{R}$, let Tu denote the truncated function $Tu(x):=\big(u(x)\wedge 1\big)\vee -1$. Then $F_\varepsilon(u,\Omega)\geq F_\varepsilon(Tu,\Omega)$ for every $\varepsilon>0$, and for every sequence (u_ε) such that $F_\varepsilon(u_\varepsilon,\Omega)$ is bounded in ε there holds $\|u_\varepsilon-Tu_\varepsilon\|_1\to 0$ as $\varepsilon\to 0$.*

Proof. The inequality $F_\varepsilon(u,\Omega)\geq F_\varepsilon(Tu,\Omega)$ is immediate.

Let us now be given a sequence (u_ε) such that $F_\varepsilon(u_\varepsilon,\Omega)\leq C$ for every ε. Since W is strictly positive and continuous out of ±1, and has growth at least linear at infinity (see paragraph 1.2), for every $\delta>0$, we may find $a>0$, $M>0$ and $b>0$ so that $W(t)\geq a$ when $1+\delta\leq|t|\leq M$ and $W(t)\geq b|t|$ when $|t|\geq M$. Then we define A_ε and B_ε as the sets of all $x\in\Omega$ where $u_\varepsilon(x)$ satisfies, respectively, $1+\delta\leq\big|u_\varepsilon(x)\big|\leq M$ and $M\leq\big|u_\varepsilon(x)\big|$. Hence

$$\|u_\varepsilon-Tu_\varepsilon\|_1\leq\delta|\Omega|+M|A_\varepsilon|+\int_{B_\varepsilon}|u_\varepsilon|\leq\delta|\Omega|+\Big(\frac{M}{a}+\frac{1}{b}\Big)\int_\Omega W(u_\varepsilon)\ .$$

Since $\int_\Omega W(u_\varepsilon)\leq C\varepsilon$, passing to the limit as $\varepsilon\to0$ we obtain that $\limsup\|u_\varepsilon-Tu_\varepsilon\|_1\leq\delta|\Omega|$, and since δ can be taken arbitrarily small, the proof is complete. □

2. Decay estimates for the locality defect

In this section we study the asymptotic behavior as ε tends to zero of the locality defect Λ_ε (see (1.3)). Roughly speaking, the goal is to show that the limit of $\Lambda_\varepsilon(u_\varepsilon,A,A')$ is determined only by the asymptotic behavior of the sequence u_ε close to the intersection of the boundaries of A and A'. The main result of this section is Theorem 2.8.

We first need to fix some additional notation. We define the auxiliary potential $\hat J$ by

$$\hat J(h):=\int_0^1 J\Big(\frac{h}{t}\Big)\Big|\frac{h}{t}\Big|\frac{dt}{t^N}\qquad\text{for every } h\in\mathbb{R}^N. \tag{2.1}$$

It follows immediately from the definition that $\hat J$ is even, non-negative, and satisfies

$$\|\hat J\|_1=\int_{\mathbb{R}^N} J(h)\,|h|\,dh<\infty\ . \tag{2.2}$$

Definition 2.1. Throughout this section Σ always denotes a subset of a Lipschitz hypersurface in $\mathbb{R}^N$ and is endowed with the Hausdorff measure $\mathscr{H}^{N-1}$ (we often omit any explicit mention to this measure). Now, let us give a set A of positive measure in $\mathbb{R}^N$, a sequence (u_n) of functions from A into $[-1,1]$, and a sequence (ε_n) of positive real numbers which tends to zero; we say that the *ε_n-traces of u_n* (relative to A) *converge on Σ to* $v:\Sigma\to[-1,1]$ when

$$\lim_{n\to\infty}\int_{y\in\Sigma}\Big[\int_{\{h:\,y+\varepsilon_n h\in A\}}\hat J(h)\,\big|u_n(y+\varepsilon_n h)-v(y)\big|\,dh\Big]\,dy=0. \tag{2.3}$$

Remark 2.2. We make no assumption on the relative position of A and Σ; in particular, they may even be far apart. Notice, moreover, that the notion of "convergence of the ε_n-traces" is introduced without defining what the ε_n-trace of a function is, and in fact, there is no such notion. This is due to the fact, that for functions in the domain of F_ε, the trace on an $(N-1)$-dimensional manifold cannot be defined (while it is defined for functions in the domain of the Γ-limit, that is, for BV functions).

In view of the definition of the locality defect, it would make more sense to replace the term $\left|u_n(y+\varepsilon_n h)-v(y)\right|$ in (2.3) with its square. But since we restrict ourselves to functions which take values in $[-1,1]$, the limit in (2.3) is independent of the power of $\left|u_n(y+\varepsilon_n h)-v(y)\right|$, and we chose the first power because this simplifies many of the following proofs.

Remark 2.3. We define the upper $\hat{J}$-density of A at the point $x\in\mathbb{R}^N$ as the upper limit

$$\limsup_{\varepsilon\to 0}\int_{\{h:\, x+\varepsilon h\in A\}}\hat{J}(h)\,dh\ ,$$

and the lower $\hat{J}$-density as the corresponding lower limit. Notice that such densities are local, that is, they do not depend upon the behavior of A out of any open neighborhood of x.

The function v which satisfies (2.3) is uniquely determined for ($\mathscr{H}^{N-1}$-) almost every point of Σ where A has positive J-upper density.

If (2.3) holds for some set A, then it is satisfied by every A' included in A. Moreover, if Σ has finite measure, then (2.3) is also satisfied by every A' such that $A'\setminus A$ has upper $\hat{J}$-density zero at almost every point of Σ. In particular, if are given sets A and A' such that the symmetric difference $A\triangle A'$ has upper $\hat{J}$-density zero at almost every point of Σ, then A satisfies (2.3) if and only if A' does.

Remark 2.4. Condition (2.3) is not easy to verify. If Σ has finite measure then (2.3) holds when

$$\lim_{n\to\infty}u_n(y+\varepsilon_n h)=v(y)\quad\text{for a.e. } y\in\Sigma \text{ and a.e. } h\in A. \tag{2.4}$$

Condition (2.4) holds, for instance, when u_n converge locally uniformly on some open neighborhood of Σ to a function which, at every point of Σ, is continuous and agrees with v.

Assume now that the functions u_n converge to u in $L^1(A)$. Unfortunately, this is not enough to deduce that the ε_n-traces of u_n converge to u on every Lipschitz hypersurface $\Sigma\subset\mathbb{R}^N$, yet this holds for "most" Σ. More precisely, we have the following proposition:

Proposition 2.5. *Take A, (ε_n) and (u_n) as in Definition 2.1; let $g:A\to\mathbb{R}$ be a Lipschitz function, and denote by Σ^t the t-level set of g for every $t\in\mathbb{R}$. If $u_n\to u$ in $L^1(A)$ then, possibly passing to a subsequence, the ε_n-traces of u_n (relative to A) converge to u on Σ^t for a.e. $t\in\mathbb{R}$.*

(Since g admits a Lipschitz extension to $\mathbb{R}^N$, Σ^t is a subset of an oriented closed Lipschitz hypersurface in $\mathbb{R}^N$ for almost every $t\in\mathbb{R}$.)

Proof. To simplify the notation we write $\varepsilon, u_\varepsilon$ instead of ε_n, u_n, we assume that g is 1-Lipschitz and $A=\mathbb{R}^N$ (the general case follows in the same way). For every $\varepsilon>0$, $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$ we set

$$\Phi_\varepsilon(x):=\int_{\mathbb{R}^N}\hat{J}(h)\left|u_\varepsilon(x+\varepsilon h)-u(x)\right|dh\qquad\text{and}\qquad g_\varepsilon(t):=\int_{\Sigma^t}\Phi_\varepsilon(x)\,dx\ . \tag{2.5}$$

By the co-area formula for Lipschitz functions (see [EG], section 3.3), we get

$$\begin{aligned}\int_{\mathbb{R}}g_\varepsilon(t)\,dt&=\int_{\mathbb{R}^N}\Phi_\varepsilon(x)\left|\nabla g(x)\right|dx\le\int_{\mathbb{R}^N}\Phi_\varepsilon(x)\,dx\\ &=\int_{\mathbb{R}^N\times\mathbb{R}^N}\hat{J}(h)\left|u_\varepsilon(x+\varepsilon h)-u(x)\right|dx\,dh\\ &\le\int_{\mathbb{R}^N\times\mathbb{R}^N}\hat{J}(h)\Big[\left|u_\varepsilon(x+\varepsilon h)-u(x+\varepsilon h)\right|+\left|u(x+\varepsilon h)-u(x)\right|\Big]\,dx\,dh\\ &\le\int_{\mathbb{R}^N}\hat{J}(h)\left[\|u_\varepsilon-u\|_1+\|\tau_{\varepsilon h}u-u\|_1\right]dh\ ,\end{aligned} \tag{2.6}$$

where $\tau_{\varepsilon h}u(x):=u(x+\varepsilon h)$.

Now $\|u_\varepsilon-u\|_1$ tends to zero by assumption and $\|\tau_{\varepsilon h}u-u\|_1$ tends to zero as $\varepsilon\to 0$ for every h, and since $\hat{J}$ is summable (cf. (2.2)), we can apply the dominated convergence theorem to the integrals in line (2.6), and we get

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}g_\varepsilon(t)\,dt=0\ .$$

Hence the functions g_ε converge to zero in $L^1(\mathbb{R})$, and passing to a subsequence we may assume that they also converge pointwise to zero for a.e. $t\in\mathbb{R}$. Since $g_\varepsilon(t)$ is equal to the double integral in (2.3) (with v replaced by u), the proof is complete. □

Definition 2.6. Let $A,A'\subset\mathbb{R}^N$ be given. We say that the set Σ *divides A and A'* when for every $x\in A$, $x'\in A'$ the segment $[x,x']$ intersects Σ. We say that Σ *strongly divides A and A'* when Σ is the (Lipschitz) boundary of some open set Ω such that $A\subset\overline{\Omega}$ and $A'\subset\mathbb{R}^N\setminus\Omega$.

Now we can state and prove the first decay estimate for the locality defect. Let disjoint sets A and A' in $\mathbb{R}^N$ be given which are divided by Σ, then take positive numbers $\varepsilon_n\to 0$ and functions $u_n:A\cup A'\to[-1,1]$ and $v,v':\Sigma\to[-1,1]$.

Lemma 2.7. *Under the above stated hypotheses, if the ε_n-traces of u_n relative to A and A' converge on Σ to v and v', respectively, then*

$$\limsup_{n\to\infty}\Lambda_{\varepsilon_n}(u_n,A,A')\le\frac{1}{2}\|\hat{J}\|_1\int_\Sigma\left|v(y)-v'(y)\right|dy\ . \tag{2.7}$$

Proof. To simplify the notation we write ε, u_ε and Λ_ε instead of ε_n, u_n, Λ_{ε_n}. By the definition of Λ_ε, and recalling that $|u_\varepsilon|\le 1$, we obtain

$$\Lambda_\varepsilon(u_\varepsilon,A,A')\le\frac{1}{2\varepsilon}\int_{\mathbb{R}^N}J(h)\,\underbrace{\Big[\int_{A_{\varepsilon h}}\left|u_\varepsilon(x+\varepsilon h)-u_\varepsilon(x)\right|dx\Big]}_{I_\varepsilon(h)}\,dh\ , \tag{2.8}$$

where $A_{\varepsilon h}$ is the set of all x such that $x\in A$ and $x+\varepsilon h\in A'$. Let us consider for the moment the integral $I_\varepsilon(h)$ defined in (2.8): for every x in the integration domain $A_{\varepsilon h}$, the segment $[x, x+\varepsilon h]$ intersects Σ, and then we can write x as $x = y - t\varepsilon h$ for some $y \in \Sigma$ and $t \in [0,1]$.

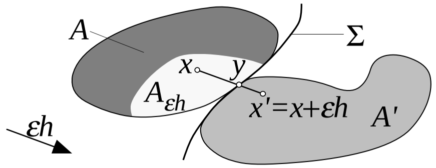

FIG.1. The set $A_{\varepsilon h}$ for given $\varepsilon > 0$ and $h \in \mathbb{R}^N$.

Since the Jacobian determinant of the map which takes $(y,t) \in \Sigma \times [0,1]$ into $y - t\varepsilon h$ does not exceed $\varepsilon|h|$, by applying the change of variable $x = y - t\varepsilon h$ we get

$$I_\varepsilon(h) \le \varepsilon|h| \int_\Sigma \Big[\int_{S_{hy}} \big|u_\varepsilon(y + (1-t)\varepsilon h) - u_\varepsilon(y - t\varepsilon h)\big|\, dt \Big]\, dy \ ,$$

where S_{hy} is the set of all $t \in [0,1]$ such that $y - t\varepsilon h \in A$ and $y + (1-t)\varepsilon h \in A'$. Hence (2.8) yields

$$\Lambda_\varepsilon(u_\varepsilon, A, A') \le \frac{1}{2} \int_{h \in \mathbb{R}^N,\, y \in \Sigma} J(h)\, |h| \Big[\int_{S_{hy}} \big|u_\varepsilon(y+(1-t)\varepsilon h) - u_\varepsilon(y - t\varepsilon h)\big|\, dt \Big]\, dy\, dh \ . \tag{2.9}$$

Now by the triangle inequality we can estimate $\big|u_\varepsilon(y+(1-t)\varepsilon h) - u_\varepsilon(y-t\varepsilon h)\big|$ by the sum of the following three terms:

$$\big|v(y) - v'(y)\big| + \big|u_\varepsilon(y - t\varepsilon h) - v'(y)\big| + \big|u_\varepsilon(y + (1-t)\varepsilon h) - v(y)\big| \ .$$

Accordingly, we estimate the double integral at the right-hand side of (2.9) by the sum of the corresponding double integrals I^1_ε, I^2_ε and I^3_ε, that is,

$$\Lambda_\varepsilon(u_\varepsilon, A, A') \le I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon \ . \tag{2.10}$$

We recall now that $|S_{hy}| \le 1$ for every h and every y, and then

$$\begin{aligned} I^1_\varepsilon :&= \frac{1}{2} \int_{h \in \mathbb{R}^N,\, y \in \Sigma} J(h)\, |h| \Big[\int_{S_{hy}} \big|v(y) - v'(y)\big|\, dt \Big]\, dy\, dh \\ &\le \frac{1}{2} \Big[\int_{\mathbb{R}^N} J(h)\, |h|\, dh \Big] \Big[\int_\Sigma \big|v(y) - v'(y)\big|\, dy \Big] \ . \end{aligned} \tag{2.11}$$

Since the first integral in line (2.11) is equal to $\|\hat{J}\|_1$ (see (2.2)), inequality (2.7) will follow from (2.10) once we have proved that I^2_ε and I^3_ε vanish as $\varepsilon \to 0$. Let us consider I^2_ε:

$$\begin{aligned} I^2_\varepsilon :&= \frac{1}{2} \int_{h \in \mathbb{R}^N,\, y \in \Sigma} J(h)\, |h| \Big[\int_{S_{hy}} \big|u_\varepsilon(y - t\varepsilon h) - v(y)\big|\, dt \Big]\, dy\, dh \\ &\le \frac{1}{2} \int_{h' \in \mathbb{R}^N,\, y \in \Sigma} \Big[\int_{S_{hy}} J\Big(\frac{h'}{t}\Big) \Big|\frac{h'}{t}\Big| \big|u_\varepsilon(y + \varepsilon h') - v(y)\big| \frac{dt}{t^N} \Big]\, dy\, dh' \\ &\le \frac{1}{2} \int_\Sigma \Big[\int_{\{h' :\, y + \varepsilon h' \in A\}} \hat{J}(h') \big|u_\varepsilon(y + \varepsilon h') - v(y)\big|\, dh' \Big]\, dy \end{aligned}$$

(the first inequality in the previous computation follows from the change of variable $h = -h'/t$, while the second one follows from (2.1), taking into account that $|S_{hy}| \le 1$ for every h, y).

Therefore, I^2_ε vanishes as $\varepsilon \to 0$ because the ε-traces of u_ε relative to A converge to v on Σ. In a similar way, one can prove that I^3_ε vanishes as $\varepsilon \to 0$. □

Now we can state the main result of this section. Let be given disjoint sets $A, A' \subset \mathbb{R}^N$, and Σ such that one of the following holds:

(a) the sets A and A' are divided by Σ (cf. Definition 2.6);
(b) the sets A and A' are strongly divided by a Lipschitz boundary S with finite measure and $\Sigma = \partial A \cap \partial A'$;
(c) either A or A' is a bounded set with Lipschitz boundary and $\Sigma = \partial A \cap \partial A'$.

Then take positive numbers $\varepsilon_n \to 0$ and functions $u_n : A \cup A' \to [-1,1]$.

Theorem 2.8. *Under the above stated hypotheses we have*

$$\limsup_{n \to \infty} \Lambda_{\varepsilon_n}(u_n, A, A') \le \|\hat{J}\|_1 \mathscr{H}^{N-1}(\Sigma) \ . \tag{2.12}$$

Moreover, if the ε_n-traces of u_n relative to A and A' converge on Σ, respectively, to v and v', then

$$\limsup_{n \to \infty} \Lambda_{\varepsilon_n}(u_n, A, A') \le \frac{1}{2} \|\hat{J}\|_1 \int_\Sigma \big|v(y) - v'(y)\big|\, dy \ . \tag{2.13}$$

Proof. Notice that (2.12) follows by applying (2.13) to the functions $\overline{u}_n$ which are equal to 1 on A and to -1 on A' (with $v := 1$ and $v' := -1$) and then using the obvious inequality $\Lambda_{\varepsilon_n}(u_n, A, A') \le \Lambda_{\varepsilon_n}(\overline{u}_n, A, A')$.

Let us prove (2.13). When (a) holds it is enough to apply Lemma 2.7, while (c) clearly implies (b). Assume that (b) holds.

First, we notice that in this case we can always modify the boundary S so that $S \cap \partial A = S \cap \partial A' = \Sigma$. Now we extend v and v' to zero in $S \setminus \Sigma$, and then the ε_n-traces of u_n relative to A and A' converge on S to v and v', respectively (use Remark 2.3, recalling that both A and A' have upper J-density zero at every point of $S \setminus \Sigma$). Now it is enough to apply Lemma 2.7 with S instead of Σ. □

3. Proof of the compactness result

The following theorem implies statement (i) of Theorem 1.4, and shows that the domain of the Γ-limit of the functionals F_ε is included in $BV(\Omega, \pm 1)$.

Theorem 3.1. *Let Ω be a regular open set and let sequences (ε_n) and (u_n) be given such that $\varepsilon_n \to 0$, $u_n : \Omega \to [-1,1]$, and $F_{\varepsilon_n}(u_n, \Omega)$ is bounded. Then the sequence (u_n) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$.*

Proof. To simplify the notation we replace as usual ε_n, u_n, and F_{ε_n} with ε, u_ε, and F_ε.

We need the following inequality, which may be proved by a direct computation: for every non-negative $g \in L^1(\mathbb{R}^N)$ and every $u : \mathbb{R}^N \to \mathbb{R}$, we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (g * g)(y) \big|u(x+y) - u(x)\big|\, dy\, dx \le 2\|g\|_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} g(y) \big|u(x+y) - u(x)\big|\, dy\, dx \ . \tag{3.1}$$

The proof of the theorem is now divided into two steps.

Step 1. We first prove the theorem under the assumption that each u_ε takes values ± 1 only.

We extend each function u_ε to 1 in $\mathbb{R}^N \setminus \Omega$, and then we observe that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} J_\varepsilon(y) \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx = O(\varepsilon) \ . \tag{3.2}$$

Indeed, the assumption $u_\varepsilon = \pm 1$ implies $\left| u_\varepsilon(x') - u_\varepsilon(x) \right| = \frac{1}{2}\big(u_\varepsilon(x') - u_\varepsilon(x)\big)^2$, and then by the definition of F_ε we obtain

$$\begin{aligned}\frac{1}{\varepsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} & J_\varepsilon(x'-x) \left| u_\varepsilon(x') - u_\varepsilon(x) \right| dx' dx = \\ &= 2F_\varepsilon(u_\varepsilon, \mathbb{R}^N) = 2F_\varepsilon(u_\varepsilon, \Omega) + 4\Lambda_\varepsilon(u_\varepsilon, \Omega, \mathbb{R}^N \setminus \Omega) \ .\end{aligned}$$

We apply inequality (2.12) with $A = \Omega$ and $A' = \mathbb{R}^N \setminus \Omega$ to show that $\Lambda_\varepsilon(u_\varepsilon, \Omega, \mathbb{R}^N \setminus \Omega)$ is uniformly bounded in ε (recall that we are considering only a subsequence ε_n which converges to zero), while $F_\varepsilon(u_\varepsilon, \Omega)$ is uniformly bounded by assumption. Hence (3.2) is proved.

Now we combine inequality (3.1), with $g := J_\varepsilon$, and inequality (3.2), and we obtain

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx = O(\varepsilon) \ . \tag{3.3}$$

Since $J * J$ is a non-negative continuous function, we may find a non-negative smooth function φ (not identically zero) with compact support such that

$$\varphi \le J * J \quad \text{and} \quad |\nabla \varphi| \le J * J \ . \tag{3.4}$$

We set $c := \int_{\mathbb{R}^N} \varphi(y)\, dy$ and for every $y \in \mathbb{R}^N$ and every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(y) := \frac{1}{c\varepsilon^N} \varphi(y/\varepsilon) \quad \text{and} \quad w_\varepsilon(y) := \varphi_\varepsilon * u_\varepsilon\,(y) \ . \tag{3.5}$$

The functions φ_ε are smooth and non-negative, have integral equal to unity, and converge to the Dirac mass centered at 0 as $\varepsilon \to 0$. We claim that the sequence (w_ε) is asymptotically equivalent to (u_ε) in $L^1(\mathbb{R}^N)$, and that the gradients ∇w_ε are uniformly bounded in $L^1(\mathbb{R}^N)$. Once this claim is proved, we could infer that the sequence (w_ε) is relatively compact in $L^1(\Omega)$ and each of its cluster points belongs to $BV(\Omega, \pm 1)$, and that the same holds for the sequence (u_ε).

Now it remains to prove the claim. We have

$$\begin{aligned}\int_{\mathbb{R}^N} |w_\varepsilon - u_\varepsilon|\, dx &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \varphi_\varepsilon(y) \big(u_\varepsilon(x+y) - u_\varepsilon(x)\big)\, dy \Big|\, dx \\ &\le \int_{\mathbb{R}^N \times \mathbb{R}^N} \left|\varphi_\varepsilon(y)\right| \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx \\ &\le \frac{1}{c} \int_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx \ = O(\varepsilon)\end{aligned}$$

(here the second inequality follows from $\varphi_\varepsilon \le \frac{1}{c} J_\varepsilon * J_\varepsilon$, cf. (3.4) and (3.5), while the last equality follows from (3.3)). Moreover

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_\varepsilon|\, dx &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y)\, u_\varepsilon(x+y)\, dy \Big|\, dx \\ &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y) \big(u_\varepsilon(x+y) - u_\varepsilon(x)\big)\, dy \Big|\, dx \\ &\le \int_{\mathbb{R}^N \times \mathbb{R}^N} \left|\nabla \varphi_\varepsilon(y)\right| \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx \\ &\le \frac{1}{c\varepsilon} \int_{\mathbb{R}^N \times \mathbb{R}^N} (J_\varepsilon * J_\varepsilon)(y) \left| u_\varepsilon(x+y) - u_\varepsilon(x) \right| dy\, dx \ = O(1)\end{aligned}$$

(to obtain the second equality we use the fact that $\int_{\mathbb{R}^N} \nabla \varphi_\varepsilon(y)\, dy = 0$ because φ_ε has compact support; the second inequality follows from $|\nabla \varphi_\varepsilon| \le \frac{1}{c\varepsilon} J_\varepsilon * J_\varepsilon$, cf. (3.4) and (3.5), while the last equality follows from (3.3)).

Step 2. We now consider the general case. For every $s \in \mathbb{R}$ we set

$$T(s) := \begin{cases} -1 & \text{if } s < 0, \\ +1 & \text{if } s \ge 0, \end{cases} \tag{3.6}$$

and then we define

$$v_\varepsilon := T(u_\varepsilon) \ . \tag{3.7}$$

The functions v_ε take values ± 1 only, and we claim that the sequence (v_ε) is asymptotically equivalent to (u_ε) in $L^1(\Omega)$ and that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded. Once we have proved this claim, the theorem will follow from Step 1.

Take δ so that $0 < \delta < 1$, and let K_ε be the set of all $x \in \Omega$ such that $u_\varepsilon(x) \in [-1+\delta, 1-\delta]$. Then $|u_\varepsilon - v_\varepsilon| \le \delta$ in $\Omega \setminus K_\varepsilon$, and we deduce

$$\int_\Omega |u_\varepsilon - v_\varepsilon|\, dx \le \delta|\Omega| + \int_{K_\varepsilon} \big(|u_\varepsilon| + |v_\varepsilon|\big)\, dx \le \delta|\Omega| + 2|K_\varepsilon| \ . \tag{3.8}$$

Since $\delta > 0$ and W is zero only at ± 1, there exists a positive constant ρ (which depends on δ) such that $W(t) \ge \rho$ for every $t \in [-1+\delta, 1-\delta]$. Hence

$$|K_\varepsilon| \le \frac{1}{\rho} \int_{K_\varepsilon} W\big(u_\varepsilon(x)\big)\, dx \le \frac{\varepsilon}{\rho} F_\varepsilon(u_\varepsilon, \Omega) = \frac{O(\varepsilon)}{\rho}. \tag{3.9}$$

The inequalities (3.8) and (3.9) imply

$$\limsup_{\varepsilon \to 0} \int_\Omega |u_\varepsilon - v_\varepsilon|\, dx \le \delta|\Omega| \ .$$

As δ is arbitrary, the sequences (u_ε) and (v_ε) are asymptotically equivalent in $L^1(\Omega)$.

It remains to prove that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded in ε. Since $\int_\Omega W(v_\varepsilon)\, dy = 0$, we have only to estimate the first integral in the definition of F_ε. Given $s_1, s_2 \in [-1, 1]$ we have that

$$\text{either } |s_1| \le 1/2 \text{ or } \big|T(s_1) - T(s_2)\big| \le 4|s_1 - s_2|.$$

Hence, if we denote by H_ε the set of all $x \in \Omega$ such that $\big|u_\varepsilon(x)\big| \le 1/2$, we deduce

$$\begin{aligned} F_\varepsilon(v_\varepsilon, \Omega) &= \frac{1}{4\varepsilon} \int_{\Omega\times\Omega} J_\varepsilon(x'-x) \big(Tu_\varepsilon(x') - Tu_\varepsilon(x)\big)^2 dx'dx \\ &\le \frac{4}{\varepsilon} \int_{\Omega\times\Omega} J_\varepsilon(x'-x) \big(u_\varepsilon(x') - u_\varepsilon(x)\big)^2 dx'dx + \frac{1}{\varepsilon} \int_{H_\varepsilon\times\Omega} J_\varepsilon(x'-x)\, dx'dx \\ &\le 16\, F_\varepsilon(u_\varepsilon, \Omega) + \frac{1}{\varepsilon} \|J\|_1 |H_\varepsilon| \ . \end{aligned} \tag{3.10}$$

By the properties of W, there exists a positive constant ρ such that $W(t) \ge \rho$ for every t such that $|t| \le 1/2$, and, reasoning as in (3.9), we get $|H_\varepsilon| = O(\varepsilon)$; together with (3.10) this proves that $F_\varepsilon(v_\varepsilon, \Omega)$ is uniformly bounded in ε. □

4. Proof of the lower bound inequality

In this section we prove statement (ii) of Theorem 1.4.

We begin with some notation. For every $\varepsilon > 0$, $A \subset \mathbb{R}^N$ and $u : \mathbb{R}^N \to [-1, 1]$ we define the rescaling of the functional $\mathscr{F}$ given in (1.7) by

$$\mathscr{F}_\varepsilon(u, A) := \frac{1}{4\varepsilon} \int_{x\in A,\, h\in\mathbb{R}^N} J_\varepsilon(h) \big(u(x+h) - u(x)\big)^2 dx\, dh + \frac{1}{\varepsilon} \int_{x\in A} W\big(u(x)\big)\, dx \ . \tag{4.1}$$

Recalling the definitions of F_ε and Λ_ε, we obtain

$$\mathscr{F}_\varepsilon(u, A) = F_\varepsilon(u, A) + \Lambda_\varepsilon(u, A, \mathbb{R}^N \setminus A) \ . \tag{4.2}$$

Let us now be given a function u defined on (a subset of) $\mathbb{R}^N$, a point $\bar{x} \in \mathbb{R}^N$ and a positive number r. We define the *blow-up of u centered at $\bar{x}$ with scaling factor r* as the function $R_{\bar{x},r}u$ given by

$$(R_{\bar{x},r}u)(x) := u(\bar{x} + rx) \ ; \tag{4.3}$$

when $\bar{x} = 0$ we write $R_r u$ instead of $R_{0,r}u$. For every set $A \subset \mathbb{R}^N$ we set, as usual, $\bar{x} + rA := \{\bar{x} + rx : x \in A\}$, and then we easily obtain the following *scaling identities*:

$$F_\varepsilon(u, \bar{x} + rA) = r^{N-1} F_{\varepsilon/r}(R_{\bar{x},r}u, A) \tag{4.4}$$

$$\mathscr{F}_\varepsilon(u, \bar{x} + rA) = r^{N-1} \mathscr{F}_{\varepsilon/r}(R_{\bar{x},r}u, A) \ . \tag{4.5}$$

In the proof we also make use of the following well-known results about the blow-up of finite perimeter sets and measures:

4.1. Some blow-up results

Let S be a rectifiable set in $\mathbb{R}^N$ with normal vector field ν; let μ be the restriction of the Hausdorff measure $\mathscr{H}^{N-1}$ to the set S, that is, $\mu := \mathscr{H}^{N-1} \llcorner S$, and let λ be a finite measure on $\mathbb{R}^N$. Then for $\mathscr{H}^{N-1}$-a.e. $\bar{x} \in S$ the density of λ with respect to μ at $\bar{x}$ is given by the following limit:

$$\frac{d\lambda}{d\mu}(\bar{x}) = \lim_{r\to 0} \frac{\lambda\big(\bar{x} + rQ\big)}{r^{N-1}} \tag{4.6}$$

where Q is any unit cube centered at 0 such that $\nu(x)$ is one of its axes.

Let u be a fixed function in $BV(\Omega, \pm 1)$. For every $\bar{x} \in Su$, we denote by $v_{\bar{x}} : \mathbb{R}^N \to \pm 1$ the step function

$$v_{\bar{x}}(x) := \begin{cases} +1 & \text{if } \langle x, \nu_u(\bar{x})\rangle \ge 0, \\ -1 & \text{if } \langle x, \nu_u(\bar{x})\rangle < 0. \end{cases} \tag{4.7}$$

Then for $\mathscr{H}^{N-1}$-a.e. $\bar{x} \in Su$, and more precisely for all $\bar{x} \in Su$ such that the density of the measure Du with respect to $|Du|$ exists and is equal to $\nu_u(\bar{x})$, we have the limit

$$R_{\bar{x},r}u \longrightarrow v_{\bar{x}} \quad \text{in } L^1_{\rm loc}(\mathbb{R}^N) \text{ as } r \to 0 \tag{4.8}$$

(if u is not defined on the whole of $\mathbb{R}^N$, we take an arbitrary extension).

4.2. Proof of statement (ii) of Theorem 1.4

We can now begin the proof of statement (ii) of Theorem 1.4. We assume, therefore, that we are given a sequence (u_ε) which converges to $u \in BV(\Omega, \pm 1)$ in $L^1(\Omega)$; we have to prove that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) \ge \int_{Su} \sigma(\nu_u)\, d\mathscr{H}^{N-1} \ . \tag{4.9}$$

In the following, u_ε and u are fixed. We shall often extract from all positive ε a subsequence (ε_n) which converges to zero; to simplify the notation, we shall keep writing ε, F_ε, and u_ε instead of ε_n, F_{ε_n}, u_{ε_n}.

First, we notice that it is enough to prove inequality (4.9) when the lower limit at the left-hand side is finite, and then, passing to a subsequence, we may also assume that it is a limit.

Now we follow the approach of [FM]; the main feature of this method consists of the reduction of the lower bound inequality (4.9) to a density estimate (see (4.13)) which has to be satisfied point-by-point. What follows, up to equation (4.18), is a straightforward adaptation of this general method (see also [BF], [BFM]).

For every $\varepsilon > 0$ we define the energy density associated with u_ε at the point $x \in \Omega$ as

$$g_\varepsilon(x) := \frac{1}{4\varepsilon} \int_\Omega J_\varepsilon(x'-x) \big(u_\varepsilon(x') - u_\varepsilon(x)\big)^2 dx' \ + \frac{1}{\varepsilon} W\big(u_\varepsilon(x)\big) \ , \tag{4.10}$$

and then we consider the corresponding energy distribution

$$\lambda_\varepsilon := g_\varepsilon \cdot \mathscr{L}_N \llcorner \Omega \ . \tag{4.11}$$

Thus the total variation $\|\lambda_\varepsilon\|$ of the measure λ_ε (on Ω) is equal to $F_\varepsilon(u_\varepsilon, \Omega)$, and since $F_\varepsilon(u_\varepsilon, \Omega)$ is equibounded with respect to ε, possibly passing to a subsequence we can assume that there exists a finite positive measure λ on Ω such that

$$\lambda_\varepsilon \rightharpoonup \lambda \quad \text{weakly* on } \Omega \text{ as } \varepsilon \to 0.$$

Since $F_\varepsilon(u_\varepsilon, \Omega) = \|\lambda_\varepsilon\|$ and $\liminf\limits_{\varepsilon\to 0} \|\lambda_\varepsilon\| \ge \|\lambda\|$, the inequality (4.9) is implied by the following:

$$\|\lambda\| \ge \int_{Su\cap\Omega} \sigma(\nu_u)\, d\mathscr{H}^{N-1} \ . \tag{4.12}$$

In fact, we prove a stronger result: the density of λ with respect to $\mu := \mathscr{H}^{N-1} \llcorner Su$ is greater than or equal to $\sigma(\nu_u)$ at $\mathscr{H}^{N-1}$-a.e. point of Su, that is

$$\frac{d\lambda}{d\mu}(\bar{x}) \geq \sigma\big(\nu_u(\bar{x})\big) \qquad \text{for } \mathscr{H}^{N-1}\text{-a.e. } \bar{x} \in Su. \tag{4.13}$$

More precisely, we have the following lemma:

Lemma 4.3. *With the previous notation, the inequality* (4.13) *holds for every* $\bar{x} \in Su$ *which satisfies* (4.6) *and* (4.8).

Proof. We fix such a point $\bar{x} \in Su$, and we denote by ν the vector $\nu_u(\bar{x})$ and by v the step function $v_{\bar{x}}$ defined in (4.7). Following the notation of paragraph 1.3 we fix an $(N-1)$-dimensional unit cube $C \in \mathscr{C}_\nu$, and we take $Q = Q_C$ and $T = T_C$ accordingly.

As the measures λ_ε converge in the weak* sense to λ on Ω as $\varepsilon \to 0$, we have that $\lambda_\varepsilon(A) \to \lambda(A)$ for every set A such that $\lambda(\partial A) = 0$. Since $\lambda\big(\bar{x} + r(\partial Q)\big) = 0$ for all positive r up to an exceptional countable set N, we deduce that $\lambda_\varepsilon(\bar{x} + rQ) \to \lambda(\bar{x} + rQ)$ for every positive $r \notin N$. Therefore, recalling (4.6), we write

$$\lim_{\substack{r\to 0\\ r\notin N}} \Big(\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon(\bar{x} + rQ)}{r^{N-1}} \Big) = \lim_{\substack{r\to 0\\ r\notin N}} \frac{\lambda(\bar{x} + rQ)}{r^{N-1}} = \frac{d\lambda}{d\mu}(\bar{x}) \ . \tag{4.14}$$

Since $u_\varepsilon \to u$ in $L^1(\Omega)$ by assumption, by (4.8) we also have

$$\lim_{r\to 0} \big(\lim_{\varepsilon \to 0} R_{\bar{x},r} u_\varepsilon \big) = \lim_{r \to 0} R_{\bar{x},r} u = v \quad \text{in } L^1(Q). \tag{4.15}$$

By a diagonal argument we can choose subsequences (r_n) and (ε_n) so that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \big(\varepsilon_n / r_n\big) = 0 \ , \tag{4.16}$$

$$\lim_{n\to\infty} \frac{\lambda_{\varepsilon_n}(\bar{x} + r_n Q)}{r_n^{N-1}} = \frac{d\lambda}{d\mu}(\bar{x}) \ , \tag{4.17}$$

$$\lim_{n\to\infty} R_{\bar{x},r_n} u_{\varepsilon_n} = v \quad \text{in } L^1(Q), \tag{4.18}$$

and then we set $\underline{\varepsilon}_n := \varepsilon_n / r_n$, $v_n := R_{\bar{x},r_n} u_{\varepsilon_n}$. To simplify the notation, in the following we write ε, $\underline{\varepsilon}$, r, u_ε and $v_{\underline{\varepsilon}}$ instead of ε_n, $\underline{\varepsilon}_n$, r_n, u_{ε_n} and v_n, respectively.

From the scaling identity (4.4) and the definition of λ_ε we infer

$$\frac{\lambda_\varepsilon(\bar{x} + rQ)}{r^{N-1}} \geq \frac{F_\varepsilon(u_\varepsilon, \bar{x} + rQ)}{r^{N-1}} = F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) \ . \tag{4.19}$$

Keeping in mind (4.17) and (4.19), we can try to prove (4.13) by establishing a precise relation between $F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q)$ and $\sigma(\nu)$ (see paragraph 1.3).

One possibility is the following: we extend $v_{\underline{\varepsilon}}$ to the strip T by setting $v_{\underline{\varepsilon}} := v$ in $T \setminus Q$, and then we take the C-periodic extension in the rest of $\mathbb{R}^N$. Now, by the scaling identity (4.5) we know that

$$\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T) \geq \sigma(\nu) \ ,$$

and then it would remain to prove that the difference between $\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T)$ and $F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q)$ vanishes as $\underline{\varepsilon} \to 0$; this difference can be written as (cf. (4.20) below)

$$\mathscr{F}_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T) - F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) = \Lambda_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, T, \mathbb{R}^N \setminus T) + 2\Lambda_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q, T \setminus Q) \ ;$$

unfortunately, we cannot use Theorem 2.8 to show that it vanishes as $\underline{\varepsilon} \to 0$ because we have no information about the convergence of the $\underline{\varepsilon}$-traces of $v_{\underline{\varepsilon}}$ on the boundaries ∂Q and ∂T.

We overcome this difficulty as follows: as $v_{\underline{\varepsilon}} \to v$ in $L^1(Q)$, Proposition 2.5 shows that for a.e. $t \in (0,1)$ the $\underline{\varepsilon}$-traces of $v_{\underline{\varepsilon}}$ converge to v on the boundary Σ^t of the cube tQ (notice that each Σ^t is the t-level set of the Lipschitz function $g(x) := 1 - \mathrm{dist}(x, \partial Q)$).

We fix for the moment such a t, and we define $\tilde{v}_{\underline{\varepsilon}}$ on the strip tT by

$$\tilde{v}_{\underline{\varepsilon}}(x) := \begin{cases} v_{\underline{\varepsilon}}(x) & \text{if } x \in tQ, \\ \\ v(x) & \text{if } x \in tT \setminus tQ, \end{cases}$$

and then we take the tC-periodic extension in the rest of $\mathbb{R}^N$. Hence $\tilde{v}_{\underline{\varepsilon}}$ belongs to $X(tC)$ (cf. paragraph 1.3), and since $\tilde{v}_{\underline{\varepsilon}} = v_{\underline{\varepsilon}}$ in tQ

$$\begin{aligned} F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) &\geq F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, tQ) = F_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tQ) = \\ &= F_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tT) - 2\Lambda_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tQ, tT \setminus tQ) \\ &= \mathscr{F}_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tT) - \underbrace{\Lambda_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tT, \mathbb{R}^N \setminus tT)}_{L^1_{\underline{\varepsilon}}} - 2 \underbrace{\Lambda_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tQ, tT \setminus tQ)}_{L^2_{\underline{\varepsilon}}} \ . \end{aligned} \tag{4.20}$$

Now we claim that both locality defects $L^1_{\underline{\varepsilon}}$ and $L^2_{\underline{\varepsilon}}$ vanish as $\underline{\varepsilon} \to 0$; once this is proved we can deduce from the previous formula that

$$\limsup_{\underline{\varepsilon}\to 0} F_{\underline{\varepsilon}}(v_{\underline{\varepsilon}}, Q) \geq \limsup_{\underline{\varepsilon}\to 0} \mathscr{F}_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tT) \ . \tag{4.21}$$

Let us consider first $L^2_{\underline{\varepsilon}}$: the sets tQ and $tT \setminus tQ$ are divided by the boundary Σ^t of tQ, and by the choice of t, the $\underline{\varepsilon}$-trace of $\tilde{v}_{\underline{\varepsilon}}$ relative to tQ converge to v on Σ^t (recall that $\tilde{v}_{\underline{\varepsilon}} = v_{\underline{\varepsilon}}$ on tQ). On the other hand, $\tilde{v}_{\underline{\varepsilon}} = v$ in $tT \setminus tQ$, and then also the $\underline{\varepsilon}$-trace relative to $tT \setminus tQ$ converge to v on Σ^t. Hence Theorem 2.8 applies, and $L^2_{\underline{\varepsilon}}$ vanishes as $\underline{\varepsilon} \to 0$. In a similar way one can prove that also $L^1_{\underline{\varepsilon}}$ vanishes as $\underline{\varepsilon} \to 0$ (in fact it is enough to verify that the $\underline{\varepsilon}$-trace of $\tilde{v}_{\underline{\varepsilon}}$ relative to $\mathbb{R}^N$ converges to v on the boundary of tT).

Eventually, we use the scaling identity (4.5) and the definition of $\sigma(\nu)$ to get

$$\mathscr{F}_{\underline{\varepsilon}}(\tilde{v}_{\underline{\varepsilon}}, tT) = \underline{\varepsilon}^{N-1} \mathscr{F}\big(R_{\underline{\varepsilon}} \tilde{v}_{\underline{\varepsilon}} \, , \tfrac{t}{\underline{\varepsilon}} T\big) \geq t^{N-1} \sigma(\nu) \ , \tag{4.22}$$

and putting together (4.17), (4.19), (4.21) and (4.22), we obtain

$$\frac{d\lambda}{d\mu}(\bar{x}) \geq t^{N-1} \sigma(\nu) \ ;$$

the proof of inequality (4.13) is thus completed by taking t arbitrarily close to unity. $\square$

5. Proof of the upper bound inequality

Throughout this section Ω is always a regular open set.

Definition 5.1. A *N-dimensional polyhedral set in* $\mathbb{R}^N$ is an open set E whose boundary is a Lipschitz manifold contained in the union of finitely many affine hyperplanes; the *faces* of E are the intersections of the boundary of E with each one of these hyperplanes, and an *edge point* of E is a point which belongs to at least two different faces (that is, a point where ∂E is not smooth). We denote by ν_E the inner normal to ∂E (defined for all points in the boundary which are not edge points).

A *k-dimensional polyhedral set in* $\mathbb{R}^N$ is a polyhedral subset of a k-dimensional affine subspace of $\mathbb{R}^N$. A *polyhedral set in* Ω is the intersection of a polyhedral set in $\mathbb{R}^N$ with Ω.

We say that $u \in BV(\Omega, \pm 1)$ is a *polyhedral function* if there exists an N-dimensional polyhedral set E in $\mathbb{R}^N$ such that ∂E is transversal to $\partial\Omega$ (that is, $\mathscr{H}^{N-1}(\partial E \cap \partial\Omega) = 0$) and $u(x) = 1$ for every $x \in \Omega \cap E$, $u(x) = -1$ for every $x \in \Omega \setminus E$.

Theorem 5.2. *Let $u \in BV(\Omega, \pm 1)$ be a polyhedral function. Then there exists a sequence of functions (u_ε) defined on Ω such that $|u_\varepsilon| \le 1$ for every ε, u_ε converge to u uniformly on every compact set $K \subset \overline{\Omega} \setminus Su$, and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \le \int_{Su} \sigma(\nu_u)\, d\mathscr{H}^{N-1} . \tag{5.1}$$

Proof. Let us fix some notation: E is the polyhedral set associated with u in Definition 5.1; we denote by S the set of all edge points of E which belong to Ω and by Σ a general face of Su (that is, a face of E). Then S is a finite union of $(N-2)$-dimensional polyhedral sets in Ω, $\partial E = Su$, and we may choose the orientation of Su so that $\nu_E = \nu_u$ (for every point in $Su \setminus S$).

Given two open sets A_1, A_2, we denote by $A_1 \sqcup A_2$ the interior of $\overline{A}_1 \cup \overline{A}_2$. We define $\mathscr{G}$ as the class of all sets A such that

(i) A is an N-dimensional polyhedral set in Ω, and ∂A and Su are transversal (that is, $\mathscr{H}^{n-1}(Su \cap \partial A) = 0$);

(ii) there exists a sequence of functions (u_ε) defined on $\overline{A}$ such that $|u_\varepsilon| \le 1$ and

$$u_\varepsilon \to u \quad \text{uniformly on every compact set } K \subset \overline{A} \setminus Su, \tag{5.2}$$

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) \ \le \int_{A \cap Su} \sigma(\nu_u)\, d\mathscr{H}^{N-1} . \tag{5.3}$$

The proof of Theorem 5.2 is achieved by showing that $\Omega \in \mathscr{G}$; this is a consequence of the following three statements:

(a) *if A is an N-dimensional polyhedral set in Ω such that $\mathscr{H}^{N-1}(\overline{A} \cap Su) = 0$, then $A \in \mathscr{G}$;*

(b) *let Σ be a face of Su and let π be the projection map on the affine hyperplane which contains Σ: if A is an N-dimensional polyhedral set in Ω such that $Su \cap \overline{A} = \Sigma$ and $\pi(A) = \Sigma$, then $A \in \mathscr{G}$;*

(c) *if A_1, A_2 belong to $\mathscr{G}$ and are disjoint, then $A_1 \sqcup A_2 \in \mathscr{G}$.*

Step 1: proof of statement (a).

In this case, $\mathscr{H}^{N-1}(\partial A \cap Su) = 0$ and $A \cap Su = \emptyset$; then u is constant (-1 or 1) in A, and it is enough to take $u_\varepsilon := u$ for every $\varepsilon > 0$.

Step 2: proof of statement (b).

Property (i) is immediate; let us prove (ii). We denote by e the constant inner normal to Σ; therefore, Σ lies on some affine hyperplane which is parallel to the orthogonal complement to e, M; we may assume that Σ lies exactly on M. Following the notation of paragraph 1.3, for every fixed $\eta > 0$, we can find $C \in \mathscr{C}_e$ and $w \in X(C)$ such that

$$|C|^{-1} \mathscr{F}(w, T_C) \le \sigma(e) + \eta , \tag{5.4}$$

and then we define

$$u_\varepsilon(x) := w(x/\varepsilon) \quad \text{for every } x \in \mathbb{R}^N. \tag{5.5}$$

Property (5.2) holds because $w(x) \to \pm 1$ as $x_e \to \pm\infty$ (see paragraph 1.3). We claim that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, A) \le \mathscr{H}^{N-1}(\Sigma) \cdot \big(\sigma(e) + \eta\big) . \tag{5.6}$$

For the sake of simplicity, we assume that C is a unit cube.

Since Σ is a polyhedral set in the $(N-1)$-dimensional space M, for every $\varepsilon > 0$ we can cover it by a finite number $h = h(\varepsilon)$ of copies of the closed cube $\varepsilon\overline{C}$ (denoted by $x_i + \varepsilon\overline{C}$, with $x_i \in M$ for $i = 1, \dots, h$) so that

$$h\varepsilon^{N-1} = \sum_i \mathscr{H}^{N-1}\big(x_i + \varepsilon\overline{C}\big) \longrightarrow \mathscr{H}^{N-1}(\Sigma) \quad \text{as } \varepsilon \to 0. \tag{5.7}$$

Notice that A is included in the union of the strips $x_i + \varepsilon\overline{T}_C$ because Σ is the projection of A on M, and then by (4.1) we have

$$F_\varepsilon(u_\varepsilon, A) \le F_\varepsilon\big(u_\varepsilon, \cup_i(x_i + \varepsilon T_C)\big) \le \mathscr{F}_\varepsilon\big(u_\varepsilon, \cup_i(x_i + \varepsilon T_C)\big) \le \sum_{i=1}^h \mathscr{F}_\varepsilon(u_\varepsilon, \varepsilon T_C) , \tag{5.8}$$

where the last inequality follows from the fact that $\mathscr{F}_\varepsilon(u_\varepsilon, \cdot)$ is translation invariant and subadditive. Applying now the scaling identity (4.5) with $\bar{x} = 0$ and $\varepsilon = r$, we get $\mathscr{F}_\varepsilon(u_\varepsilon, \varepsilon T_C) = \varepsilon^{N-1} \mathscr{F}(w, T_C)$, so that by (5.8) and (5.4) we deduce

$$F_\varepsilon(u_\varepsilon, A) \le h\varepsilon^{N-1}\big(\sigma(e) + \eta\big) .$$

Taking into account (5.7) we get (5.6).

Since e coincides with ν_u in $\Sigma = Su \cap A$, (5.3) follows from inequality (5.6) by a simple diagonal argument, and the proof of statement (b) is complete.

Step 3: proof of statement (c).

Given disjoint $A_1, A_2 \in \mathscr{G}$, we set $A := A_1 \sqcup A_2$ and we take sequences (u^1_ε), (u^2_ε) which satisfy property (ii) for A_1 and A_2, respectively. Then we set

$$u_\varepsilon(x) := \begin{cases} u^1_\varepsilon(x) & \text{if } x \in A_1, \\[2ex] u^2_\varepsilon(x) & \text{if } x \in A_2. \end{cases}$$

One can check that properties (i) and (5.2) are satisfied, and that (5.3) reduces to

$$\lim_{\varepsilon\to 0} \Lambda_\varepsilon(u_\varepsilon, A_1, A_2) = 0 \ .$$

Notice that by (5.2) the ε-traces of u^i_ε relative to A_i converge to u on every Lipschitz hypersurface $\Sigma \subset \overline{A}_i$ such that $\mathscr{H}^{N-1}(\Sigma \cap Su) = 0$ for $i = 1, 2$ (cf. Remark 2.4); in particular, this holds true for $\Sigma = \partial A$. Hence the previous identity follows from Theorem 2.8.

Step 4: proof of Theorem 5.2.
It is clear that Ω can be written as $\Omega = \sqcup A_i$ with finitely many sets A_i which are pairwise disjoint and satisfy either the assumptions of statement (a) or of statement (b) above. Therefore Ω belongs to $\mathscr{G}$ by statement (c), and Theorem 5.2 follows from property (ii). □

To complete the proof of Theorem 1.4 we need the following lemma:

Lemma 5.3. *The function σ defined in paragraph 1.3 is upper semicontinuous on the unit sphere of $\mathbb{R}^N$.*

Proof. Fix a unit vector ν in $\mathbb{R}^N$, and for every linear isometry I of $\mathbb{R}^N$ set

$$\hat\sigma(I) := \inf\big\{|C|^{-1}\mathscr{F}(u\circ I, T_C) : \, C \in \mathscr{C}_\nu, \, u \in X(C)\big\} \tag{5.9}$$

(here we follow the notation of paragraph 1.3). One easily verifies that for every $u \in X(C)$ the map $I \mapsto \mathscr{F}(u \circ I, T_C)$ is continuous on the space $\mathscr{I}$ of all linear isometries of $\mathbb{R}^N$, and therefore $\hat\sigma$ is upper semicontinuous on $\mathscr{I}$ because it is defined in (5.9) as an infimum of continuous functions. We deduce the thesis by remarking that $\sigma(e) = \hat\sigma(I)$ whenever $e = I\nu$. □

5.4. Proof of statement (iii) of Theorem 1.4

For every $\mathbb{R}^N$-valued Borel measure μ on Ω we set

$$G(\mu) := \int_\Omega \sigma\big(\mu/|\mu|\big)\, d|\mu| \ , \tag{5.10}$$

where $\mu/|\mu|$ stands for the density of μ with respect to its total variation. Now statement (iii) of Theorem 1.4 reads as follows: for every function $u \in BV\big(\Omega, \pm 1\big)$ there exists a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $\limsup F_\varepsilon(u_\varepsilon, \Omega) \le G(Du)$.

By Theorem 5.2 this is true when u is a polyhedral function, and then the general case follows by a simple diagonal argument once we have proved that every function $u \in BV\big(\Omega, \pm 1\big)$ can be approximated (in $L^1(\Omega)$) by a sequence of polyhedral functions (u_n) so that $\limsup G(Du_n) \le G(Du)$. Now, every $u \in BV\big(\Omega, \pm 1\big)$ can be approximated by polyhedral functions (u_n) in variation, that is, $u_n \to u$ in $L^1(\Omega)$ and $\|Du_n\| \to \|Du\|$ (in fact, when Ω is regular, every set of finite perimeter can be approximated in variation by smooth sets, and hence also by polyhedral sets, see, for instance, [Gi], Theorem 1.24), and then it is enough to prove that G is upper semicontinuous with respect to convergence in variation of measures. Since σ is a non-negative upper semicontinuous function on the unit sphere of $\mathbb{R}^N$ (Lemma 5.3), this follows by a well-known result due to Reshetnyak (see, for instance, the appendix of [LM]). □

Conclusions

In this paper we have proved that the variational limit $F(u)$ of the rescaled functionals $F_\varepsilon(u)$ defined in (1.1) is finite only when u is equal to ± 1 a.e., and in this case is equal to the area of the (measure theoretic) interface which separates the sets $\{u = +1\}$ and $\{u = -1\}$ weighted by a positive anisotropic function σ which depends only on the normal to the interface (see formula (1.10) and Theorem 1.4).

The functionals F_ε can be viewed as rescalings of the free energy of a continuum Ising system with two stable phases $u = \pm 1$, and our result shows that in the thermodynamic limit the classical surface tension model for phase separation is recovered. The extension to the multi-phase case is also briefly sketched (see paragraph 1.12).

A key rôle in the proof is played by the ferromagnetic assumption, namely that the interaction potential J in (1.1) is positive (cf. paragraph 1.13). Even if our result holds for potentials J with a small negative part (see [AB2]), the problem of understanding what happens in the general case seems still wide open; indeed a quite different asymptotic behavior is expected, since in particular the ground states associated with the unscaled energy may be not constant.

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