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Local Mappings on Spaces of Differentiable Functions

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Abstract. We study conditions under which a functional $F(u,B)$ defined for every $u \in C^k(\Omega;\mathbb{R}^m)$ and every Borel subset B of Ω admits the integral representation

$$F(u,B) = \int_B f\big(x, D^k u(x)\big)\, d\mu(x)$$

for a suitable measure μ.

1. Introduction and Statement of the Main Result

The problem of representing in a suitable integral form local mappings defined on various spaces of functions has been widely studied in the literature (see references); however, an important difference has to be remarked between the case of spaces "without derivatives" as $L^p(\Omega;\mathbb{R}^m)$ or $\mathscr{M}(\Omega;\mathbb{R}^m)$, and the case of spaces "with derivatives" as $W^{1,p}(\Omega;\mathbb{R}^m)$ or $BV(\Omega;\mathbb{R}^m)$. Actually, in the first case the results are very general, and only locality and lower semicontinuity are required to get an integral representation (see for instance Buttazzo & Dal Maso [7] and Bouchitté & Buttazzo [4]), whereas in the second case additional hypotheses as growth conditions are usually imposed (see for instance Buttazzo & Dal Maso [8], Bouchitté & Dal Maso [5], Alberti & Buttazzo [2], Alberti [1]).

In this paper we consider mappings defined on spaces of differentiable functions as $C^k(\Omega;\mathbb{R}^m)$ and we prove (see Theorem 1.2) that under very mild assumptions they are representable in the form

$$\int_\Omega f\big(x, D^k u(x)\big)\, d\mu(x)$$

for a suitable measure μ.

Throughout the whole paper Ω denotes a bounded open subset of $\mathbb{R}^n$, k a nonnegative integer, $\mathscr{A}(\Omega)$ the collection of all open subsets of the open set Ω, $\mathscr{B}(\Omega)$ the σ-field of all Borel subsets of Ω, $\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and $|B| = \mathscr{L}^n(B)$ the Lebesgue measure of the Borel set B.

We refer to customary functional analysis notation (see Brezis [6], Chapter IX, and Rudin [10], Chapter 6); to avoid any ambiguity, however, we use greek letters for multi-indexes $\alpha = (a_1, \ldots, a_n)$ with norm $|\alpha| = a_1 + \ldots + a_n$, and we denote by D^α the partial derivative $(\partial/\partial x_1)^{a_1} \ldots (\partial/\partial x_n)^{a_n}$.

Let $I(k)$ be the set of all multi-indexes α with $|\alpha| = k$; if u is a function from Ω into $\mathbb{R}^m$ we denote by $D^k u$ the k-th derivative of u, i.e., the function of Ω into $(\mathbb{R}^m)^{I(k)}$ defined by $\big(D^k u(x)\big)_\alpha = D^\alpha u(x)$ for all $\alpha \in I(k)$ and $x \in \Omega$.

In the paper the following function spaces are used.

$C^k(\Omega, \mathbb{R}^m)$ the space of all functions of Ω into $\mathbb{R}^m$ such that $D_\alpha u$ is a continuous function for every α with $|\alpha| \le k$; $C^k(\Omega, \mathbb{R}^m)$ is usually endowed with the structure of Fréchet space induced by the seminorms $\phi_{\alpha,K}$ given by

$$\phi_{\alpha,K}(u) = \sup_{x \in K} \big|D_\alpha u(x)\big| \quad \text{for all } u,$$

where α is a multi-index with norm $|\alpha| \le k$ and K is a compact subset of Ω.

$C^k_c(\Omega, \mathbb{R}^m)$ the space of all functions $u \in C^k(\Omega, \mathbb{R}^m)$ with compact support in Ω.

$C^k_0(\Omega, \mathbb{R}^m)$ the space of all functions u such that $D_\alpha u$ is a continuous function which vanish at infinity, i.e., such that for every $\varepsilon > 0$ there exists a compact set K which satisfies $|D_\alpha u(x)| < \varepsilon$ for all $x \in \Omega \setminus K$ for every multi-index α with $|\alpha| \le k$.

$C^k(\overline{\Omega}, \mathbb{R}^m)$ the space of all u such that $D_\alpha u$ admits a continuous extension to the set $\overline{\Omega}$ for all α with $|\alpha| \le k$. Both $C^k_0(\Omega, \mathbb{R}^m)$ and $C^k(\overline{\Omega}, \mathbb{R}^m)$ are closed subspace of the Sobolev space $W^{k,\infty}(\Omega, \mathbb{R}^m)$ and are usually endowed with the norm

$$\|u\|_{W^{k,\infty}} = \sum_{\alpha \in I(k)} \|D_\alpha u\|_\infty.$$

Let us recall now some definitions which will be used in the following.

By measure on Ω we mean any σ-additive set function defined on the σ-field $\mathscr{B}(\Omega)$ with values in $]-\infty, +\infty]$. If λ is a measure on Ω, $|\lambda|$ denotes its total variation and λ^+, λ^- denote its positive and negative variations respectively. We say that a measure μ is absolutely continuous with respect to λ (and we write $\mu \ll \lambda$) if $|\mu|(E) = 0$ whenever $|\lambda|(E) = 0$. We say that $\lambda \le \mu$ if $\lambda(B) \le \mu(B)$

for all Borel sets B. If λ is a measure on Ω and $B \in \mathscr{B}(\Omega)$, we denote by $\lambda \llcorner B$ the measure given by

$$\big[\lambda \llcorner B\big](E) = \lambda(B \cap E) \qquad \forall E \in \mathscr{B}(\Omega).$$

The supremum of a collection $\mathscr{F} = \{\lambda_i\}_{i \in I}$ of measures on Ω is defined by

$$\Big[\bigvee_{\lambda \in \mathscr{F}} \lambda\Big](E) = \sup\Big\{\sum_{j \in J} \lambda_j(E_j)\Big\} \qquad \forall E \in \mathscr{B}(\Omega) \tag{1.1}$$

where the supremum is taken over all finite subsets J of I and all Borel partitions $\{E_j\}_{j \in J}$ of E. It may be easily proved that it is always a measure; moreover, if every λ is a positive measure and A is an open set,

$$\Big[\bigvee_{\lambda \in \mathscr{F}} \lambda\Big](A) = \sup\Big\{\sum_{i=1}^h \lambda_i(A_i)\Big\} \tag{1.2}$$

where the supremum is taken over all families $\{A_1, \ldots, A_h\}$ of pairwise disjoint open subsets of A and $\{\lambda_1, \ldots, \lambda_h\}$ of measures in $\mathscr{F}$.

We say that a subspace X of $C(\Omega, \mathbb{R}^m)$ is a local space if it is closed under multiplication by functions in $C^\infty_c(\Omega)$. Any mapping $F : X \times \mathscr{B}(\Omega) \to]-\infty, +\infty]$ will be called a functional on X.

Definition 1.1. *Let X be a local subset of $C^k(\Omega, \mathbb{R}^m)$ endowed with a topology τ and let F be a functional on X. Then we say that:*

(i) *F is a (finite, positive) measure if $F(u, \cdot)$ is a (finite, positive) measure on Ω for all $u \in X$; F is absolutely continuous with respect to a positive measure λ if $F(u, \cdot)$ is absolutely continuous with respect to λ for all $u \in X$;*

(ii) *F is D^k-local if $F(u, B) = F(v, B)$ whenever $D^k u$ and $D^k v$ agrees on an open set which include B; notice that when F is finite, F is D^k-local means that $F(u, A) = F(v, A)$ whenever A is an open set and u and v are functions whose k-th derivatives agree on A;*

(iii) *F is (sequentially) τ-l.s.c. on $\mathscr{A}$ (resp., on $\mathscr{B}$) if $F(\cdot, B)$ is a (sequentially) τ-l.s.c. function of X for every open (resp., Borel) sets $B \subset \Omega$;*

(iv) *F is (k, ∞)-bounded if, for every real number $r > 0$, there exist a finite set E and a positive finite measure ϕ (both depending on r) such that $|F(u, \cdot)| \llcorner \Omega \setminus E \le \phi$ for all $u \in X$ with $\|D^k u\|_\infty \le r$ and this means that*

$$|F(u, B)| \le \phi(B) \qquad \text{whenever } B \subset \Omega \setminus E \text{ and } \|D^k u\|_\infty \le r.$$

We can now state the main result of this paper.

Theorem 1.2. *Let X be one of the spaces $C^k(\Omega,\mathbb{R}^m)$, $C_0^k(\Omega,\mathbb{R}^m)$, $C^k(\overline{\Omega},\mathbb{R}^m)$, and let F be a functional on X which satisfies the following conditions:*

(i) *F is a finite measure,*

(ii) *F is (strongly) l.s.c. on open sets,*

(iii) *F is D^k-local.*

Then there exists a finite positive measure λ on Ω and a Borel function $f:\Omega\times(\mathbb{R}^m)^{I(k)}\to\mathbb{R}$ which is l.s.c. with respect to second variable and satisfies

$$F(u,B)=\int_B f\big(x,D^k u(x)\big)\,d\lambda(x)\qquad \text{for all } u\in X \text{ and } B\in\mathscr{B}(\Omega). \tag{1.3}$$

Moreover f is unique in the following sense: if f' is a Borel function which satisfies representation formula (1.3), then there exists a λ negligible set $N\subset\Omega$ such that $f'(x,s)=f(x,s)$ for all $x\in\Omega\setminus N$ and all $s\in(\mathbb{R}^m)^{I(k)}$. It follows immediately from this fact that if F is continuous, $f(x,\cdot)$ is continuous for λ-a.e. $x\in\Omega$.

Further details and generalizations of this statement are are illustrated in Section 4 (cf. Theorem 3.3 and Remark 3.4). In the proofs we shall need the following results.

Theorem 1.3. (Besicovitch Covering Lemma, see Morgan [9], Theorem 2.7) *Suppose λ is a finite positive measure on $\mathbb{R}^n$, E is a Borel subset of Ω and $\mathscr{F}$ is a collection of non trivial closed balls such that* $\inf\big\{r\ :\ B(x,r)\in\mathscr{F}\big\}=0$ *for all $x\in E$. Then for every $\varepsilon>0$ there exists a finite disjoint collection $\mathscr{F}'\subset\mathscr{F}$ such that*

$$\lambda\Bigg(E\setminus\bigcup_{B\in\mathscr{F}'}B\Bigg)>\varepsilon.$$

Theorem 1.4. (see Alberti [1], Theorem 5.8) *Let λ be a positive finite measure on $\mathbb{R}^n$. Then there exists a Borel set E such that $\lambda(\mathbb{R}^n\setminus E)=0$ and*

$$\limsup_{r\to 0}\frac{\lambda(B(x,ar))}{\lambda(B(x,r))}\geq a^n\qquad \forall x\in E,\ \forall a\in]0,1[.$$

Lemma 1.5. (Glueing Lemma, see Alberti [1], Lemma 5.10) *Let X be a local subspace of $C^k(\Omega,\mathbb{R}^m)$, let $B(x_1,r_1),\dots,B(x_h,r_h)$ be pairwise disjoint open balls, and let $u_1,\dots u_h\in X$. Then, for every $\varepsilon>0$ there exists $u\in X$ such that*

(i) $u=0$ on $\Omega\setminus\big(\cup B(x_i,r_i)\big)$;

(ii) $D^k u=D^k u_i$ on $B(x_i,(1-\varepsilon)r_i)$ for all i;

(iii) $\|D^k u\|_{L^\infty(B_i)}\leq C\varepsilon^{-k}\|D^k u_i\|_{L^\infty(B_i)}$ for all i, where C is a constant which depends on n and k only.

Lemma 1.6. (see Alberti [1], Lemma 3.8) *Let (T,d) be a separable metric space and let $f:T\to[-\infty,+\infty]$ be a l.s.c. function. Then there exists a countable set $S\subset T$ such that f is the relaxation on X of its restriction to S, i.e.,*

$$f(x)=\liminf_{\substack{y\to x\\ y\in S}} f(y)\qquad \forall x\in T.$$

2. Finite Functionals

Unless differently stated, throughout this section X is the space $C^k\big(\overline{\Omega},\mathbb{R}^m\big)$ and F is a functional on X which is D^k-local and measure.

Definition 2.1. *For every $u\in X$ and for every positive continuous function g on Ω we set*

$$\begin{aligned}
&P(u,g)=\big\{x\ :\ |D^k u(x)|<g(x)\big\}\\
&\lambda_{u,g}=F(u,\cdot)\llcorner P(u,g)\\
&\lambda_g=\bigvee_{u\in X}|\lambda_{u,g}|\\
&S_g=\big\{x\ :\ \lambda_g(\{x\})=+\infty\big\}.
\end{aligned}$$

Notice that F is (k,∞)-bounded if S_r and λ_r are finite for every positive constant r.

Theorem 2.2. *Assume F is a positive functional and g is a bounded continuous function. If F is finite, then S_g is finite and $\lambda_g(\Omega\setminus S_g)<+\infty$. In particular F is (k,∞)-bounded.*

Proof. We assume by contradiction that S_g is not a finite set or $\lambda_g(\Omega\setminus S_g)=+\infty$ and then we prove that there exists a function u such that $F(u,\Omega)=+\infty$. This proof is divided in four steps. In the following, we say that a sequence of open sets $\{B_h\}$ is strictly increasing if $\overline{B}_h\subset B_{h+1}$ for every h.

Step 1: *there exists a countable collection $\{A_i\ :\ i\in I\}$ of pairwise disjoint open sets such that $\lambda_g(A_i)>1$ for all i.*

If S_g is infinite, the proof is trivial. Suppose that $\lambda_g(\Omega \setminus S_g) = +\infty$ and set $\lambda = \lambda_g \llcorner B \setminus S_g$. By Lemma 2.3 below there exists a strictly increasing sequence of open sets $\{B_h\}$ such that $\lambda(\Omega \setminus \cup B_h) < +\infty$ and $\lambda(\Omega \setminus B_h) = +\infty$ for all h. We may suppose that $B_0 = \emptyset$. By induction on h, we choose integers m_h so that $m_0 = 0$ and $\lambda\big(B_{m_h} \setminus B_{m_{h-1}}\big) > 1$ for all $h \geq 1$. Let $h \geq 1$ be a fixed integer and let m_{h-1} be chosen. By the choice of B_h we have

$$\lim_{m\to\infty} \lambda\big(B_m \setminus B_{m_{h-1}+1}\big) = \lambda\big(\Omega \setminus B_{m_{h-1}+1}\big) = +\infty$$

and then there exists an integer m_h such that $\lambda\big(B_{m_h} \setminus B_{m_{h-1}+1}\big) > 1$. Set $A_h = B_{m_h} \setminus \overline{B}_{m_{h-1}}$ for all $h \in \mathbb{N}$. Recalling that B_{m-1} is relatively compact in B_m for all m and that $\lambda_g \geq \lambda$, it may easily be proved that the collection $\{A_h \;:\; h \in \mathbb{N}\}$ satisfies our thesis.

Step 2: *there exist a countable collections $\{A_i \;:\; i \in I\}$ of pairwise disjoint open sets and a countable collection $\{v_i \;:\; i \in I\}$ of functions in X such that for all $i \in I$*

$$\sum_i F(v_i, A_i) = +\infty \tag{2.1}$$

$$\|D^k v_i\|_{L^\infty(A_i)} \leq \|g\|_{L^\infty(A_i)}. \tag{2.2}$$

By Step 1 we may find countably many pairwise disjoint open sets $E_i \subset \Omega$ such that $\lambda_g(E_i) > 1$ for every $i \in I$. Let i be fixed. Since $\lambda_g(E_i) > 1$, by definition of λ_g (cf. formulas (1.1) and (1.2)) we may find functions $v_{i,1}, \ldots, v_{i,h}$ and pairwise disjoint open sets $B_{i,1} \ldots B_{i,h} \subset E_m$ such that

$$\sum_{j=1}^{h} F\big(v_{i,j}, B_{i,j} \cap P(v_{i,j}, g)\big) > 1$$

and then it is enough to consider the collection of all functions $v_{i,j}$ and the collection of all open sets $A_{i,j} = B_{i,j} \cap P(v_{i,j}, r)$.

Step 3: *there exist a point $s \in (\mathbb{R}^m)^{I(k)}$ and, for all $h \in \mathbb{N}$, pairwise disjoint open sets $A_h \subset \Omega$ and functions $v_h \in X$ so that*

$$\sum_h F(v_h, A_h) = +\infty \tag{2.3}$$

$$\lim_{h\to+\infty} \|D^k v_h - s\|_{L^\infty(A_h)} = 0. \tag{2.4}$$

By Step 2, there exist, for all $h \in \mathbb{N}$, pairwise disjoint open sets $E_h \subset \Omega$ and functions $v_h \in X$ such that $\sum_h F(v_h, E_h) = +\infty$ and $\|D^k v_h\|_{L^\infty(E_h)} \leq \|g\|_{L^\infty(E_h)}$ for all h. We want to find open sets $A_h \subset E_h$ so that (2.3) and (2.4) hold. For every h, set $\mu_h = F(v_h, \cdot) \llcorner E_h$; each μ_h is a positive finite measure. We build, by induction on m, a decreasing sequence of bounded closed sets $C_m \subset (\mathbb{R}^m)^{I(k)}$ and an increasing sequence of integers h_m such that, for all m

$$\operatorname{diam} C_m \leq 1/m \tag{2.5}$$

$$\sum_{h=0}^{\infty} \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_m\big\}\Big) = +\infty \tag{2.6}$$

$$\sum_{h=h_{m-1}}^{h_m-1} \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_m\big\}\Big) > 1. \tag{2.7}$$

Set $C_0 = \big\{s \in (\mathbb{R}^m)^{I(k)} \;:\; |s| \leq \|g\|_\infty\big\}$. It is obvious that (2.5) and (2.6) hold and by (2.6) there exists h_0 such that (2.7) holds (if we take $h_{-1} = 0$). Let $m > 0$ be a fixed integer and suppose that C_{m-1} and h_{m-1} have already been chosen. Since C_{m-1} is bounded, it may be covered by finitely many closed balls $B_p \subset (\mathbb{R}^m)^{I(k)}$ with radius less than $1/m$ and then, by (2.6),

$$\sum_p \Big[\sum_h \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_{m-1} \cap B_p\big\}\Big)\Big] \geq \sum_h \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_{m-1}\big\}\Big) = +\infty$$

and then there exists $\bar{p}$ such that

$$\sum_h \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_{m-1} \cap B_{\bar{p}}\big\}\Big) = +\infty.$$

Hence (2.5) and (2.6) holds if we set $C_m = C_{m-1} \cap B_{\bar{p}}$. Moreover,

$$\sum_{h=h_{m-1}}^{\infty} \mu_h\Big(\big\{x \;:\; D^k v_h(x) \in C_{m-1} \cap B_{\bar{p}}\big\}\Big) = +\infty$$

and then there exists an integer h_m such that (2.7) holds. Now, let A_h be the open set given by

$$A_h = \Big\{x \in E_h \;:\; \operatorname{dist}\big(D^k v_h(x), C_m\big) < 1/m\Big\}$$

for all m and for all h with $h_{m-1} \le h < h_m$. By (2.7) and by the definition of μ_h we get

$$\sum_h F(v_h, A_h) \ge \sum_{m=0}^{\infty} \left[\sum_{h=m_{h-1}}^{m_h - 1} \mu_h\Big(\big\{x \ :\ D^k v_h(x) \in C_m\big\}\Big)\right] = +\infty$$

and (2.3) is proved. By (2.5) the intersection of all C_m contains just one element which we denote by s and then (2.5) yields (2.4).

Step 4: *there exists a function $u \in X$ such that $F(u,\Omega) = +\infty$.*
Take $s \in (\mathbb{R}^m)^{I(k)}$, pairwise disjoint open sets $A_h \subset \Omega$ and functions $v_h \in X$ such that (2.3) and (2.4) hold. Let $p : \mathbb{R}^n \to \mathbb{R}^m$ be the homogeneous polynomial function which satisfy $D^k p = s$ everywhere and apply Lemma 2.4 below to find functions u_h with compact supports in A_h such that, for all h

(a) $\qquad F(p + u_h, A_h) \ge 2^{-(3n+1)} F(v_h, A_h),$

(b) $\qquad \|D^k u_h\|_\infty \le C 2^k \|D^k v_h - s\|_{L^\infty(A_h)}.$

Taking into account Poincaré inequality, (b) and the fact that the support of each u_h is included in A_h, we obtain that the series $\sum u_h$ converges in norm to a function in X and we write $u = p + \sum u_h$. Hence (a) and the fact that F is D^k-local yields

$$F(u,\Omega) \ge \sum_h F(u, A_h) = \sum_h F(p + u_h, A_h) = \sum_h F(v_h, A_h) = +\infty. \ \square$$

Lemma 2.3. *Let λ be a positive measure on Ω such that every point has finite measure. If λ is not finite there exists a strictly increasing sequence of open sets $\{B_h\}$ such that $\lambda(\Omega \setminus \cup B_h) < +\infty$ and $\lambda(\Omega \setminus B_h) = +\infty$ for all integers h.*

Proof. Let $\{\Omega_h\}$ be a strictly increasing sequence of open sets which cover Ω and set

$$A = \big\{x \ :\ \lambda\big(B(x,r)\big) = +\infty \ \forall r > 0\big\}.$$

If $A = \emptyset$, every compact set $K \subset \Omega$ can be covered by a finite collection of open balls with finite measure and then it has finite measure. Hence it is enough to take $B_h = \Omega_h$ for all h. If $A \ne \emptyset$, there exists $x \in \Omega$ such that $\lambda\big(B(x,r)\big) = +\infty$ for all $r > 0$. Hence it is enough to take $B_h = \Omega_h \setminus \overline{B(x, 1/h)}$ for all integers h. $\square$

Lemma 2.4. *Let p, v be functions in X and let $A \subset \Omega$ be an open set. Then there exists a function $u \in X$ with compact support in A such that*

(i) $\qquad F(u + p, A) \ge 2^{-(3n+1)} F(v, A)$

(ii) $\qquad \|D^k u\|_\infty \le C 2^k \|D^k(v - p)\|_{L^\infty(A)}$

where C is the same constant of Lemma 1.5.

Proof. Let λ be the positive finite measure $F(v,\cdot)$. By Theorem 1.4 there exists a Borel set $E \subset A$ such that $\lambda(A \setminus E) = 0$ and, for all $x \in E$,

$$\limsup_{r \to 0} \frac{\lambda\big(B(x,ar)\big)}{\lambda\big(B(x,r)\big)} \ge a^n \qquad \forall a \in]0,1[. \tag{2.8}$$

Let $\mathscr{F}$ be the collection of all closed balls $\overline{B(x,r)} \subset A$ which satisfy

$$\frac{\lambda\big(B(x,r/2)\big)}{\lambda\big(B(x,2r)\big)} \ge 2^{-(2n+1)}. \tag{2.9}$$

By (2.8) for all $x \in E$ there exist closed balls $\overline{B(x,r)}$ in $\mathscr{F}$ with r arbitrary small. Hence we may apply Theorem 1.3 to find closed balls $\overline{B}_i = \overline{B(x_i, r_i)} \in \mathscr{F}$ with $i = 1, \dots, n$ such that

$$\lambda\big(\cup \overline{B}_i\big) \ge \frac{1}{2}\lambda(E) = \frac{1}{2}\lambda(A) \tag{2.10}$$

and by Glueing Lemma 1.5 there exists $u \in X$ such that

(a) $u = 0$ out of the union of all B_i,

(b) $D^k u = D^k(v - p)$ within $\overline{B(x_i, r_i/2)}$ for all i,

(c) $\|D^k u\|_{L^\infty(B_i)} \le C 2^k \|D^k(v-p)\|_{L^\infty(B_i)}$ for all i, where C is a constant which depends on n and k only.

By (a) we obtain that u has compact support in A. By (b), (2.9), (2.10) and by the fact that F is D^k-local we obtain

$$\begin{aligned} F(p+u, A) &\ge \sum_i F\big(p + u, B(x_i, r_i/2)\big) = \sum_i F\big(v, B(x_i, r_i/2)\big) \\ &= \sum_i \lambda\big(B(x_i, r_i/2)\big) \ge \sum_i 2^{-(2n+1)} \lambda\big(B(x_i, 2r_i)\big) \\ &\ge \sum_i 2^{-(2n+1)} \lambda\big(\overline{B(x_i, r_i)}\big) \ge 2^{-(2n+2)} \lambda(A) \end{aligned}$$

and (i) is proved. Finally (a), (c) and the fact that the balls $\overline{B}_i$ were chosen pairwise disjoint and included in A yield

$$\|D^k u\|_\infty = \sup_i \|D^k u\|_{L^\infty(B_i)} \le \sup_i C 2^k \|D^k(v-p)\|_{L^\infty(B_i)} \le C 2^k \|D^k(v-p)\|_\infty$$

and (ii) is proved. $\square$

Corollary 2.5. *Let F be a functional on X which is D^k-local. Then the following three statements are equivalent:*

(i) *F is finite;*

(ii) *S_g is finite and $\lambda_g(\Omega \setminus S_g) < +\infty$ for every bounded positive continuous function g;*

(iii) *F is (k,∞)-bounded.*

Proof. Implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial. Let us prove (i) $\Rightarrow$ (ii). For every $u \in X$, let $F^+(u,\cdot)$ and $F^-(u,\cdot)$ be the positive and negative variations of $F(u,\cdot)$ respectively. It is obvious that both F^+ and F^- are functionals on X which are D^k-local positive finite measures. Then it is enough to apply Theorem 2.2 to F^+ and F^-. □

Remark 2.6. Of course, statements very similar to Theorem 2.2 (and Corollary 2.5) hold when we consider functionals which are defined on other spaces of continuously differentiable functions. In particular, when X is $C^k(\Omega,\mathbb{R}^m)$, we have that F is finite if and only if S_g is finite and $\lambda_g(\Omega \setminus S_g) < +\infty$ for every positive continuous function g (and then a finite functional is (k,∞)-bounded). When X is $C_0^k(\Omega,\mathbb{R}^m)$, F is finite if and only if S_g is finite and $\lambda_g(\Omega \setminus S_g) < +\infty$ for every positive $g \in C_0(\Omega)$ (in this case a finite functional may be not (k,∞)-bounded).

In general, optimal results of this kind hold for a lot of spaces but they do not follow directly from Theorem 2.2 and Corollary 2.5. However, the following generalization of Corollary 2.5 is straightforward: when X is a subset of $C^k(\Omega,\mathbb{R}^m)$ which contains $C_c^k(\Omega,\mathbb{R}^m)$ and F is a functional on X which is D^k-local and is a (locally) finite measure, then F is locally (k,∞)-bounded, and this means that for every positive real number r and every compact set $K \subset \Omega$, there exists a finite set E and a finite measure ϕ such that $|F(u,\cdot)| \llcorner K \setminus E \le \phi$ whenever $\|D^k u\|_{L^\infty(K)} \le r$.

Examples may be given of linear spaces X which do not contain $C_c^k(\Omega,\mathbb{R}^m)$ and of functionals F on X which are D^k-local finite measures but are not locally (k,∞)-bounded.

Corollary 2.7. *Suppose that F is a functional on X which is a D^k-local finite measure. Then there exists a positive finite measure λ such that F is absolutely continuous with respect to λ.*

Proof. Since F is (k,∞)-bounded, we may find, for every integer h, finite positive measures ϕ_h on Ω and finite sets $E_h \subset \Omega$ such that $|F(u,\cdot)| \llcorner \Omega \setminus E_h \le \phi_h$ for every u with $\|D^k u(x)\|_\infty \le h$ and then $F(u,\cdot) \ll \lambda$ for every $u \in X$ if we take

$$\lambda = \sum_h \frac{1}{2^h \|\phi_h\|}\phi_h + \sum_h \frac{1}{2^h \mathscr{H}_0(E_h)} \mathscr{H}_0 \llcorner E_h. \ \square$$

Remark 2.8. Of course, Corollary 2.7 holds even if X is a subset of $C^k(\Omega,\mathbb{R}^m)$ which contains $C_c^k(\Omega,\mathbb{R}^m)$ and F is a functional on X which is a finite measure and D^k-local (cf. Remark 2.6).

When X is a a subset of $C^k(\Omega,\mathbb{R}^m)$ which does not contain $C_c^k(\Omega,\mathbb{R}^m)$, we have the following result: if F is a functional on X which is a positive finite measure, l.s.c. with respect to a separable topology τ, then there exists a finite measure λ such that F is absolutely continuous with respect to λ. If we drop either lower semicontinuity or positivity, examples may be given of functionals which are D^k-local finite measures but which are not a.c. with respect to any finite measure λ.

Theorem 2.9. *Let F be a functional on X which is a measure, (k,∞)-bounded and l.s.c. on $\mathscr{A}$. Then F is l.s.c. on $\mathscr{B}$.*

Proof. Let $\{u_h\}$ be a sequence of functions in the space $C^k(\overline{\Omega},\mathbb{R}^m)$ which converges to u in norm and let B be a Borel subset of Ω. We want to prove that $\liminf_{h\to+\infty} F(u_h,B) \ge F(u,B)$.

Notice that there exists r such that $r \ge \|D^k u_h\|_\infty$ for every h. Since F is (k,∞)-bounded, there exists a finite set E and a positive finite measure ϕ such that $|F(u,B)| \le \phi(B)$ whenever $B \cap E = \emptyset$ and $\|D^k u\|_\infty \le r$. Then, for every h and for every open set A which satisfies $A \supset B$ and $(A \setminus B) \cap E = \emptyset$, $F(u_h,B) \ge F(u_h,A) - \phi(A \setminus B)$ and $F(u,A) \ge F(u,B) - \phi(A \setminus B)$ and then, taking into account that F is l.s.c. on $\mathscr{A}$,

$$\begin{aligned}\liminf_{h\to+\infty} F(u_h,B) &\ge \liminf_{h\to+\infty} F(u_h,A) - \phi(A \setminus B)\\ &\ge F(u,A) - \phi(A \setminus B) \ge F(u,B) - 2\phi(A \setminus B).\end{aligned}$$

Since S is finite, we may find open sets A which satisfy the conditions above so that $\phi(A \setminus B)$ is arbitrary small and then the proof is complete. □

Corollary 2.10. *Let F be a functional on X which is a finite measure, D^k-local and l.s.c. on $\mathscr{A}$. Then F is l.s.c. on $\mathscr{B}$.*

Proof. It is enough to apply Corollary 2.5 and Theorem 2.9. □

Remark 2.11. Let X be a subset of $C^k(\Omega,\mathbb{R}^m)$ and let τ be a topology on X. We say that τ satisfies a condition (C_k) when:

(C_k) *every $u \in X$ admits an open neighborhoud $A \in \tau$ so that for every compact set $K \subset \Omega$ there exists a positive real number r and $|D^k v(x)| \leq r$ for all $v \in A$ and all $x \in K$.*

Condition (C_k) is verified when X is (a subset of) $C^k(\Omega,\mathbb{R}^m)$ or $C^k(\overline{\Omega},\mathbb{R}^m)$ or $C_0^k(\Omega,\mathbb{R}^m)$ endowed with the usual strong or weak topologies. But it is not verified, for example, when X is $C^k(\Omega,\mathbb{R}^m)$ endowed with the strong topology of $C^h(\Omega,\mathbb{R}^m)$ with $h < k$.

Theorem 2.9 may be generalized in the following form. Let X be a subset of $C^k(\Omega,\mathbb{R}^m)$ and let τ be a topology on X which satisfies condition (C_k). Let F be a functional on X which is a measure, locally (k,∞)-bounded and (sequentially) τ-l.s.c. on $\mathscr{A}$. Then F is (sequentially) τ-l.s.c. on $\mathscr{B}$.

If we drop the (C_k) condition, examples may be given of topologies τ and functionals F which are measures, locally (k,∞)-bounded, τ-l.s.c on $\mathscr{A}$ but not τ-l.s.c. on $\mathscr{B}$.

Taking into account Remark 2.7, Corollary 2.10 may be generalized in the following form. Let X be a subset of $C^k(\Omega,\mathbb{R}^m)$ which contains $C_c^k(\Omega,\mathbb{R}^m)$ and let τ be a topology on X which satisfies condition (C_k). Let F be a functional on X which is a finite measure, D^k-local and (sequentially) τ-l.s.c. on $\mathscr{A}$. Then F is (sequentially) τ-l.s.c. on $\mathscr{B}$.

3. Representation Theorem

Unless differently stated, throughout this section X is $C^k(\overline{\Omega},\mathbb{R}^m)$, λ is a finite positive measure on Ω and F is a fixed functional on X which is a finite measure absolutely continuous with respect to λ, D^k-local on $\mathscr{A}$ and strongly l.s.c. on $\mathscr{B}$.

We recall that by Corollary 2.5 and 2.10, for every functional F which is a finite measure, D^k-local on $\mathscr{A}$ and strongly l.s.c. on $\mathscr{A}$ there exists a finite positive measure λ so that the conditions above are fulfilled.

Definition 3.1. *Since for all $u \in X$, $F(u,\cdot)$ is a finite measure on Ω which is absolutely continuous with respect λ, it may be represented by a Borel function in $L^1(\lambda)$ that we denote by f_u. In other words we have*

$$F(u,B) = \int_B f_u\,dx \quad \text{for all } B \in \mathscr{B}(\Omega). \tag{3.1}$$

Let $\mathscr{S}$ be a countable subset of X. For all $x \in \Omega$ and all $s \in (\mathbb{R}^m)^{I(k)}$ set

$$f_{\mathscr{S}}(x,s) = \sup_{\varepsilon>0}\Big[\inf\Big\{f_v(x)\ :\ v \in \mathscr{S},\ |D^k v(x) - s| \leq \varepsilon\Big\}\Big] \tag{3.2}$$

where the infimum is taken $+\infty$ when the set is empty.

In other words we have that

$$f_{\mathscr{S}}(x,s) = \inf\big[\liminf_{h\to+\infty} f_{v_h}(x)\big],$$

where the infimum is taken over all sequences $\{v_h\}$ in $\mathscr{S}$ so that $\lim_h D^k v_h(x) = s$ and is $+\infty$ when s does not belong to the closure of the set $\{D^k v(x) : v \in \mathscr{S}\}$. Then f has the following properties.

Proposition 3.2. *For every countable set $\mathscr{S}$, $f_{\mathscr{S}}$ is a real Borel function of $\Omega \times (\mathbb{R}^m)^{I(k)}$ which is l.s.c. with respect to second variable for every $x \in \Omega$ and $f\big(x, D^k v(x)\big) \leq f_v(x)$ for all x and all $v \in \mathscr{S}$. Moreover, if E is the set of all points $x \in \Omega$ with $\lambda\big(\{x\}\big) > 0$, for every $r > 0$ there exists a function $g_r \in L^1(\lambda)$ such that*

$$f_{\mathscr{S}}(x,s) \geq g_r(x) \tag{3.3}$$

for all $x \in \Omega \setminus E$ and all $s \in (\mathbb{R}^m)^{I(k)}$ with $|s| < r$.

Proof. For every positive integer h and every $v \in \mathscr{S}$, let $f_{h,v}$ be the function on $\Omega \times \mathbb{R}^m$ which is given by

$$f_{h,v}(x,s) = \begin{cases} f_v(x) & \text{if } \big|s - D^k v(x)\big| \leq 1/h \\ +\infty & \text{otherwise,} \end{cases}$$

and notice that for all x, s,

$$f_{\mathscr{S}}(x,s) = \sup_h \inf_{v\in\mathscr{S}} f_{h,v}(x,s).$$

Since each $f_{h,v}$ is a Borel function, $f_{\mathscr{S}}$ is a Borel function. It is obvious that $f_{\mathscr{S}}$ is l.s.c. with respect to second variable and that

$$f_{\mathscr{S}}\big(x, D^k v(x)\big) \leq f_v(x) \qquad \forall (x,v) \in \Omega \times \mathscr{S}.$$

Moreover, as F is finite, by Corollary 2.5 for every $r > 0$ the measure λ_r and the set S_r given in Definition 2.1 are finite and then, since F is absolutely continuous with respect to λ, we may find a function g_r in $L^1(\lambda)$ such that $f_v(x,r) \geq g_r(x)$ for all $v \in \mathscr{S}$ and all $x \in \Omega \setminus E$ with $\big|D^k v(x)\big| < r$ and then (3.3) is proved. □

The main result of this section is the following theorem.

Theorem 3.3. *There exists a countable set $\mathscr{S} \subset X$ such that $f_{\mathscr{S}}$ represents F that is*

$$F(u,B) = \int_B f_{\mathscr{S}}\big(x, D^k u(x)\big)\, d\lambda(x) \qquad \textit{for all } u, B. \tag{3.4}$$

Proof. We want to show that there exists a countable set $\mathscr{S} \subset X$ such that, for every $u \in X$, $f_{\mathscr{S}}\big(x, D^k u(x)\big) = f_u(x)$ for λ-a.e. $x \in \Omega$. This proof will be divided in two steps.

Step 1: *For every countable set $\mathscr{S} \subset X$ and every $u \in X$, $f_{\mathscr{S}}\big(x, D^k u(x)\big) \geq f_u(x)$ for λ-a.e. $x \in \Omega$.*

Assume by contradiction that the statement of Step 1 does not hold for some $u \in X$. With no loss in generality we may assume that $u = 0$. Then we may find a positive c and a Borel set E with $\lambda(E) > 0$ such that

$$f_0(x) \geq f_{\mathscr{S}}(x, 0) + 2c \qquad \forall x \in E. \tag{3.5}$$

By Lemma 3.7, for every integer $h > 0$ we may find functions w_h and open sets A_h such that

(a) $f_{\mathscr{S}}(x,0) + c \geq f_{w_h}(x)$ for λ-a.e. $x \in A_h$,
(b) $\lambda(E \setminus A_h) \leq 2^{-(h+1)}\lambda(E)$,
(c) $\|w_h\|_{W^{k,\infty}} \leq 2^{-h}$.

Set $B = E \cap (\cap A_h)$: (b) yields $\lambda(B) \geq \lambda(E)/2 > 0$ and then, for every integer h, taking into account (a) and (3.5),

$$F(0,B) \geq \int_B \big[f_{\mathscr{S}}(x,0) + 2c\big]\, d\lambda(x) \geq F(w_h, B) + c\lambda(B)$$

and this is impossible because (c) yields that w_h converge to 0 in the strong topology of $C^k\big(\overline{\Omega}, \mathbb{R}^m\big)$ and F is assumed l.s.c. on $\mathscr{B}$.

Step 2: *There exists a countable set $\mathscr{S} \subset X$ such that, for every $u \in X$, $f_{\mathscr{S}}\big(x, D^k u(x)\big) \leq f_u(x)$ for λ-a.e. $x \in \Omega$.*

As in Proposition 3.2, let E be the set of all x such that $\lambda(\{x\}) > 0$. Consider the class $\mathscr{B}$ of all Borel subsets of $\Omega \setminus E$ as a subset of $L^1(\Omega \setminus E)$. Since L^1 is a separable metric space, we may find a countable collection of Borel sets $\mathscr{H}'$ which is dense in $\mathscr{B}$ and then let $\mathscr{H}$ be the collection which contains every set in $\mathscr{H}'$ and every set of the form $\{x\}$ with $x \in E$. Since $F(\cdot, B)$ is l.s.c. for every Borel set B and X is separable, by Lemma 1.6 we may find a countable set $\mathscr{S} \subset X$ such that

$$F(u,B) = \liminf_{v \in \mathscr{S},\ v \to u} F(v,B) \qquad \forall u \in X,\ B \in \mathscr{H}. \tag{3.6}$$

Let $u \in X$ be fixed: we want to show that for λ a.e. x

$$f_u(x) \geq f_{\mathscr{S}}\big(x, D^k u(x)\big)\ . \tag{3.7}$$

To begin with, suppose that x belongs to E and set $c = \lambda(\{x\})$. In this case $\{x\} \in \mathscr{H}$ and taking into account (3.6) and Proposition 3.2 we obtain that

$$\begin{aligned} f_u(x) = \frac{1}{c} F(u, \{x\}) &= \liminf_{v \in \mathscr{S}, v \to u} \frac{1}{c} F(v,B) \\ &\geq \liminf_{v \in \mathscr{S}, v \to u} f_{\mathscr{S}}\big(x, D^k v(x)\big) \geq f_{\mathscr{S}}\big(x, D^k u(x)\big) \end{aligned}$$

and then it is enough to prove that (3.7) holds for λ-a.e. $x \in \Omega \setminus E$.

By Proposition 3.2 we have that for all Borel set $B \subset \Omega \setminus E$ and all $v \in \mathscr{S}$

$$F(v,B) \geq \int_B f_{\mathscr{S}}\big(x, D^k v(x)\big)\, d\lambda(x). \tag{3.8}$$

Take $r > \|D^k u\|_\infty$ and take g_r as in Proposition 3.2. Then $f_{\mathscr{S}}\big(x, D^k v(x)\big) \geq g_r(x)$ for all $x \in \Omega \setminus E$ and all v such that $\|D^k v\|_\infty < r$ and this is an open neighborhood of u. Hence we may apply Fatou's lemma to obtain that

$$\liminf_{v \in \mathscr{S},\ v \to u} \int_B f_{\mathscr{S}}\big(x, D^k v(x)\big)\, d\lambda(x) \geq \int_B \liminf_{v \in \mathscr{S},\ v \to u} f_{\mathscr{S}}\big(x, D^k v(x)\big)\, d\lambda(x). \tag{3.9}$$

Since $f_{\mathscr{S}}$ is l.s.c. with respect to second variable (Proposition 3.2) we have that for every $x \in \Omega$

$$\liminf_{v \in \mathscr{S},\ v \to u} f_{\mathscr{S}}\big(x, D^k v(x)\big) \geq f_{\mathscr{S}}\big(x, D^k u(x)\big). \tag{3.10}$$

From (3.6), (3.9) and (3.10) we obtain that for all $B \in \mathscr{H}'$

$$F(u,B) \geq \int_B f_{\mathscr{S}}\big(x, D^k u(x)\big)\, d\lambda(x).$$

Since $\mathscr{H}'$ is dense in $\mathscr{B}$, this inequality holds for all Borel sets $B \subset \Omega \setminus E$ and the proof is complete. $\square$

Remark 3.4. In general, suppose that X is a subset of $C^k(\Omega, \mathbb{R}^m)$ endowed with a topology τ and let F be a functional on X which is a measure. If $\mathscr{S}$ is a countable subset of X, we may take $f_{\mathscr{S}}$ as in Definition 3.1 and it turns out to be a Borel function l.s.c. with respect to the second variable. Then there exists a countable set $\mathscr{S}$ such that $f_{\mathscr{S}}$ represents F if the following general hypothesis on the functional F and the space X are fulfilled:

(i) X is a local space,

(ii) if u_h is a sequence in X such that there exists a compact set $K \subset \Omega$ and a function $u \in X$ which satisfy $\operatorname{supp}(u_h - u) \subset K$ for all h and $\|u_h - u\|_{W^{k,\infty}}$ tends to 0, then u_h converge to $u \in X$ in the τ topology,

(iii) F is absolutely continuous with respect to some positive finite measure λ,

(iv) F is D^k-local,

(v) F is sequentially τ-l.s.c. on $\mathscr{B}$.

Notice that (i) and (ii) are fulfilled by (any local subspace of) $C^k(\Omega, \mathbb{R}^m)$ or $C^k(\overline{\Omega}, \mathbb{R}^m)$ endowed with the usual strong (or weak) topologies. Notice that finiteness is not an essential hypothesis in Theorem 3.3, but it allows a simpler proof than the general case. We want to point out that (i) plays an essential role (with (iv) and (v)) in the proof of Step 1, while (ii) and (iv) are essential in the proof of Step 2. If we drop (i) or (ii), examples may be given of functionals which are (k,∞)-bounded, a.c. with respect to some positive finite measure, D^k-local and τ-l.s.c. but cannot be represented by any Borel function f.

In order to prove the first step of the proof of Theorem 3.3, we need three lemmas.

Lemma 3.5. *Let λ be a finite positive measure on Ω. Let $E \subset \Omega$ be a Borel set which is covered by a countable family of Borel sets $\mathscr{F}$. Then, for every $\varepsilon > 0$ there exist $E_1, \dots, E_m \in \mathscr{F}$ and pairwise disjoint open sets $B_1, \dots, B_m$ such that*

$$\lambda\Big(E \setminus \bigcup_h (E_h \cap B_h)\Big) \le \lambda(E)\varepsilon$$

Proof. It is enough to choose $E_1, \dots, E_m$ so that $\lambda\big(E \setminus \cup E_h\big) \le \lambda(E)\varepsilon/2$ and take into account that λ is an outer regular measure. □

Lemma 3.6. *Suppose that v is a function in $C^k(\Omega, \mathbb{R}^m)$, λ is a finite positive measure on Ω. Then, for every $\varepsilon > 0$, there exists a function $w \in C^k_c(\Omega, \mathbb{R}^m)$ and an open set $A \subset \Omega$ such that*

(i) *$D^k w = D^k v$ everywhere in A;*

(ii) *$\lambda(\Omega \setminus A) \le (2n+2)\lambda(\Omega)\varepsilon$;*

(iii) *$\|w\|_{W^{k,\infty}} \le C\varepsilon^{-k}\|D^k v\|_\infty$ where C is a constant which depends on h and k only.*

Proof. By Theorem 1.4, there exists a Borel set E such that $\lambda(\Omega \setminus E) = 0$ and, for every $x \in E$,

$$\limsup_{r \to 0} \frac{\lambda(B(x,ar))}{\lambda(B(x,r))} \ge a^n \qquad \forall a \in]0,1[. \tag{3.11}$$

Let $\mathscr{F}$ be the collection of all closed balls $\overline{B} = \overline{B(x,r)} \subset \Omega$ such that $x \in E$ and

$$\frac{\lambda\big[B\big(x,(1-\varepsilon)r\big)\big]}{\lambda\big[\overline{B(x,r)}\big]} \ge \frac{\lambda\big[B\big(x,(1-\varepsilon)r\big)\big]}{\lambda\big[B\big(x,(1-\varepsilon)^{-1}r\big)\big]} \ge (1-\varepsilon)^{2n+1}. \tag{3.12}$$

Formula (3.11) shows that for all $x \in E$ there exist closed balls $\overline{B} \in \mathscr{F}$ with center x and arbitrary small radius and then we may apply Theorem 1.3 to obtain disjoint closed balls $\overline{B}_i = \overline{B(x_i, r_i)}$ in $\mathscr{F}$ for $i = 1, \dots, n$ such that

$$\lambda\Big[E \setminus \bigcup_i \overline{B(x_i, r_i)}\Big] \le \lambda(\Omega)\varepsilon.$$

Set $A = \bigcup B\big(x_i, (1-\varepsilon)r_i\big)$. Taking into account (3.12), we obtain

$$\begin{aligned}
\lambda(\Omega \setminus A) = \lambda(E \setminus A) &= \lambda\Big[E \setminus \bigcup_i \overline{B(x_i, r_i)}\Big] + \sum_i \lambda\Big[\overline{B(x_i, r_i)} \setminus B\big(x_i, (1-\varepsilon)r_i\big)\Big] \\
&\le \lambda(\Omega)\varepsilon + \sum_i \big[1 - (1-\varepsilon)^{2n+1}\big]\lambda(\overline{B}_i) \le \lambda(\Omega)\varepsilon + \sum_i (2n+1)\varepsilon\lambda(\overline{B}_i) \\
&\le (2n+2)\lambda(\Omega)\varepsilon
\end{aligned}$$

and (ii) is proved. Since $B_1, \dots, B_h$ are disjoint, we may apply the Glueing Lemma 1.5 to obtain a function $w \in C^k(\Omega, \mathbb{R}^m)$ such that

(a) $w = 0$ out of the union of all B_i.

(b) $D^k w = D^k v$ everywhere in $B(x_i, (1-\varepsilon)r_i)$ for all i.

(c) $\|D^k w\|_{L^\infty(B_i)} \le C_1\varepsilon^{-k}\|D^k v\|_{L^\infty(B_i)}$ for all i where C_1 is a constant which depends on h and k only.

Then (b) yields (i) while (a), (c) and $\overline{B}_i \subset \Omega$ for all i yield

$$\|D^k w\|_\infty = \sup_i \|D^k w\|_{L^\infty(B_i)} \le \sup_i C_1\varepsilon^{-k}\|D^k v\|_{L^\infty(B_i)} \le C_1\varepsilon^{-k}\|D^k v\|_\infty.$$

Since every ball in $\mathscr{F}$ was chosen relatively compact in Ω, (a) implies that w has compact support in Ω and we may apply Poincaré inequality to obtain

$$\|w\|_{W^{k,\infty}} \le C_2\|D^k w\|_\infty \le C_2 C_1\varepsilon^{-k}\|D^k v\|_\infty$$

where C_2 is a constant which depends on h and k only, and (iii) is proved. □

Lemma 3.7. *Let $\mathscr{S}$ be a countable subset of X. Let c be a positive real number and let E be a Borel set such that $f_{\mathscr{S}}(x,0) < +\infty$ for all $x \in E$. Then, for every $\varepsilon > 0$ there exists a function $w \in C^k_c(\Omega, \mathbb{R}^m)$ and an open set $A \subset \Omega$ such that*

(i) $f_{\mathscr{S}}(x,0)+c \geq f_w(x)$ *for* λ*-a.e.* $x \in A$*;*

(ii) $\lambda(E \setminus A) \leq (2n+3)\,\lambda(E)\,\varepsilon$*;*

(iii) $\|w\|_{W^{k,\infty}} \leq C\varepsilon$ *where* C *is the same constant of Lemma 3.6.*

Proof. Let $\varepsilon > 0$ be fixed. For every $v \in \mathscr{S}$, let E_v be the set of all points $x \in \Omega$ such that $f_{\mathscr{S}}(x,0)+c \geq f_v(x)$ and $|D^k v(x)| < \varepsilon^{k+1}$.

By the definition of $f_{\mathscr{S}}$ we have that E is covered by the sets E_v with $v \in \mathscr{S}$ and then we may apply Lemma 3.5 to find $v_1,\dots,v_m$ and pairwise disjoint open sets $B_1,\dots B_m$ so that setting $E_h = E_{v_h}$ for every h we have

$$\lambda\big(E \setminus \bigcup_h (E_h \cap B_h)\big) \leq \lambda(E)\varepsilon.$$

Let Ω_h be the open set of all points $x \in B_h$ such that $|D^k v_h(x)| < \varepsilon^{k+1}$. By definition of E_h we have that $E_h \cap B_h = E_h \cap \Omega_h$ and then

$$\lambda\big(E \setminus \bigcup_h (E_h \cap \Omega_h)\big) \leq \lambda(E)\varepsilon. \tag{3.13}$$

Hence we may apply Lemma 3.6 to obtain functions $w_h \in C_c^k(\Omega_h,\mathbb{R}^m)$ and an open sets $A_h \subset \Omega_h$ such that

(a) $D^k w_h = D^k v_h$ everywhere in A_h,

(b) $\lambda(\Omega_h \setminus A_h) \leq (2n+2)\lambda(\Omega_h)\varepsilon$,

(c) $\|w_h\|_{W^{k,\infty}} \leq C\varepsilon^{-k}\|D^k v_h\|_{L^\infty(\Omega_h)}$.

We set $w = \sum w_h$ and $A = \cup A_h$: w has compact support because is a finite sum of functions with compact support. Recalling that the sets Ω_h are pairwise disjoint and F is D^k-local, (a) yields $f_w(x) = f_{w_h}(x) = f_{v_h}(x)$ for all h and all $x \in \mathscr{A}_h$, then $f_{\mathscr{S}}(x,0)+c \geq f_w(x)$ for all $x \in \cup(E_h \cap A_h)$ and (i) is proved. (ii) and (iii) immediately follow from (b) and (c) taking into account (3.13) and the fact that the supports of w_h are included in the pairwise disjoint sets Ω_h. □

Proof of Theorem 1.2. We begin with proving the existence of a function f which represents F. If X is the space $C^k\big(\overline{\Omega},\mathbb{R}^m\big)$, by Corollaries 2.5 and 2.10, F satisfies the hypothesis of Theorem 3.3 and the proof is complete. If X is $C^k(\Omega,\mathbb{R}^m)$ or $C_0^k(\Omega,\mathbb{R}^m)$ and A is an open subset of Ω which is relatively compact and has boundary of class C^∞, we have that the space $\{f \llcorner A : f \in X\}$ agrees with $C^k\big(\overline{A},\mathbb{R}^m\big)$ and then it is enough to apply Theorem 3.3 to the restriction of F to the space $C^k\big(\overline{A},\mathbb{R}^m\big)$ for countably many open sets A which cover Ω.

The uniqueness of f immediately follows from Theorem 3.11 of Alberti [1]. □

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