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Rank one property for derivatives of functions with bounded variation

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Synopsis

In this paper we introduce a new tool in geometric measure theory and then we apply it to study the rank properties of the derivatives of vector functions with bounded variation.

Introduction

When u is a function of bounded variation of the open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^m$, Du is a measure on Ω which takes values in the space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices such that $(Du)_{i,j} = \partial u_i/\partial x_j$ for every i,j with $1 \le i \le m$, $1 \le j \le n$ and we denote by $D_S u$ the part of the measure Du which is singular with respect to Lebesgue measure.

We recall that the rank of a matrix $A \in \mathbb{R}^{m\times n}$ is dimension of the space spanned by its columns (in $\mathbb{R}^m$) or by its rows (in $\mathbb{R}^n$) and in particular the rank of A is one if and only if there exist $e \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$ with $e \neq 0$, $\eta \neq 0$, so that $A = e \otimes \eta$ i.e. $A_{i,j} = e_i \eta_j$ for all i,j (cf. Remark 1.7). When λ is a measure which takes values in the space $\mathbb{R}^{m\times n}$ and $\big[d\lambda/d|\lambda|\big]$ denotes the Radon–Nikodym derivative of λ with respect to the total variation $|\lambda|$, we say that λ has rank one when the rank of the matrix $\big[d\lambda/d|\lambda|\big](x)$ is one for $|\lambda|$ almost every x.

In this paper we prove we prove the following statement (see Theorem 4.5, Corollary 4.6 and Remark 4.9):

Let Ω be an open subset of $\mathbb{R}^n$, let u be a function of bounded variation of Ω into $\mathbb{R}^m$ and denote by $D_S u$ the part of Du which is singular with respect to Lebesgue measure. Then $D_S u$ is a rank one measure and this means that for $|D_S u|$ almost all x the matrix $\big[dD_S u/d|D_S u|\big](x)$ has rank one.

This property of the singular parts of derivatives was conjectured by Ambrosio and De Giorgi in a note about some variational problems which involve vector valued functions with bounded variation (see [5]), and it is very useful in dealing with quasiconvex integral functionals with linear growth. In particular it has been used to study the relaxation on the space $BV(\Omega, \mathbb{R}^m)$ (that is, the space of all functions of Ω into $\mathbb{R}^m$ with bounded variation) with respect to the strong topology of $L^1(\Omega, \mathbb{R}^m)$ of the functional

$$F(u) = \int_\Omega f\big(x, u(x), Du(x)\big)dx, \quad u \in C^1(\Omega, \mathbb{R}^m),$$

where f is a quasiconvex function of the third variable (see [4] for the case $f(x,u,Du) = g(Du)$ and [11] and [12] for the general case).

When u is a function of bounded variation of Ω into $\mathbb{R}^m$, an important decomposition of the derivative is $Du = f_u \cdot \mathscr{L}_n + D_S u$ where $\mathscr{L}_n$ is Lebesgue measure, f_u is a Lebesgue summable function of Ω which takes values in $\mathbb{R}^{m\times n}$ and $D_S u$ is a measure on Ω which takes values in $\mathbb{R}^{m\times n}$ and is singular with respect Lebesgue measure. Moreover coarea formula (cf. Theorem 1.12) yields $|D_S u|(B) = 0$ for every Borel set B such that $\mathscr{H}^{n-1}(B) = 0$ ($\mathscr{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure) and then we may write $D_S u = D_C u + g_u \cdot \mathscr{H}^{n-1}$ where g_u is an $\mathscr{H}^{n-1}$ summable function of Ω into $\mathbb{R}^{m\times n}$ and $D_C u$ is a measure on Ω which takes values in $\mathbb{R}^{m\times n}$, is singular with respect to Lebesgue measure and satisfies $|D_C u|(B) = 0$ for all Borel sets B with $\mathscr{H}^{n-1}(B) < \infty$. Hence (cf. [2], section 3, and [3], Proposition 2.4)

$$Du = f_u \cdot \mathscr{L}_n + D_C u + g_u \cdot \mathscr{H}^{n-1}.$$

When S is an $(n-1)$–dimensional submanifold of Ω of class C^1, we may choose a vector $\eta(x)$ with $|\eta(x)| = 1$ which is orthogonal to the tangent space of S in x for every x. It is well–known that $Du \llcorner S$ (namely, the restriction of the measure Du to the set S) may be written in the form $g \cdot \mathscr{H}^{n-1} \llcorner S$ where, for $\mathscr{H}^{n-1}$ almost all x, $g(x)$ is a matrix of the form $e(x) \otimes \eta(x)$ for some $e(x) \in \mathbb{R}^m$ (in particular $e(x) = u^+(x) - u^-(x)$, where $u^+(x)$ and $u^-(x)$ are the approximate limits of u in x in the semispaces $\big\{y : \langle \eta(x); y - x\rangle \geq 0\big\}$ and $\big\{y : \langle \eta(x); y - x\rangle \leq 0\big\}$ respectively), and this means that $\big[g(x)\big]_{i,j} = \big[e(x)\big]_i \big[\eta(x)\big]_j$ for every i, j (cf. Proposition 1.9).

In particular, if S is the set of all x such that $g_u(x) \neq 0$, S is a rectifiable set (see Definition 1.6), that is, there exist countably many $(n-1)$–dimensional manifolds of Ω of class C^1 which cover $\mathscr{H}^{n-1}$ almost all of S and then we obtain that $g_u(x)$ is a rank one matrix for $\mathscr{H}^{n-1}$ almost all x and $g_u \cdot \mathscr{H}^{n-1}$ is a rank one measure.

Taking into account this fact, Ambrosio and De Giorgi conjectured in [5] that also $D_S u$ is a rank one measure. We shall prove this statement in the following way: when μ is a positive measure on $\mathbb{R}^n$ and x is a point of $\mathbb{R}^n$, we define $E(\mu, x)$ as the set of all vectors v such that there exists a real function of bounded variation u and

$$\lim_{r\to 0} \frac{\big|Du - v \cdot \mu\big|(B(x,r))}{\mu(B(x,r))} = 0$$

(see Definitions 2.1 and 2.3). It follows immediately from definition that $E(\mu, x)$ is always a linear subspace of $\mathbb{R}^n$ and we show that when μ is a singular measure, the dimension of $E(\mu, x)$ is 0 or 1 for μ almost all x (Theorem 3.1). We say that a singular measure μ is rectifiable when the dimension of $E(\mu, x)$ is one for μ almost every x (Definition 4.1). In this case we may choose a vector $\eta(x)$ so that $|\eta(x)| = 1$ and $E(\mu, x)$ is the span of $\eta(x)$ for μ almost all x and we prove (see Theorem 4.5 and Remark 4.9) that if u is a function of bounded variation and we decompose Du as $Du = f \cdot \mu + \theta$ with $\theta \perp \mu$, then, for μ almost all x, $f(x)$ is a matrix of the form $e(x) \otimes \eta(x)$ for some $e(x) \in \mathbb{R}^m$ and in particular its rank is 1 or 0.

Since $|D_S u|$ is always a rectfiable measure (cf. statement (v) of Proposition 4.4), if we take $\mu = |D_S u|$ we obtain that the rank of the matrix $\big[dD_S u / d|D_S u|\big](x)$ is one for $|D_S u|$ almost all x and $D_S u$ is a rank one measure (see Corollary 4.6 and Remark 4.9). In Theorem 4.13 and Corollary 4.14 we extend this result to higher order derivatives.

Eventually we want to make some important remarks.

About the definition of rectifiable measure, notice that if μ is a measure of the form $\mathscr{H}^{n-1} \llcorner S$, then μ is rectfiable if and only if S is a rectifiable set and in this case $E(\mu, x)$ agrees with the approximate normal space of S in x for $\mathscr{H}^{n-1}$ almost all $x \in S$ (see statements (iii) and (iv) of Proposition 4.4).

Notice that the rank theorem may be extended to higher order derivatives and in particular a partial positive answer has already been given in [6] in the case of second order derivatives of scalar functions (see Corollary 4.14 and Remarks 4.16 and 4.17).

$E(\mu, x)$ has already been introduced in [7], with a slightly different definition, and turns out to be a useful tool in studying the integral representation of convex functionals on the space $BV(\Omega, \mathbb{R}^m)$.

For technical reasons we shall consider functions with bounded variation which are defined in general on (open subsets of) n–dimensional linear subspaces N of some $\mathbb{R}^k$ instead of $\mathbb{R}^n$ and take values in a generic finite dimensional Banach space E instead of $\mathbb{R}^m$. Hence derivatives will be measures (or functions) which take values in the space $\mathscr{L}(N, E)$ of all linear application of N into E instead of the space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices. We shall consider apart the meanings of the main results (Theorems 4.5 and 4.13 and Corollaries 4.6 and 4.14) when $N = \mathbb{R}^n$ and $E = \mathbb{R}^m$ (see Remarks 4.9, 4.16 and 4.17).

This paper is divided in five sections. In section 1 we give some basic notation and some preliminary results about measures and BV functions, almost every statement in this section is a well–known result but sometimes it is not given in the standard form.

In section 2 we give the definition of $E(\mu, x)$ (Definition 2.3), we describe some of its main properties (essentially Proposition 2.6 and Theorem 2.12) and then we consider some concrete examples.

Section 3 is dedicated to prove the key result of this paper (Theorem 3.1): when μ is a singular measure, the dimension of $E(\mu, x)$ is 1 or 0 for μ almost all x. Except for Theorem 3.1, the content of this section is not necessary for the comprehension of section 4.

In section 4 we give the definition and a characterization of rectifiable measure (Definition 4.1 and Proposition 4.2) and eventually we prove the rank conjecture in its full generality (Theorem 4.5 and Corollary 4.6). Then we generalize these results to higher order derivatives and we give some open problems (Remark 4.18). All results in this section are almost immediate corollaries of Theorem 3.1.

Section 5 is devoted to the proofs of some technical results given in the previous sections.

1. Basic Notation and Preliminary Results

In this paper we refer to customary measure theory and functional analysis notation whenever possible. Unusual definition are given in two particular cases: the standard domain we deal with and the definition of derivatives (see 1.7 and 1.8). Regarding the definitions and the statements in this section, we give proofs (when needed) and references in section 5.

Unless differently stated, throughout this paper N is always a linear subspace with dimension n of some euclidean space, Ω is an open subset of N and E is a finite dimensional Banach space. In general we deal with (weakly differentiable) functions and measures which are defined on (open subsets of) N instead of $\mathbb{R}^n$ only: this notation is necessary because sometimes we have to study traces of measures and functions on a generic subspace of the original domain (see for instance Propositions 1.5, 1.10 and 3.6).

$\mathbb{R}^n$ and all the linear subspaces of $\mathbb{R}^n$ are always endowed with their Hilbert space structure by the standard scalar product $\langle\ ;\ \rangle$. When e is a vector in N we denote by e^* the linear functional on N given by $x \mapsto \langle e; x\rangle$. Unless differently stated, we do not identify an Hilbert space and its dual space.

When S is a subset of N, the annihilator of S is the linear subspace $S^\perp$ of all linear functionals $v \in N^*$ such that $v\,x = 0$ for all $x \in S$. When S is a subset of N^*, the annihilator of S is the linear subspace ${}^\perp S$ of all $x \in N$ such that $v\,x = 0$ for all $v \in S$.

When A is a subset of N and f is a function the domain of which includes A, $f \llcorner A$ is the function defined by $\big[f \llcorner A\big](x) = f(x)$ when $x \in A$ and 0 when $x \in N \setminus A$.

We say that a subset X of E is a cone if $ax \in X$ whenever $x \in X$ and a is a positive real number. When E is a subspace of some euclidean space (in general, an Hilbert space), $e \in E$, $|e| = 1$, and $a \in]0,1[$, we denote by $X(e,a)$ and $X^*(e,a)$ the closed convex cones respectively in E and in the dual of E given by

$$X(e,a) = \big\{x \in E : \langle e; x\rangle \geq a\,|x|\big\} \qquad \text{and} \qquad X^*(e,a) = \big\{v \in E^* : v\,e \geq a\,|v|\big\}. \tag{1.1}$$

For every integer (real) k, $\mathscr{H}^k$ is the k–dimensional Hausdorff measure on any metric space M. We write $\mathscr{H}^k \llcorner M$ when we want to point out the metric space. To simplify the notation, we denote also Lebesgue measure in $\mathbb{R}^n$ by $\mathscr{H}^n \llcorner \mathbb{R}^n$.

By positive measure we mean any locally finite positive measure on the σ–field of all Borel sets, by real or vector measure we mean any real or vector valued measure on the σ–field of all Borel sets with finite total variation. We denote by $\mathscr{M}(\Omega, E)$, the Banach space of all E valued measures on Ω with finite total variation, $|\mu|$ is the total variation of μ and $\|\mu\| = |\mu|(\Omega)$ is the norm of μ. Of course, every measure μ admits a natural extension to the σ–field of all $|\mu|$ measurable sets: we denote also this extension with μ.

When X is a subset of E, $\mathscr{M}(\Omega, X)$ is the set of all measures $\psi \in \mathscr{M}(\Omega, E)$ which take values in X.

When μ is an element of $\mathscr{M}(\Omega, E)$ and ϕ is an element of $C_0(\Omega, E^*)$ (the space of all continuous function of Ω into E^* which vanishes at infinity, endowed with the supremum norm), we write

$$\langle \mu; \phi\rangle = \int_\Omega \Big[\phi(x) \cdot \frac{d\mu}{d|\mu|}(x)\Big]\, d|\mu|(x) \tag{1.2}$$

(notice that for every x, $\phi(x) \in E^*$, $\big[d\mu/d|\mu|\big](x) \in E$ and then $\phi(x) \cdot \big[d\mu/d|\mu|\big](x) \in \mathbb{R}$). For every μ, the application $I\mu : C_0(\Omega, E^*) \to \mathbb{R}$ given by $I\mu : \phi \mapsto \langle \mu; \phi\rangle$ is a continuous linear functional on $C_0(\Omega, E^*)$ and I is an isometric immersion of $\mathscr{M}(\Omega, E)$ into the dual of $C_0(\Omega, E^*)$ and is surjective and then we identify $\mathscr{M}(\Omega, E)$ and the dual of $C_0(\Omega, E^*)$.

When μ is a positive measure on Ω, we denote by $L^p(\mu, E)$ the Banach space of all μ measurable functions $f : \Omega \to E$ such that $\int |f|^p d\mu < \infty$. We write f

either for the function or for the equivalence class, the distinction being clear from the context, and when needed we always take f Borel measurable instead of μ measurable.

Sometimes in the following Ω happens to be a generic separable locally compact topological space and E an infinite dimensional Banach space (some cares must be taken in this cases). When $E = \mathbb{R}$, we omit to write it. For the general properties of Borel measures we refer to [10], [15], [16], and [17].

We write $\lambda \perp \mu$ when the measures $|\lambda|$ and $|\mu|$ are mutually singular, $\lambda \ll \mu$ when $|\lambda|$ is absolutely continuous with respect to $|\mu|$, $\lambda \leq \mu$ when λ and μ are real measures and $\lambda(B) \leq \mu(B)$ for all Borel sets B. As usual, $[d\lambda/d\mu]$ is the Radon–Nikodym derivative with respect to μ of the part of λ which is absolutely continuous with respect to μ. We say that a measure μ is concentrated in a ($|\mu|$ measurable) set B when $|\mu|(B^C) = 0$.

When μ is a measure on Ω, f is a Borel function of Ω into Ω', $f^{\#}\mu$ is the measure on $\mathscr{M}(\Omega', E)$ given by $\big[f^{\#}\mu\big](B) = \mu\big(f^{-1}(B)\big)$ for all Borel sets B. When μ is a positive measure and $f \in L^1(\mu, E)$, $f \cdot \mu$ is the measure defined by $\big[f \cdot \mu\big](B) = \int_B f\, d\mu$ for all Borel sets $B \subset \Omega$. When μ is a measure and A is a $|\mu|$ measurable set, $\mu \llcorner A$ is the measure defined by $\big[\mu \llcorner A\big](B) = \mu(A \cap B)$ for all Borel sets B.

Remark 1.1. It is well–known that the space $\mathscr{M}(\Omega, E)$ is a non–separable Banach space which admits a strong topology s and a weak* topology w^* because it is the dual of $C_0(\Omega, E^*)$, and $s \supset w^*$. Since $\mathscr{M}(\Omega, E)$ is non–separable, the Borel σ–fields generated by these topologies are not the same. In the following we shall consider $\mathscr{M}(\Omega, E)$ endowed with the Borel σ–field $\mathscr{B}$ generated by w^* only. In particular, when $(X, \mathscr{E})$ is a measurable space and $f : E \to \mathscr{M}(\Omega, E)$, f is measurable when $f^{-1}(B) \in \mathscr{E}$ for every set $B \in \mathscr{B}$ or, it is the same, when the function $t \to \langle f(t); \phi\rangle$ (see (1.2)) is measurable for every ϕ in (a dense subset of) $C_0(\Omega, E^*)$. Hence, when λ is a positive measure on a separable locally compact space M, the functions in $L^p\big(\lambda, \mathscr{M}(\Omega, E)\big)$ are Borel measurable or μ measurable in the above sense.

It is important to notice that the topology induced by w^* on every closed ball in $\mathscr{M}(\Omega, E)$ is compact and metrizable and then the σ–field of all Borel subsets of a closed ball is induced by a complete separable metric. The same remark holds when we consider spaces such as $BV(\Omega, E)$, $\mathscr{M}_{\mathrm{loc}}(\Omega, E)$ or $BV_{\mathrm{loc}}(\Omega, E)$ (see for instance Definitions 1.8 and 1.13).

Definition 1.2. (Integration of Measures, see [9], nos. 70 to 74). Let M be a separable locally compact topological space. Let λ be a positive measure on M and suppose that $t \mapsto \mu_t$ is a function in $L^1\big(\lambda, \mathscr{M}(\Omega, E)\big)$. We denote by $\mu = \int \mu_t\, d\lambda(t)$ the E valued measure on Ω which corresponds to the continuous linear functional on $C_0(\Omega, E^*)$ given by

$$\phi \longmapsto \int_M \langle \mu_t; \phi\rangle d\lambda(t)$$

where $\langle \mu_t; \phi\rangle$ is given in (1.2).

Consider the measures $|\mu|$ and $\int |\mu_t|\, d\lambda(t)$: it may be easily proved that $|\mu| \leq \int |\mu_t|\, d\lambda(t)$ but in general they are not equal (of course, they are equal if and only if $\|\mu\| = \int \|\mu_t\|\, d\lambda(t)$). Notice that equality holds, for example, in the following two particular cases: (λ almost) all the measures μ_t are real and positive or (λ almost) all the measures μ_t are pairwise mutually singular.

When f is a function in $L^1\big(\int|\mu_t|d\lambda(t)\big)$, f belongs to $L^1\big(|\mu_t|\big)$ for λ a.a. t, $t\mapsto\int f\,d\mu_t$ belongs to $L^1(\lambda,E)$ and

$$\int_\Omega f\,d\mu=\int_M\Big[\int_\Omega f\,d\mu_t\Big]d\lambda(t) \tag{1.3}$$

Equality (1.3) does not make sense in general for every function f in $L^1\big(|\mu|\big)$ when $\|\mu\|<\int\|\mu_t\|\,d\lambda(t)$, but it always holds when f is a bounded Borel function, and in particular $\mu(B)=\int\mu_t(B)\,d\lambda(t)$ for every Borel set $B\subset\Omega$.

In the following we often use measures which take values in a given closed convex cone and we need some of their properties.

Proposition 1.3. *Let Ω be an open subset of N and let M be a separable locally compact topological space. Let E be a subspace of some euclidean space and X a closed convex cone in E. Let ψ be a measure in $\mathscr{M}(\Omega,E)$, λ a positive measure on M and $t\mapsto\psi_t$ a function in $L^1\big(\lambda,\mathscr{M}(\Omega,E)\big)$ so that $\psi=\int\psi_t d\lambda(t)$ (see. Definition 1.2). Then*

(i) *ψ belongs to $\mathscr{M}(\Omega,X)$ if and only if $\big[d\psi/d|\psi|\big](x)\in X$ for $|\psi|$ a.a. x,*

(ii) *$\mathscr{M}(\Omega,X)$ is a weak* closed convex cone in $\mathscr{M}(\Omega,E)$ and in particular this means that a series of elements of $\mathscr{M}(\Omega,X)$ may converge to an element of $\mathscr{M}(\Omega,X)$ only,*

(iii) *$f\cdot\psi\in\mathscr{M}(\Omega,X)$ whenever $\psi\in\mathscr{M}(\Omega,X)$ and f is a non–negative function in $L^1(|\psi|)$,*

(iv) *$\psi\in\mathscr{M}(\Omega,X)$ whenever $\psi_t\in\mathscr{M}(\Omega,X)$ for λ a.a. $t\in M$,*

(ii) *$\psi_t\in\mathscr{M}(\Omega,X)$ for λ a.a. $t\in M$ whenever $\psi\in\mathscr{M}(\Omega,X)$ and $\|\psi\|=\int\|\psi_t\|d\lambda(t)$ (notice that this hypothesis cannot be dropped).*

Remark 1.4. We are also interested in a particular case of integration of measures.

Let M and N be two linear subspaces of an euclidean space with dimensions m and n respectively. Let π_M and π_N be the standard projections of $M\times N$ onto M and N respectively.

For every $t\in M$, let $\epsilon_t\in\mathscr{M}(M)$ be Dirac delta mass concentrated in t (i.e. $\epsilon_t(B)=1$ if $t\in B$ and 0 otherwise). For every set $B\subset M\times N$ and every $t\in M$, set $B_t=\{s\in N:(t,s)\in B\}$. For every function f of $M\times N$ and every $t\in M$, f_t is the function of N defined by $f_t(s)=f(t,s)$ for all $s\in N$.

Let λ be a positive measure on M and let $t\mapsto\mu_t$ be a function in $L^1\big(\lambda,\mathscr{M}(N,E)\big)$. Then, for every t, $\big[\epsilon_t\otimes\mu_t\big]$ is the product measure on $M\times N$ which is given by $\big[\epsilon_t\otimes\mu_t\big](B)=\mu_t(B_t)$ for all Borel sets $B\subset M\times N$, $\mu=\int\big[\epsilon_t\otimes\mu_t\big]\,d\lambda(t)$ is an E valued measure on $M\times N$, $|\mu|=\int\big[\epsilon_t\otimes|\mu_t|\big]\,d\lambda(t)$, and, for every $f\in L^1\big(|\mu|\big)$, f_t belongs to $L^1\big(|\mu_t|\big)$ for λ a.a. t, $t\mapsto\int f_t d\mu_t$ belongs to $L^1(\lambda,E)$ and

$$\int_{M\times N} f\,d\mu=\int_M\Big[\int_N f_t d\mu_t(B_t)\Big]d\lambda(t) \tag{1.4}$$

and in particular $\mu(B)=\int\mu_t(B_t)\,d\lambda(t)$ for every Borel set B in $M\times N$.

Let $\lambda=\lambda_A+\lambda_S$ be the Lebesgue decomposition of λ with respect to $\mathscr{H}^m\llcorner M$. Then $\int\big[\epsilon_t\otimes\mu_t\big]\,d\lambda(t)$ is absolutely continuous with respect to $\mathscr{H}^{m+n}\llcorner M\times N$ if and only if $\mu_t=0$ for λ_S a.a. $t\in M$ and $\mu_t\ll\mathscr{H}^n\llcorner N$ for λ_A a.a. $t\in M$.

PROPOSITION 1.5. (Disintegration of Measures). *When μ is a measure in $\mathscr{M}(M \times N, E)$ and λ is a positive measure on M such that $\pi_M^{\#}|\mu| \ll \lambda$, there exists a function $t \mapsto \mu_t$ in $L^1\big(\lambda, \mathscr{M}(N,E)\big)$ such that*

$$\mu = \int_M \big[\epsilon_t \otimes \mu_t\big] \, d\lambda(t).$$

The function $t \mapsto \mu_t$ is λ essentially unique and we refer to it as the disintegration of μ with respect to λ.

DEFINITION 1.6. (Rectifiable and Purely Unrectifiable Sets). We say that a Borel set $S \subset N$ is rectifiable when there exist $(n-1)$–dimensional submanifolds $M_n \subset N$ of class C^1 for $n = 1, 2, \ldots$ which cover $\mathscr{H}^{n-1}$ almost all of S, i.e. $\mathscr{H}^{n-1}\big(S \setminus \cup M_n\big) = 0$ (in this paper we deal with manifolds and rectifiable sets with codimension 1 only and we simply write "rectifiable" instead of the standard "countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable").

In order to define the tangent bundle of a rectifiable set, we recall the following well–known proposition: when M and M' are two $(n-1)$–dimensional submanifold of N, the tangent spaces of M and M' in x agree for $\mathscr{H}^{n-1}$ almost all $x \in M \cap M'$.

Let $G(N)$ be the Grassman manifold of all $(n-1)$–dimensional subspaces of N and let S be a rectifiable set in N. Taking into account the proposition above, it is not difficult to show that there exists a Borel function $x \mapsto \mathrm{Tan}(S,x)$ which take N into $G(N)$ and satisfies the following property: for every $(n-1)$–dimensional submanifold M of N of class C^1, $\mathrm{Tan}(S,x)$ is the usual tangent space of M in x for $\mathscr{H}^{n-1}$ almost all $x \in S \cap M$.

This function is unique up to sets which are negligible for the measure $\mathscr{H}^{n-1} \llcorner S$, and we refer to it as the tangent bundle of S. It follows immediately that when S and S' are rectifiable sets,

$$\mathrm{Tan}(S,x) = \mathrm{Tan}(S',x) \quad \text{for } \mathscr{H}^{n-1} \text{ almost all } x \in S \cap S'.$$

We say that a Borel function $\eta : N \to N$ is an orientation of S if $|\eta(x)| = 1$ and $\eta(x) \perp \mathrm{Tan}(S,x)$ for $\mathscr{H}^{n-1}$ a.a. $x \in S$. A Borel set $T \subset N$ is purely unrectifiable when $\mathscr{H}^{n-1}(M \cap T) = 0$ for all $(n-1)$–dimensional submanifold M of N of class C^1. Notice that every Borel set S which is σ–finite with respect to $\mathscr{H}^{n-1}$ may be written as the union of a rectifiable set and a purely unrectifiable set.

Suppose that S is a set such that $\mathscr{H}^{n-1}(S \cap K) < \infty$ for every compact set K. In the usual approach to rectifiability (see [16], chapter 3), a definition is given of ($(n-1)$–dimensional) approximate tangent space of S in a point x and it may be shown that S is rectifiable if and only if there exist approximate tangent spaces for $\mathscr{H}^{n-1}$ almost all points of S. Of course, $\mathrm{Tan}(S,x)$ agrees with the approximate tangent space for $\mathscr{H}^{n-1}$ almost every $x \in S$ and then our definition of tangent bundle is consistent with the usual one. Since we are not interested in the pointwise differentiability of rectifiable sets and we need the tangent space for all rectifiable sets, we have preferred an "axiomatic" definition instead of the standard one.

Remark 1.7. Let N and E be taken as usual. We denote by $\mathscr{L}(N,E)$ the Banach space of all linear applications of N into E, endowed with the usual norm

$\|T\| = \sup\{|Ty| : y \in N,\ |y| = 1\}$ for every $T \in \mathscr{L}(N,E)$. As usual we write N^* instead of $\mathscr{L}(N,\mathbb{R})$, we do not identify N and N^*.

The rank of a linear application $T \in \mathscr{L}(N,E)$ is the dimension of the space $T(N)$, that is, the codimension of the kernel of T.

Suppose that the rank of T is 1 and let M be the kernel of T. Then the dimension of M is $n-1$ and there exists a vector η with norm 1 which is orthogonal to M. Set $e = T\eta$ and let T' be the element of $\mathscr{L}(N,E)$ given by $T'y = \langle \eta; y\rangle\, e$ for every $y \in N$. Then $T'\eta = \langle \eta; \eta\rangle\, e = T\eta$ and $T'y = Ty = 0$ for every $y \in M$. As N is the span of M and η, $T' = T$.

Hence we have proved that the rank of T is one if and only if there exist $\eta \in N$ with $|\eta| = 1$ and $e \in E$, $e \neq 0$, such that

$$Ty = \langle \eta; y\rangle\, e \quad \text{for every } y \in N. \tag{1.5}$$

In general, when Ω is an open subset of N, the derivative (in any sense) of a function $f : \Omega \to E$ always takes values in the space $\mathscr{L}(N,E)$ (see Definition 1.8). This notation is not very easy when we consider real valued functions, but it is essential when either we deal with vector valued functions or we consider higher order derivatives (see Definitions 4.10 and 4.12).

Definition 1.8. (Functions of Bounded Variation). Let N and E be taken as usual and let Ω be an open subset of N. We recall that a function $u \in L^1(\mathscr{H}^n \llcorner \Omega, E)$ is a function of bounded variation, and we write $u \in BV(\Omega,E)$, when its distributional derivative is (represented by) a measure in $\mathscr{M}\big(\Omega, \mathscr{L}(N,E)\big)$. This means that

$$\int_\Omega \big[D\phi(x)\cdot y\big]\, u(x)\, d\mathscr{H}^n(x) = -\int_\Omega \phi(x)\, d\big[Du\cdot y\big](x) \qquad \text{for all } y \in N \text{ and all } \phi \in C^\infty_C(\Omega), \tag{1.6}$$

where the measure $\big[Du\cdot y\big]$ is defined by $\big[Du\cdot y\big](B) = Du(B)\cdot y$ for all Borel sets B.

When M is a subspace of N we denote by $D_M u$ the derivative of u with respect to M, i.e. the measure in $\mathscr{M}\big(\Omega, \mathscr{L}(M,E)\big)$ given by $\big[D_M u\big](B) = D_u(B)\llcorner M$ for all Borel sets B. When e is a vector in N, $\big[\partial u/\partial e\big]$ is the directional derivative of u with respect to e, that is the E valued measure given by

$$\left[\frac{\partial u}{\partial e}\right](B) = \big[Du(B)\big]\cdot e \quad \text{for all Borel sets } B.$$

For the general properties of BV functions, we refer to [10], [14] and [17].

Now we give some well–known results about functions of bounded variation which we need in this paper. An essential result is the coarea formula; in Theorem 1.12 we state a particular version of this theorem.

Proposition 1.9. (Behaviour of Derivatives on Rectifiable Sets). *Let u be a function in $BV(\Omega,E)$ and let (S,η) be an oriented rectifiable set (see Definition 1.6). For every $x \in S$, let $B^+(x,r)$ and $B^-(x,r)$ be the sets of all points $y \in B(x,r)$*

such that $\langle \eta(x); y-x\rangle > 0$ *and* $\langle \eta(x); y-x\rangle < 0$ *respectively. Then, for* $\mathscr{H}^{n-1}$ *almost all* $x \in S$ *there exists* $u^+(x)$ *and* $u^-(x)$ *in* E *such that*

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B^+(x,r)} \big|u(y) - u^+(x)\big|\, d\mathscr{H}^n(y) = 0,$$
$$\lim_{r\to 0} \frac{1}{r^n} \int_{B^-(x,r)} \big|u(y) - u^-(x)\big|\, d\mathscr{H}^n(y) = 0.$$

We denote the functions u^+ *and* u^- *as the traces of* u *on* (S,η)*, they belong to* $L^1(\mathscr{H}^{n-1} \llcorner S, E)$ *and moreover*

$$Du \llcorner S = (u^+ - u^-)\, \eta^*\, \mathscr{H}^{n-1} \llcorner S \tag{1.7}$$

and this means that for every Borel set $B \subset S$ *and every* $y \in N$

$$\big[Du(B)\big] : y \longmapsto \int_B \big[u^+(x) - u^-(x)\big] \langle \eta(x); y\rangle\, d\mathscr{H}^{n-1}(x).$$

Proposition 1.10. (Disintegration of BV Functions). *Let* M *and* N *be two linear subspaces of an euclidean space with dimensions* m *and* n *respectively (cf. Remark 1.4). We identify* M *and* N *with* $M \times \{0\}$ *and* $\{0\} \times N$*. Let* u *be a real function of bounded variation of* $M \times N$ *and let* $D_N u$ *be the derivative of* u *with respect to* N *(see Definition 1.8), then* $\pi_M^\#(D_N u)$ *is absolutely continuous with respect to* $\mathscr{H}^m \llcorner M$*. Let* $t \mapsto (D_N u)_t$ *be the disintegration of* $D_N u$ *with respect to* $\mathscr{H}^m \llcorner M$ *(see Proposition 1.5), then, for* $\mathscr{H}^m$ *a.a.* $t \in M$*,* u_t *is a function of bounded variation of* N *and* $Du_t = (D_N u)_t$*.*

Definition 1.11. (Sets with Finite Perimeter). A set $E \subset \Omega$ has finite perimeter in Ω when the derivative of its characteristic function v is (represented by) a measure in $\mathscr{M}(\Omega, N^*)$. We recall that by structure theorem (see [14], Theorem 3.4, or [17], section 5.8 or [10], section 5.7.3) when E is a set with finite perimeter there exists an oriented rectifiable set (S,η) such that

$$Dv = \eta^* \cdot \mathscr{H}^{n-1} \llcorner S. \tag{1.8}$$

Notice that the characteristic function of a set with finite perimeter E belongs to $BV(\Omega)$ if and only if $\mathscr{H}^n(E) < \infty$.

Theorem 1.12. (Coarea Formula). *Let* u *be a function in* $L^1(\mathscr{H}^n \llcorner \Omega)$*. For every* $t \in \mathbb{R}$*, let* v_t *be the characteristic function of the set* $\{x : u(x) > t\}$*. Then*

(i) $t \mapsto v_t$ *is a* $\mathscr{H}^1$ *measurable function of* $\mathbb{R}$ *into* $L^1(\mathscr{H}^n \llcorner \Omega)$*, in particular the set* B *of all* t *such that* F_t *has finite perimeter is* $\mathscr{H}^1$ *measurable and* $t \mapsto Dv_t$ *is an* $\mathscr{H}^1$ *measurable function of* B *into* $\mathscr{M}(\Omega, N)$*. By formula* (1.8) *there exist, for all* $t \in B$*, oriented rectifiable sets* (S_t, η_t) *such that*

$$Dv_t = \eta_t^* \cdot \mathscr{H}^{n-1} \llcorner S_t.$$

(ii) *u belongs to $BV(\Omega)$ if and only if $\mathscr{H}^1(\mathbb{R}\setminus B)=0$ and $\int_{\mathbb{R}}\|Dv_t\|\,d\mathscr{H}^1(t)<\infty$. Moreover (cf. Definition 1.2)*

$$Du = \int_{\mathbb{R}} Dv_t \; d\mathscr{H}^1(t) = \int_{\mathbb{R}} \eta_t^* \cdot \left[\mathscr{H}^{n-1} \llcorner S_t\right] d\mathscr{H}^1(t) \tag{1.9}$$

$$|Du| = \int_{\mathbb{R}} |Dv_t|\, d\mathscr{H}^1(t) = \int_{\mathbb{R}} \left[\mathscr{H}^{n-1} \llcorner S_t\right] d\mathscr{H}^1(t). \tag{1.10}$$

Definition 1.13. Sometimes the following generalized spaces may occur. We extend to these spaces many of the results we have just stated (for example, Propositions 1.3, 1.5 and 1.10, and Theorem 1.12).

$L^p_{\mathrm{loc}}(\mu, E)$ is the Fréchet space of all functions $f : \Omega \to E$ which belong to $L^p(\mu\llcorner A, E)$ for all open sets A which are relatively compact in Ω.

$\mathscr{M}_{\mathrm{loc}}(\Omega, E)$ is the Fréchet space of all E valued set functions μ defined on the family of all relatively compact Borel subsets of Ω such that the restriction of μ to the family of all Borel subsets of A belongs to $\mathscr{M}(A, E)$ for all open sets A which are relatively compact in Ω. When X is a subset of E, $\mathscr{M}_{\mathrm{loc}}(\Omega, X)$ is the set of all measures in $\mathscr{M}_{\mathrm{loc}}(\Omega, E)$ which takes values in X.

We denote by $BV_{\mathrm{loc}}(\Omega, E)$ the Fréchet space of all functions $u : \Omega \to E$ which belong to $BV(A, E)$ for all open sets $\mathscr{A}$ which are relatively compact in Ω. Strictly speaking, the derivative of a function in $BV_{\mathrm{loc}}(\Omega, E)$ is an element of $\mathscr{M}_{\mathrm{loc}}\big(\Omega, \mathscr{L}(N, E)\big)$.

Remark 1.14. (Measurable Multifunctions). In many technical steps, we need some measurable selection theorems for measurable multifunctions (namely functions which take values in the class of all closed subsets of a given space). Of course, they are not essential to the comprehension of the general ideas but we must spend some words about them.

Let X be a locally compact complete metric space, let $\mathscr{F}(X)$ be the collection of all closed subsets of X endowed with the usual Hausdorff metric δ. Following [8] we say that when $(E, \mathscr{E})$ is a measurable space and F is (multi)function of E into $\mathscr{F}(X)$, then F is measurable if one of the following equivalent statements holds:

(a) for every open set $A \subset X$, $\big\{t \in E : F(t) \cap A \neq \emptyset\big\} \in \mathscr{E}$,

(b) for every $x \in X$, $t \mapsto d\big(x, F(t)\big)$ is a measurable real function,

(c) for every compact set $K \subset X$, $t \mapsto F(t) \cap K$ is measurable if considered as a map of $(E, \mathscr{E})$ into the metric space $\mathscr{F}(X)$.

When μ is a positive measure on N and F is a function of N into $\mathscr{F}(X)$, we say that F is μ approximately lower semicontinuous when for every open set $A \subset X$, the set $S(A)$ of all points t such that $F(t) \cap A \neq \emptyset$ has μ density 1 in everyone of its points, and this means that

$$\lim_{r\to 0} \frac{\mu\big(B(t,r) \cap S(A)_*\big)}{\mu\big(B(t,r)\big)} = 1 \qquad \text{for every } t \in S(A), \tag{1.11}$$

where $S(A)_*$ is a μ maximal Borel set included in $S(A)$, i.e. $\mu\big(B \setminus S(A)_*\big) = 0$ for every Borel set $B \subset S(A)$. It may be proved that every μ approximately lower semicontinuous function is μ measurable (for further details, see section 5 and [8], chapters II and III).

THEOREM 1.15. (Measurable Selection Theorem). *Let X be a (closed subset of) a finite dimensional Banach space, M a locally compact space and μ a locally finite positive measure on M. Let $F : M \to \mathscr{F}(X)$ be a μ measurable function such that $F(x) \neq \emptyset$ for μ almost all x.*

Then there exist Borel functions $f_n : M \to X$ for $n = 1, 2, \dots$ such that $F(x)$ is the closure of the set $\{f_n(x) : n = 1, 2, \dots\}$ for μ almost all x. In particular there always exists a Borel function $f : M \to X$ such that $f(x) \in F(x)$ for μ almost all x. Moreover, if $F(x)$ is a always a linear space, we may take each f_n in $L^1(\mu, X)$.

2. Definition and Main Properties of $E(\mu, x)$

In this section we give the definition of $E(\mu, x)$ (see 2.1, 2.3 and 2.4). Since $E(\mu, x)$ is the essential tool of this paper, it is important to make clear its meaning; we show what $E(\mu, x)$ really is in some particular cases (Remarks 2.7, 2.8, 2.9 and 2.10) and we state some of its general properties (essentially Proposition 2.6 and Theorem 2.12) without detailed proofs (which can be found in section 5).

DEFINITION 2.1. Let ψ and λ be two measures in $\mathscr{M}(\Omega, E)$ (or in $\mathscr{M}_{\mathrm{loc}}(\Omega, E)$) and let x be a point of Ω. We say that λ is tangent to ψ in x when $x \in \operatorname{supp} \psi$ and

$$\lim_{r \to 0} \frac{|\lambda - \psi|(B(x,r))}{|\psi|(B(x,r))} = 0. \tag{2.1}$$

Remark 2.2. When λ is tangent to ψ in x, the inequalities $|\psi| - |\lambda - \psi| \leq |\lambda| \leq |\psi| + |\lambda - \psi|$ yield

$$\lim_{r \to 0} \frac{|\lambda|(B(x,r))}{|\psi|(B(x,r))} = 1 \tag{2.2}$$

and in particular $x \in \operatorname{supp} \lambda$. It follows immediately from (2.2) that tangency in x is an equivalence relation and in particular that λ is tangent to ψ in x if and only if ψ is tangent to λ in x.

Suppose that λ belongs to $\mathscr{M}(\Omega, E)$ and let μ be a positive measure. By Radon–Nikodym theorem we may write $\lambda = f \cdot \mu + \theta$ where $f \in L^1(\mu, E)$ and $\theta \perp \mu$. Then, λ is tangent to $v \cdot \mu$ with $v \neq 0$ in a point $x \in \operatorname{supp} \mu$ if an only if

$$\lim_{r \to 0} \frac{|\theta|(B(x,r))}{\mu(B(x,r))} = 0 \quad \text{and} \quad \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \bigl| f(t) - v \bigr| \, d\mu(t) = 0 \tag{2.3}$$

and well–known theorems about derivation of measures and approximate continuity ensure that (2.3) holds with $v = f(x)$ for μ a.a. x with $f(x) \neq 0$ (see for instance [10], sections 1.6 and 1.7, or [16], section 1.4, or [17], section 1.3). When $|\lambda|$ and $|\psi|$ are mutually singular they are tangent in no points.

DEFINITION 2.3. Let μ be a positive measure on N and let x be a point of N. We define $E(\mu, x)$ as the set which contains 0 and all linear functionals $v \in N^*$ such that there exists a function $u \in BV(N)$ the derivative of which is tangent to $v \cdot \mu$ in x.

Remark 2.4. Notice that in Definition 2.3 it makes no difference to take $u \in BV_{\mathrm{loc}}(A)$, where A is any open set of N which contains x, instead of $u \in BV(N)$.

DEFINITION 2.5. We denote by $\mathscr{D}(N)$ the class of all vector measures $\psi \in \mathscr{M}(N, N^*)$ such that

$$\left[\frac{d\psi}{d|\psi|}\right](x) \in E\big(|\psi|, x\big) \qquad \text{for } \psi \text{ almost all } x \in N. \tag{2.4}$$

This class of measures is relevant because it turns out to be exactly the class of all vector measures which may be written as $Du \llcorner B$ with suitable $u \in BV(\Omega)$ and $B \subset \Omega$ (see Corollary 2.13 and Remark 2.14).

PROPOSITION 2.6. *Let μ and λ be positive measures on N, ψ a measure in $\mathscr{M}(N, N^*)$ and u a function in $BV(N)$. Then*

(i) *$E(\mu, x)$ is a linear space for all x and $E(\mu, x) = \{0\}$ for every $x \notin \operatorname{supp} \mu$,*

(ii) *$E(\lambda, x) = E(\mu, x)$ whenever λ is tangent to $c \cdot \mu$ in x for some $c > 0$,*

(iii) *$E(\lambda, x) = E(\mu, x)$ for λ almost all x whenever $\lambda \ll \mu$,*

(iv) *$\big[dDu/d|Du|\big](x) \in E(\mu, x)$ for μ almost all x whenever $\mu \ll Du$, and in particular $\big[dDu/d|Du|\big](x) \in E\big(|Du|, x\big)$ for $|Du|$ almost all x,*

(v) *$\psi \in \mathscr{D}(N)$ whenever $\psi = f \cdot Du$ with $f \in L^1(|Du|)$, in particular $\psi \in \mathscr{D}(N)$ whenever $\psi = Du \llcorner B$,*

(vi) *$\mathscr{D}(N)$ is a strongly closed subspace of $\mathscr{M}(N, N^*)$,*

Now we want to show what happens in some particular cases.

Remark 2.7. (N has dimension 1). When N has dimension 1 (roughly speaking, when $N = \mathbb{R}$), every measure in $\mathscr{M}(N, N^*)$ is the derivative of a function in $BV_{\mathrm{loc}}(N)$ and then we have that $\mathscr{D}(N) = \mathscr{M}(N, N^*)$ and $E(\mu, x) = N^*$ for every positive measure μ on N and every point $x \in \operatorname{supp} \mu$.

Remark 2.8. (Measures of the Form $f \cdot \mathscr{H}^n \llcorner N$). Let v be a linear functional on N; v is a function in $BV_{\mathrm{loc}}(N)$ with derivative $v \cdot \mathscr{H}^n \llcorner N$. Then $v \in E(\mathscr{H}^n \llcorner N, x)$ for all points x and this yields $E(\mathscr{H}^n \llcorner N, x) = N^*$ for all x. By Proposition 2.6(iii), $E(\mu, x) = N^*$ for μ almost all x whenever μ is a positive measure on N which is absolutely continuous with respect to Lebesgue measure.

In other words we have that every vector measure $\psi \in \mathscr{M}(N, N^*)$ which is absolutely continuous with respect to $\mathscr{H}^n \llcorner N$ belongs to $\mathscr{D}(N)$ (cf. Remark 2.14).

Remark 2.9. (Measures of the form $f \cdot \mathscr{H}^{n-1} \llcorner N$). Let A be a bounded open subset of N with boundary ∂A of class C^1. It is well–known that the characteristic function of A belongs to $BV(N)$ (i.e. A is a set of finite perimeter) and in particular we have that its derivative is $\eta^* \cdot \mathscr{H}^{n-1} \llcorner \partial A$ where η is the inner normal of ∂A (see for instance [10], sections 5.8 and 5.11, or [14], Example 3.4, or [17], Remark 5.8.3). Hence $\eta(x)^* \in E(\mathscr{H}^{n-1} \llcorner \partial A, x)$ for all points x.

Moreover, for every function $u \in BV(N)$, the following formula holds (cf. (1.7)):

$$Du \llcorner \partial A = (u^+ - u^-)\eta^* \cdot \mathscr{H}^{n-1} \llcorner \partial A. \tag{2.5}$$

Hence, if Du is tangent in x to $v \cdot \mathscr{H}^{n-1} \llcorner \partial A$ for some v, we have that $v = |v|\eta(x)^*$ or $-|v|\eta(x)^*$ (see Remark 2.2) and then $E(\mathscr{H}^{n-1} \llcorner \partial A, x)$ is included in the span of $\eta(x)^*$ for all x. By previous observation, we have just proved that for every bounded open set A with boundary of class C^1

$$E(\mathscr{H}^{n-1} \llcorner \partial A, x) = \operatorname{Span}\{\eta(x)^*\} \qquad \text{for all } x \in \partial A. \tag{2.6}$$

It follows immediately that the same holds when ∂A is replaced by a $(n-1)$–dimensional manifold M of class C^1 with locally finite $\mathscr{H}^{n-1}$ measure because for every point x of M we may find a bounded open set A with a C^1 boundary such that $M = \partial A$ in a neighborhood of x.

Suppose that $\mu = f \cdot \mathscr{H}^{n-1}$ where f is a non–negative function in $L^1(\mathscr{H}^{n-1} \llcorner N)$ and $S = \{x : f(x) \neq 0\}$ is a rectifiable set (for example, take $\mu = \mathscr{H}^{n-1} \llcorner S$ where S is a rectifiable set with $\mathscr{H}^{n-1}(S) < \infty$). Let η be an orientation of S, then Definition 1.6, Proposition 2.6(iii) and (2.6) yield

$$E(\mu, x) = \operatorname{Span}\{\eta(x)^*\} \qquad \text{for } \mu \text{ a.a. } x. \tag{2.7}$$

Remark 2.10. (Measures Concentrated on Purely Unrectifiable Sets). Suppose that μ is a positive real measure on N which is concentrated in a purely unrectifiable set T. By Definition 1.6, we obtain that $\mathscr{H}^{n-1}(S \cap T) = 0$ for every rectifiable set S and then, when u is a function in $BV(N)$, taking into account coarea formula (1.10) we get

$$|Du|(T) = \int_{\mathbb{R}} \mathscr{H}^{n-1}(S_t \cap T) d\mathscr{H}^1(t) = 0.$$

Then $Du \perp \mu$ and Du is tangent to $v \cdot \mu$ in no points (Remark 2.2). Hence $E(\mu, x) = \{0\}$ for all points x.

In particular, taking into account previous remark, a measure $f \cdot \mathscr{H}^{n-1}$ with $f \in L^1(\mathscr{H}^{n-1} \llcorner N, N^*)$ belongs to $\mathscr{D}(N)$ if and only if the set $S = \{x : f(x) \neq 0\}$ is rectifiable and $\operatorname{Ker}[f(x)] = \operatorname{Tan}(S, x)$ for $\mathscr{H}^{n-1}$ a.a. $x \in S$ (cf. Remark 2.14).

In many technical steps, we shall need some measurability properties of the (multi)function $x \mapsto E(\mu, x)$

Proposition 2.11. (Regularity Property of $x \mapsto E(\mu, x)$). *When μ is a positive measure on N, $x \mapsto E(\mu, x)$ is a function of N into $\mathscr{F}(N^*)$ which is μ approximately lower semicontinuous and then it is μ measurable. Hence also $x \mapsto \dim E(\mu, x)$ is a μ approximately lower semicontinuous and μ measurable real function of N (cf. Remark 1.14).*

We have just given all the immediate properties of $E(\mu, x)$. Now we want to state a deeper theorem which is one of the basic ingredients of our method.

We recall that Proposition 2.6(iv) may be formulated as follows: when u is a function in $BV(N)$ and μ is a positive measure on N, we may write $Du = f \cdot \mu + \theta$ where f belongs to $L^1(\mu, N^*)$ and θ is a vector measure such that $\theta \perp \mu$. Then $f(x)$ belongs to $E(\mu, x)$ for μ a.a. x. The next theorem is a converse of this statement.

Theorem 2.12. *Let μ be a positive measure on N and suppose that $f \in L^1(\mu, N^*)$ is a function such that*

$$f(x) \in E(\mu, x) \quad \textit{for } \mu \textit{ almost all } x. \tag{2.8}$$

Then there exist $u \in BV(\Omega)$ and $\theta \in \mathscr{M}(\Omega, N^*)$ such that $\theta \perp \mu$ and $Du = f \cdot \mu + \theta$. Moreover u may be taken so that $\|u\|_{BV} \le C \|f\|_{L^1(\mu)}$ where C is a constant which depends on the dimension of N only.

An equivalent formulation of the first part of Theorem 2.12 is the following.

COROLLARY 2.13. *Let ψ be a measure in $\mathscr{M}(N, N^*)$, then ψ belongs to $\mathscr{D}(N)$ if and only if there exists $\theta \in \mathscr{M}(N, N^*)$ such that $\theta \perp \psi$ and $\psi + \theta$ is the derivative of some BV function of N.*

Remark 2.14. It is interesting to notice what happens if we apply Theorem 2.12 in two particular cases. In the following examples $N = \mathbb{R}^n$ and we identify $\mathbb{R}^n$ with its dual space.

Suppose that f is a vector function in $L^1(\mathscr{H}^n \llcorner \mathbb{R}^n, \mathbb{R}^n)$. In Remark 2.8 we proved that $f \cdot \mathscr{H}^n \llcorner \mathbb{R}^n \in \mathscr{D}(\mathbb{R}^n)$ and by the previous corollary there exists a function $u \in BV(\mathbb{R}^n)$ so that the Radon–Nikodym derivative of Du with respect to Lebesgue measure agrees with f almost everywhere. This means that the approximate gradient of u agrees with f for $\mathscr{H}^n$ almost all points of $\mathbb{R}^n$ (cf. [1], Theorem 3).

Suppose that f is a vector function in $L^1(\mathscr{H}^{n-1} \llcorner \mathbb{R}^n, \mathbb{R}^n)$. In Remarks 2.9 and 2.10 we showed that the measure $f \cdot \mathscr{H}^{n-1}$ belongs to $\mathscr{D}(\mathbb{R}^n)$ if and only if the set $S = \{x : f(x) \neq 0\}$ is rectifiable and $f(x) \perp \mathrm{Tan}(S, x)$ for $\mathscr{H}^{n-1}$ a.a. $x \in S$. By Corollary 2.13, these two statements and the following are equivalent: there exists $u \in BV(\mathbb{R}^n)$ such that $Du \llcorner S = f \cdot \mathscr{H}^{n-1}$.

This results is reminiscent of a well–known result about traces of BV functions due to Gagliardo (see [13] or [14], Theorem 2.16).

3. The Dimension of $\mathbf{E}(\mu, \mathbf{x})$

This section is devoted to the proof of the following result.

THEOREM 3.1. *Let N be an n–dimensional linear subspace of some euclidean space and let μ be a positive measure on N which is singular with respect to $\mathscr{H}^n \llcorner N$. Then the dimension of $E(\mu, x)$ is 1 or 0 for μ almost all x.*

Remark 3.2. Notice that by Proposition 2.11, the dimension of $E(\mu, x)$ is a μ approximately lower semicontinuous function of x and then Theorem 3.1 yields that the dimension of $E(\mu, x)$ is 0 or 1 everywhere and not only μ almost everywhere.

Theorem 3.1 is the essential result of this paper and both the definition of rectifiable measures and the rank theorems in the following section are essentially based upon it.

To begin with, we state the following particular result.

LEMMA 3.3. *Let N be a 2–dimensional linear subspace of some euclidean space, $\{e_1, e_2\}$ an orthonormal basis of N and μ a positive measure on N. Suppose that there exists a bounded function $u = (u^1, u^2)$ in $BV_{\mathrm{loc}}(N, \mathbb{R}^2)$ such that, for $i = 1, 2$, $Du^i \in \mathscr{M}\big(N, X^*(e_i, \sqrt{3}/2)\big)$ (see (1.1)) and $\mu \ll Du^i$. Then $\mu \ll \mathscr{H}^2 \llcorner N$.*

This is the key lemma in the proof of Theorem 3.1. We apply it to study the dimension of $E(\mu, x)$ in the following way: suppose that N has dimension 2 and μ is a measure on N such that $E(\mu, x)$ has dimension 2 for μ almost all points, by Theorem 3.5(v) and definition 3.4 there exists a function u which satisfies the hypothesis of Lemma 3.3 and then μ is absolutely continuous with respect to $\mathscr{H}^2 \llcorner N$. Hence $E(\mu, x)$ has dimension at most 1 whenever N has dimension 2 and μ is a measure on N which is singular with respect to $\mathscr{H}^2 \llcorner N$. We prove that the same holds when the dimension of N is arbitrary studying the disintegration behaviour of $E(\mu, x)$ (cf. Proposition 3.6).

The proof of Lemma 3.3 is very simple and disclose the essential idea of the main results, so we like better to give it immediately. Then we state the technical results (Theorem 3.5 and Proposition 3.6) that we need together with Lemma 3.3 in the proof of Theorem 3.1 (their proofs can be found in section 5) and eventually we prove it.

Proof of lemma 3.3. Possibly replacing u with the function $v = (v^1, v^2)$ given by $v^i(x) = u^i(x) + \arctan\langle x; e_1\rangle$ for all $x \in N$ and $i = 1, 2$, we may suppose that the support of Du^i is N for $i = 1, 2$.

Let ρ_ε denote as usual some positive mollifiers of class C^∞ on N, set $u_\varepsilon = (u^1_\varepsilon, u^2_\varepsilon) = u * \rho_\varepsilon$ for every $\varepsilon > 0$, and notice that the following facts hold:

(a) Since u is bounded, there exists $r > 0$ so that $|u(x)| \le r$ for every x. For $i = 1, 2$, $Du^i \in \mathscr{M}\big(N, X^*(e_i, \sqrt{3}/2)\big)$ yields

$$\partial u^i/\partial e_i \ge \sqrt{3}/2 \cdot |Du^i| \tag{3.1}$$

and then $\mu \ll \partial u^i/\partial e_i$. Moreover $\partial u^i/\partial e_i$ is a positive measure with support N because the support of Du^i is N.

(b) u_ε is a function of class C^∞, and taking into account (a), for $i = 1, 2$ and every $x \in N$, $|u(x)| \le r$,

$$\frac{\partial u^i_\varepsilon}{\partial e_i}(x) > 0 \quad \text{and} \quad Du^i_\varepsilon(x) \in X^*(e_i, \sqrt{3}/2) \tag{3.2}$$

and then
$$\frac{1}{\sqrt{3}}\frac{\partial u^1_\varepsilon}{\partial e_1}(x) \ge \left|\frac{\partial u^1_\varepsilon}{\partial e_2}(x)\right| , \qquad \frac{1}{\sqrt{3}}\frac{\partial u^2_\varepsilon}{\partial e_2}(x) \ge \left|\frac{\partial u^2_\varepsilon}{\partial e_1}(x)\right| \tag{3.3}$$

(c) u_ε is an injective function: indeed, let $x, y \in N$ with $y \ne 0$, then

$$u_\varepsilon(x+y) - u_\varepsilon(x) = \int_0^1 Du_\varepsilon(x+ty)\cdot y\, d\mathscr{H}^1(t). \tag{3.4}$$

Now, since

$$X(e_1, \sqrt{2}/2) \cup X(-e_1, \sqrt{2}/2) \cup X(e_2, \sqrt{2}/2) \cup X(e_2, \sqrt{2}/2) = N$$

(see formula (1.1)), y has to belong to one at least of these four cones. If it belongs to the first one, we have that $vy > 0$ for all $v \in X^*(e_1, \sqrt{3}/2)$ with $v \ne 0$ and in particular, taking into account (3.2), $Du^1_\varepsilon(z)\cdot y > 0$ for all $z \in N$. Hence (3.4) yields $u^1_\varepsilon(x+y) > u^1_\varepsilon(x)$ and then $u_\varepsilon(x+y) \ne u_\varepsilon(x)$. In a similar way it may be proved that $u_\varepsilon(x+y) \ne u_\varepsilon(x)$ when y belongs to anyone of the other three cones. Hence we have proved that u_ε is injective.

(d) As u_ε is an injective function of class C^∞, the area formula yields

$$\frac{3\pi r^2}{2} \geq \int_N \frac{\partial u_\varepsilon^1}{\partial e_1} \frac{\partial u_\varepsilon^2}{\partial e_2} \, d\mathscr{H}^2. \tag{3.5}$$

Indeed, taking into account (b), (c) and (3.3),

$$\begin{aligned}\pi r^2 \geq \mathscr{H}^2(\operatorname{Im} u_\varepsilon) = \int_N J u_\varepsilon d\mathscr{H}^2 &= \int_N \left| \frac{\partial u_\varepsilon^1}{\partial e_1} \frac{\partial u_\varepsilon^2}{\partial e_2} - \frac{\partial u_\varepsilon^1}{\partial e_2} \frac{\partial u_\varepsilon^2}{\partial e_1} \right| d\mathscr{H}^2 \\ &\geq \frac{2}{3} \int_N \frac{\partial u_\varepsilon^1}{\partial e_1} \frac{\partial u_\varepsilon^2}{\partial e_2} \, d\mathscr{H}^2.\end{aligned}$$

Now, for every $\delta > 0$, set

$$S_\delta = \left\{ x : \ \left[d\Big(\frac{\partial u^i}{\partial e_i}\Big) \Big/ d\mu \right](x) \geq \delta \ \text{ for } i = 1, 2 \right\} \tag{3.6}$$

and set $\lambda = \delta \cdot \mu \llcorner S_\delta$. Hence $\dfrac{\partial u^i}{\partial e_i} \geq \lambda$ and $\dfrac{\partial u_\varepsilon^i}{\partial e_i}(x) \geq \lambda * \rho_\varepsilon(x)$ for $i = 1, 2$, for every $\varepsilon > 0$ and every $x \in N$, and then (3.5) yields

$$\frac{3\pi r^2}{2} \geq \int_N \frac{\partial u_\varepsilon^1}{\partial e_1} \frac{\partial u_\varepsilon^2}{\partial e_2} \, d\mathscr{H}^2 \geq \int_N (\lambda * \rho_\varepsilon)^2 d\mathscr{H}^2.$$

This implies that λ is (represented by) a function in $L^2(\mathscr{H}^2 \llcorner N)$ and in particular $\mu \llcorner S_\delta \ll \mathscr{H}^2 \llcorner N$ for all positive δ. The proof is complete because $\mu \ll \partial u^i / \partial e_i$ for $i = 1, 2$ (see (a)) and then (cf. (3.6))

$$\mu\Big(N \setminus \bigcup_{n=1}^{\infty} S_{1/n}\Big) = 0. \qquad \square$$

Definition 3.4. Let X be a strongly closed convex cone in N^*. We define $\mathscr{E}(N, X)$ as the set of all finite positive measures μ on N such that there exists a function u in $BV_{\text{loc}}(N)$ which satisfies

(a) $\|u\|_\infty \leq 1$ (we mean the $L^\infty(\mathscr{H}^n \llcorner N)$ norm),

(b) $Du \in \mathscr{M}_{\text{loc}}(N, X)$ and $\mu \ll Du$.

We remark that when u is a function which fulfills (b) but not (a), then $v = (2/\pi) \arctan \circ u$ is a function in $BV_{\text{loc}}(N)$ which satisfies both (a) and (b) (this fact may be proved using coarea formula (Theorem 1.12) and statements (iv) and (v) of Propositions 1.3) and then (a) does not play an essential role in this definition but it is useful in the proofs of Theorem 3.5 and Lemma 3.8.

Theorem 3.5. (Properties of $\mathscr{E}(N, X)$). *Let X be a closed convex cone in N^* of the form $X = X^*(e, a)$ (see formula* (1.1) *) and let M be a separable locally compact topological space. Let (S, η) be an oriented rectifiable set in N with $\mathscr{H}^{n-1}(S) < \infty$, μ a finite positive measure on N, λ a positive measure on M and $t \mapsto \mu_t$ a function in $L^1\big(\lambda, \mathscr{M}(N, N^*)\big)$ so that $\mu = \int \mu_t d\lambda(t)$ (see Definition 1.2). Then*

(i) *$\mathscr{E}(N,X)$ is a strongly closed convex cone in $\mathscr{M}(N)$ and in particular a series of elements of $\mathscr{E}(N,X)$ may converge in norm to an element of $\mathscr{E}(N,X)$ only,*

(ii) *$\mu \in \mathscr{E}(N,X)$ whenever μ is absolutely continuous with respect to a measure in $\mathscr{E}(N,X)$,*

(iii) *$\mu \in \mathscr{E}(N,X)$ whenever $\mu_t \in \mathscr{E}(N,X)$ for λ a.a. $t \in M$,*

(iv) *$\mathscr{H}^{n-1} \llcorner S \in \mathscr{E}(N,X)$ whenever $\eta(x)$ belongs to the interior of $X(e,a)$ for $\mathscr{H}^{n-1}$ a.a $x \in S$,*

(v) *$\mu \in \mathscr{E}(N,X)$ whenever the intersection of $E(\mu,x)$ and the interior of X is not empty for μ a.a. x.*

Proposition 3.6. (Disintegration Property of $E(\mu,x)$). *Let μ be a finite positive measure on $M \times N$, λ a finite positive measure on M such that $\pi^{\#}\mu \ll \lambda$ and let $t \mapsto \mu_t$ be the disintegration of μ with respect to λ (see Proposition 1.5). Let $\lambda = \lambda_A + \lambda_S$ be the Lebesgue decomposition of λ with respect to $\mathscr{H}^m \llcorner M$. Then*

(i) *for λ_A a.a. $t \in M$ and for μ_t a.a. $s \in N$, $v \llcorner N \in E(\mu_t,s)$ for all $v \in E\big(\mu,(t,s)\big)$,*

(ii) *for λ_S a.a. $t \in M$ and for μ_t a.a. $s \in N$, $v \llcorner N = 0$ for all $v \in E\big(\mu,(t,s)\big)$.*

Remark 3.7. In Proposition 3.6(i) we have that for λ_A a.a. t and μ_t a.a. s,

$$E(\mu_t,s) \supset \Big\{ v \llcorner N : v \in E\big(\mu,(t,s)\big)\Big\}.$$

In general equality does not hold. Take for example $N = \mathbb{R}$ and let $f : M \to \mathbb{R}$ be a Borel function the graph of which, Γf, is a purely unrectifiable subset of $M \times \mathbb{R}$. For all $t \in M$, let $\mu_t \in \mathscr{M}(\mathbb{R})$ be Dirac delta mass concentrated in the point $f(t)$ (i.e. the measure given by $\mu_t(B) = 1$ if $f(t) \in B$ and 0 otherwise) and set

$$\mu = \int_M \big[\epsilon_t \otimes \mu_t\big]\, d\mathscr{H}^n(t)$$

(see Remark 1.4). Then $\pi_M^{\#}\mu = \mathscr{H}^m \llcorner M$ and μ is concentrated in the set Γf. Hence $E\big(\mu,(t,s)\big) = \{0\}$ for all $(t,s) \in M \times \mathbb{R}$ (Remark 2.10) but $E(\mu_t,s) = \mathbb{R}^*$ for all $t \in M$ and μ_t a.a. $s \in \mathbb{R}$ (Remark 2.7).

Taking into account Theorem 3.5 and Proposition 3.6, we may apply Lemma 3.3 to prove the essential result of this section.

Lemma 3.8. *Let N be an n–dimensional linear subspace of some euclidean space and let μ be a positive measure on N such that $E(\mu,x)$ has dimension greater than 1 for μ a.a. x. Then μ is absolutely continuous with respect to $\mathscr{H}^n \llcorner N$.*

Proof. With no loss in generality we may suppose that μ is finite. The proof of this lemma is divided in two cases.

Case 1: $n = 2$.

Let $\{e_1,e_2\}$ be a orthonormal basis of N. Since $E(\mu,x)$ has dimension greater than 1, $E(\mu,x) = N^*$ for μ a.a. x and then Theorem 3.5(v) yields $\mu \in \mathscr{E}\big(N,X^*(e,a)\big)$ for all $e \in N$ with $|e| = 1$ and $a \in]0,1[$. In particular, for $i = 1,2$

we may find $\mathscr{H}^2$ essentially bounded functions u^i in $BV_{\mathrm{loc}}(N)$ such that $\mu \ll Du^i$ and $Du^i \in \mathscr{M}\big(N, X^*(e_i, \sqrt{3}/2)\big)$ (cf. Definition 3.4) and then $\mu \ll \mathscr{H}^2 \llcorner N$ by Lemma 3.3.

Case 1: n is arbitrary.

(a) Let P be a 2–dimensional subspace of N and let M be the orthogonal complement of P, we identify $M \times P$ and N. Let B be the set of all x such that $\big\{v \llcorner P : v \in E(\mu, x)\big\}$ has dimension 2 and set $\mu' = \mu \llcorner B$, $\lambda = \pi_M^{\#}\mu'$. By Proposition 2.6(iii), $\big\{v \llcorner P : v \in E(\mu', x)\big\}$ has dimension 2 for μ' a.a. x and Proposition 3.6 implies that λ is absolutely continuous with respect to $\mathscr{H}^{n-2} \llcorner M$ and

$$\Big\{v \llcorner P : v \in E\big(\mu', (t, y)\big)\Big\} \subset E(\mu'_t, y) \quad \text{for } \lambda \text{ a.a. } t \in M \text{ and } \mu'_t \text{ a.a. } y \in P.$$

By Case 1 we have that $\mu'_t \ll \mathscr{H}^2 \llcorner P$ for λ a.a. t. and then $\mu \llcorner B = \mu' \ll \mathscr{H}^n \llcorner N$ (cf. Remark 1.4).

(b) Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of N and let $P_{i,j}$ be the span of e_i, e_j for all $i, j = 1, \dots, n$. It is well–known from linear algebra that two linear functionals $v_1, v_2 \in N^*$ are linearly dependent if and only if their restrictions on $P_{i,j}$ are linearly dependent for all i, j. This means that a linear subspace M of N^* has dimension greater than 1 if and only if there exists at least one pair i, j such that $\big\{v \llcorner P_{i,j} : v \in M\big\}$ is a subspace of dimension 2 of $P^*_{i,j}$.

Let $B_{i,j}$ be the set of all x such that $\big\{v \llcorner P_{i,j} : v \in E(\mu, x)\big\}$ has dimension 2. (a) yields $\mu \llcorner B_{i,j} \ll \mathscr{H}^n \llcorner N$ for all i, j and the proof is complete since (b) yields

$$\mu\Big(N \setminus \bigcup_{i,j} B_{i,j}\Big) = 0. \qquad \square$$

Proof of Theorem 3.1. Let B be the set of all points x such that $E(\mu, x)$ has dimension greater than 1 and set $\mu' = \mu \llcorner B$. By Proposition 2.6(iii) we have that $E(\mu', x) = E(\mu, x)$ has dimension greater than 1 for μ' a.a. x and Lemma 3.8 yields $\mu' \ll \mathscr{H}^n \llcorner N$. Hence $\mu(B) = 0$. $\square$

4. Rectifiable Measures and Rank One Property of Derivatives

In this section we give the definition and a characterization of rectifiable measures (Definitions 4.1 and Proposition 4.2) and some of their properties (Proposition 4.3 and 4.4). In Theorem 4.5 we show that when μ is a rectifiable measure and u is a (vector valued) function with bounded variation, the structure of the part of Du which is absolutely continuous with respect to μ is tightly connected to $E(\mu, x)$ (formula (4.3)) and in particular we show that the singular part of a derivative has rank one (Corollary 4.6). In Theorem 4.13 and Corollary 4.14 we extend the previous results to higher order derivatives. We end this section with some open questions (Remark 4.18). Every result in this section is almost a straightforward corollary of Theorem 3.1.

Let μ is a positive measure on N of the form $f \cdot \mathscr{H}^{n-1}$ and set $S = \big\{x : f(x) \neq 0\big\}$. In Remark 2.9 we showed that when S is a rectifiable set, then $E(\mu, x)$ is the annihilator of $\mathrm{Tan}(S,x)$ for μ a.a. x while $E(\mu,x) = \{0\}$ for (μ almost) all x whenever S is a purely unrectifiable set (Remark 2.10). In general, when μ is a positive measure measure on N which is singular with respect to $\mathscr{H}^n \llcorner N$, by Theorem 3.1 we have that the dimension of $E(\mu,x)$ is 0 or 1 for μ almost every x.

These considerations suggest to use $E(\mu,x)$ to give a definition of rectifiable and unrectifiable for those positive measures μ which are singular with respect to $\mathscr{H}^n \llcorner N$.

Definition 4.1. (Rectifiable and Purely Unrectifiable Measures). Let μ be a positive measure on N which is singular with respect to $\mathscr{H}^n \llcorner N$. We say that μ is rectifiable when $E(\mu,x)$ has dimension 1 for μ a.a. x (taking into account Theorem 3.1, this means that μ is rectifiable when $E(\mu,x) \neq \{0\}$ for μ a.a. x).

We say that a Borel function $\eta : N \to N$ is an orientation of μ when $\big|\eta(x)\big| = 1$ and $E(\mu,x)$ is the span of $\eta(x)$ for μ almost all x.

We say that μ is purely unrectifiable whenever $E(\mu,x)$ has dimension 0 for μ a.a. x.

Using Theorems 2.12 and 3.5, in the following proposition we give a characterization of those finite positive measure μ such that $E(\mu,x)$ has dimension at least 1 for μ almost all x and in particular of rectifiable measures. The same result holds, with suitable, modifications if we take μ locally finite instead of finite.

Proposition 4.2. *Let μ be a finite positive measure in N and let E be a finite dimensional Banach space as usual. Then the following statements are equivalent:*

(i) $\dim E(\mu,x) > 0$ *for μ almost all x.*

(ii) $\mu = |Du| \llcorner B$ *for some function $u \in BV(N)$ and some Borel set $B \subset N$,*

(ii') $\mu \ll Du$ *for some function $u \in BV(N,E)$,*

(iii) *there exists a function $t \mapsto \mu_t$ in $L^1\big(\mathscr{H}^1 \llcorner \mathbb{R}, \mathscr{M}(N)\big)$ such that $\mu = \int_{\mathbb{R}} \mu_t d\mathscr{H}^1(t)$ and, for $\mathscr{H}^1$ almost all $t \in \mathbb{R}$, $\mu_t = \mathscr{H}^{n-1} \llcorner S_t$ where S_t is a rectifiable set.*

(iii') there exist a separable locally compact space M, a positive measure λ on M and a function $t \mapsto \mu_t$ in $L^1\big(\lambda, \mathscr{M}(N)\big)$ such that $\mu = \int_M \mu_t d\lambda(t)$ and, for λ almost all $t \in M$, $\dim E(\mu_t,x) > 0$ for μ_t a.a. x.

Proof. We prove the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii') $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iii') $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): set $C = \big\{v \in N^* : |v| = 1\big\}$; as $\dim E(\mu,x) > 0$, $E(\mu,x) \cap C \neq \emptyset$ for μ a.a x and by Proposition 2.11 and Remark 1.14, $x \mapsto E(\mu,x) \cap C$ is a μ measurable function of N into $\mathscr{F}(N^*)$. By Theorem 1.15 we may find a function $f \in L^1(\mu, N^*)$ such that $f(x) \in E(\mu,x)$ and $|f(x)| = 1$ for μ a.a x and then it is enough to apply Theorem 2.12.

(ii) $\Rightarrow$ (ii'): trivial.

(ii') $\Rightarrow$ (i): let $\{p_1, \ldots, p_m\}$ be a basis of E^*, for every i with $1 \le i \le m$ let u_i be the function given by $u_i(x) = p_i\big(u(x)\big)$ for every $x \in N$. Each u_i is a function in $BV(N)$ and $\dim E\big(|Du_i|, x\big) > 0$ for $|Du_i|$ almost all x (cf. statement (iv) of Proposition 2.6). Taking into account statement (iii) of Proposition 2.6 and the fact that $\mu \ll |Du_1| + \ldots + |Du_m|$, we obtain that (i) holds.

(ii) $\Rightarrow$ (iii): apply the coarea formula (Theorem 1.12).

(iii) $\Rightarrow$ (iii'): trivial.

(iii) $\Rightarrow$ (i): let $X_1, \dots X_n$ be closed convex cones of the form $X^*(e,a)$ the interiors of which cover $N^* \setminus \{0\}$. Let i be fixed. For every $t \in M$, let B^i_t be the set of all points x such that $E(\mu_t, x) \cap \mathrm{Int}(X_i) \neq \{0\}$, and set

$$\mu^i_t = \mu_t \llcorner B^i_t \quad \text{and} \qquad \mu^i = \int_M \mu^i_t d\lambda(t).$$

Then $\mu \ll \sum_i \mu^i$ and by Proposition 2.6(iii), for every t, $E(\mu^i_t, x) \cap \mathrm{Int}(X_i) \neq \{0\}$ for μ^i_t almost all x. Then $\mu^i_t \in \mathscr{E}\big(N, X_i\big)$ for every t by Theorem 3.5(v) and $\mu^i \in \mathscr{E}\big(N, X_i\big)$ by Theorem 3.5(iii). In particular $E(\mu^i, x) \neq \{0\}$ for μ^i a.a. x and then $E(\mu, x) \neq \{0\}$ for μ a.a. x because $\mu \ll \sum \mu^i$ (cf. Proposition 2.6). □

In the following proposition we show that every positive singular measure is the sum of a rectifiable measure and a purely unrectifiable measure and moreover this decomposition is unique. In Proposition 4.4 we recall some immediate properties of rectifiable and purely unrectifiable measures.

Proposition 4.3. (Decomposition of Singular Measures). *For every positive measure μ on N which is singular with respect to $\mathscr{H}^n \llcorner N$, let B be the set of all x such that $E(\mu, x)$ has dimension 1 and set*

$$R\mu = \mu \llcorner B \ , \qquad S\mu = \mu \llcorner (N \setminus B). \tag{4.1}$$

Then

(i) *$R\mu$ is rectifiable, $S\mu$ is purely unrectifiable, $R\mu \perp S\mu$ and $\mu = R\mu + S\mu$,*

(ii) *the above decomposition is unique in the following sense: when $\mu = \mu_1 + \mu_2$ with μ_1 rectifiable and μ_2 purely unrectifiable, $\mu_1 = R\mu$ and $\mu_2 = S\mu$.*

Proof. Taking into account Proposition 2.6(iii) and Definition 4.1, (i) immediately follows and moreover we obtain $\mu_2 \llcorner B = 0$ and $\mu_1 \llcorner (N \setminus B) = 0$ and then $\mu = \mu_1 + \mu_2$ yields $\mu_1 = \mu \llcorner B$ and $\mu_2 = \mu \llcorner (N \setminus B)$. □

Proposition 4.4. *Let E be a finite dimensional Banach space, let μ and λ be positive measures on N which are singular with respect to $\mathscr{H}^n \llcorner N$ and $\lambda \ll \mu$. Then*

(i) *$R\lambda \ll R\mu$ and $S\lambda \ll S\mu$ and $E(R\lambda, x) = E(R\mu, x)$ for $R\lambda$ almost all x,*

(ii) *in particular, when μ is rectifiable, also λ is rectifiable and $E(\lambda, x) = E(\mu, x)$ for λ almost all x,*

(iii) *when $\mu \ll \mathscr{H}^{n-1} \llcorner S$ and S is a rectifiable set (see Definition 1.6), μ is rectifiable and $E(\mu, x) = \mathrm{Tan}(S, x)^\perp$ for μ almost all x,*

(iv) *when μ is concentrated in a purely unrectifiable set (see Definition 1.6), μ is purely unrectifiable,*

(v) *when $\mu \ll Du$ for some function $u \in BV(N, E)$, μ is rectifiable and $E(\mu, x)$ is the annihilator of the kernel of $\big[dDu/d\mu\big](x)$ for μ almost all x and in particular the dimension of the kernel of $\big[dDu/d\mu\big](x)$ is $n-1$ for μ almost all x,*

(vi) *when μ is a purely unrectifiable measure, $\mu \perp Du$ for every function $u \in BV(N, E)$.*

Proof. (i) follows from Propositions 4.2 and 2.6(iii) and (ii) follows from (i).

(iii) follows from Remark 2.9 and (iv) follows from Remark 2.10.

(v): write $Du = f\mu + \theta$ with $\theta \perp \mu$ and let $\{p_1, \dots, p_m\}$ be a basis of E^*. For every i with $1 \leq i \leq m$ let u_i be the function given by $u_i(x) = p_i\big(u(x)\big)$ for every $x \in N$. Each u_i is a function in $BV(N)$ and $Du_i = f_i \cdot \mu + \theta_i$ where $f_i \in L^1_{\mathrm{loc}}(\mu, N^*)$, $\theta_i \in \mathscr{M}(N, N^*)$, $\theta_i \perp \mu$ and for every x, $f_i(x)$ is the element of N^* given by the superposition of $f(x)$ and p_i.

By Proposition 2.6(iv), for μ almost every x and every i we have that $f_i(x) \in E(\mu, x)$ and this means that $E(\mu, x)$ includes the annihilator of the kernel of $f_i(x)$. Hence, taking into account that $\{p_1, \dots, p_m\}$ is a basis of E^*, we obtain that for μ almost all x

$$\begin{aligned} E(\mu, x) &\supset \mathrm{Span}\Big\{\mathrm{Ker}\big[f_1(x)\big]^\perp, \dots, \mathrm{Ker}\big[f_m(x)\big]^\perp\Big\} \\ &= \Big[\bigcap_{i=1}^m \mathrm{Ker}\big[f_i(x)\big]\Big]^\perp = \mathrm{Ker}\big[f(x)\big]^\perp. \end{aligned} \tag{4.2}$$

But $f(x) \neq 0$ for μ almost all x because $\mu \ll Du$, hence the dimension of $\mathrm{Ker}\big[f(x)\big]$ is lower or equal to $n-1$ and the dimension of its annihilator is greater or equal to 1. Since the dimension of $E(\mu, x)$ is 1 or 0 for μ almost every x, by inclusion (4.2) we obtain that $E(\mu, x)$ is the annihilator of the kernel of $f(x)$ and its dimension is 1 for μ almost all x and (v) is proved.

(vi) follows form (v). □

Suppose that B is a Borel set with $\mathscr{H}^{n-1}(B) < \infty$, and let B_1 and B_2 be the rectifiable and purely unrectifiable part of B respectively. Set $\mu = \mathscr{H}^{n-1} \llcorner B$: taking into account previous proposition, we obtain that $R\mu = \mathscr{H}^{n-1} \llcorner B_1$ and $S\mu = \mathscr{H}^{n-1} \llcorner B_2$ and so the decomposition given in Proposition 4.3 corresponds in this case to the usual decomposition of an $\mathscr{H}^{n-1}$ finite set as union of a rectifiable and a purely unrectifiable set (cf. Definition 1.6).

Now we have the essential result of this section.

THEOREM 4.5. *Let μ be a rectifiable measure on N and let η be an orientation of μ. Let E be a finite dimensional Banach space and let u be a function in $BV(N, E)$. As usual we write $Du = f \cdot \mu + \theta$ where $f \in L^1\big(\mu, \mathscr{L}(N, E)\big)$, $\theta \in \mathscr{M}\big(N, \mathscr{L}(N, E)\big)$ and $\theta \perp \mu$ (cf. Definition 1.8).*

Then, for μ almost all x, $f(x)$ is a linear map of N into E of the form $f(x) = e(x)\eta(x)^*$, i.e.

$$\big[f(x)\big] : y \longmapsto \langle \eta(x); y\rangle\, e(x) \quad \textit{for every } y \in N, \tag{4.3}$$

where e is the function in $L^1(\mu, E)$ given by

$$e(x) = f(x)\eta(x) \quad \textit{for } \mu \textit{ a.a. } x \in B \tag{4.4}$$

(we recall that for every x, $f(x) \in \mathscr{L}(N, E)$, $\eta(x) \in N$ and then $f(x)\eta(x)$ is an element of E).

Proof. Set $B = \{x : f(x) \neq 0\}$: it is enough to show that (4.3) holds for μ almost all $x \in B$. Set $\mu' = \mu \llcorner B$ and notice that $\mu' \ll Du$ and $\bigl[dDu/d\mu'\bigr](x) = f(x)$ for μ' almost all x. By Proposition 4.4(v), for μ almost all x the the dimension of the kernel of $f(x)$ is $n-1$, i.e. the rank of $f(x)$ is one, and $\eta(x)$ is orthogonal to $\mathrm{Ker}\bigl[f(x)\bigr]$ and then (4.3) follows from Remark 1.7. □

Suppose that μ is a finite positive measure on N which is singular with respect to $\mathscr{H}^{n-1} \llcorner N$, take $R\mu$ and $S\mu$ as in Proposition 4.3 and let η be an orientation of $R\mu$. Notice that when u is a function in $BV(N,E)$, we may write $Du = f \cdot \mu + \theta$ as in Theorem 4.5 and then $f(x) = 0$ for $S\mu$ almost all x because $Du \perp S\mu$ (statement (vi) of Proposition 4.4) while formula (4.3) holds for $R\mu$ almost all x. Hence we have the following corollary of Theorem 4.5.

Corollary 4.6. (Rank One Property of Derivatives). *Let μ be a positive measure on N which is singular with respect to $\mathscr{H}^n \llcorner N$. Let E be a finite dimensional Banach space and let u be a function in $BV(N,E)$. Then*

$$\bigl[dDu/d\mu\bigr](x) \quad \text{has rank 1 or 0 for } \mu \text{ a.a. } x \in N.$$

In particular, if λ is the singular part Du with respect to $\mathscr{H}^n \llcorner N$, $\bigl[d\lambda/d|\lambda|\bigr](x)$ has rank 1 for $|\lambda|$ a.a. $x \in N$, i.e. the singular part of a derivative has rank one.

Remark 4.7. Of course Theorem 4.5 and Corollary 4.6 hold, with suitable modifications, even if we take u in $BV_{\mathrm{loc}}(\Omega,E)$, where Ω is an open subset of N, instead of $BV(N,E)$.

Remark 4.8. Theorem 4.5 admits the following converse (cf. Theorem 2.12): when μ is a rectifiable measure on N, η is an orientation of μ, E is a finite dimensional Banach space and $e \in L^1(\mu,E)$, then there exist a function $u \in BV(N,E)$ and a measure $\theta \in \mathscr{M}\bigl(N, \mathscr{L}(N,E)\bigr)$ such that $\theta \perp \mu$ and $Du = e\,\eta^* \cdot \mu + \theta$. This fact may be proved as a corollary of Theorem 2.12 or by a straightforward generalization of the proof of Theorem 2.12.

Remark 4.9. (The Case $N = \mathbb{R}^n$ and $E = \mathbb{R}^m$). When $N = \mathbb{R}^n$ and $E = \mathbb{R}^m$, $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ is usually identified with the space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices. In particular, when Ω is an open subset of $\mathbb{R}^n$ and u is a function in $BV(\Omega,\mathbb{R}^m)$, Du is the measure in $\mathscr{M}\bigl(\Omega, \mathbb{R}^{m\times n}\bigr)$ given by $(Du)_{i,j} = \partial u_i/\partial x_j$ for every i and j with $1 \leq i \leq m$, $1 \leq j \leq n$.

In this case Theorem 4.5 may be written in the following form: when μ is a rectifiable measure on $\mathbb{R}^n$, η is an orientation of μ and u is a function in $BV(\Omega,\mathbb{R}^m)$, we may write $Du = f \cdot \mu + \theta$ where f is a function which takes values in $\mathbb{R}^{m\times n}$ and $\theta \perp \mu$ and for μ almost all x, $f(x) = e(x) \otimes \eta(x)$ and this means that $f_{i,j}(x) = e_i(x)\,\eta_j(x)$ for all i,j, where $e_i(x) = \sum_j f_{i,j}(x)\,\eta_j(x)$ for all i.

Eventually we want to show what happens when we consider higher order derivatives. In order to do this, we need some preliminary definitions. As usual, let Ω be an open subset of N, E a finite dimensional Banach space and k a positive integer.

Definition 4.10. (Symmetric k–linear Applications). We denote by $\mathrm{Sym}_k(N,E)$ the Banach space of all E valued symmetric k–linear applications on N, i.e. the space of all maps $\omega : N^k \to E$ which are separately linear with respect to every variable and satisfy

$$\omega\big(y_1, \ldots, y_k\big) = \omega\big(y_{\sigma(1)}, \ldots, y_{\sigma(k)}\big)$$

for every $\big(y_1, \ldots, y_k\big) \in N^k$ and every permutation σ of the indices. Then $\mathrm{Sym}_1(N, E) = \mathscr{L}(N, E)$. Moreover we set $\mathrm{Sym}_0(N, E) = E$. We denote by $\mathbf{y} = \big(y_1, \ldots, y_k\big)$ a generic element of N^k and sometimes we write $\omega\,\mathbf{y}$ or $\omega \cdot \mathbf{y}$ instead of $\omega(\mathbf{y})$. The norm of an element ω in $\mathrm{Sym}_k(N, E)$ is the supremum of all $|\omega\,\mathbf{y}|$ where $\mathbf{y}$ is taken so that $|y_i| \leq 1$ for $i = 1, \ldots, k$.

When $k > 0$, there exists a standard immersion of $\mathrm{Sym}_k(N, E)$ into the space $\mathscr{L}\big(N, \mathrm{Sym}_{k-1}(N, E)\big)$: for every $\omega \in \mathrm{Sym}_k(N, E)$ we may consider the element $I\omega$ in $\mathscr{L}\big(N, \mathrm{Sym}_{k-1}(N, E)\big)$ which is given by, for all $y \in N$,

$$[I\omega]y : \big(y_1, \ldots, y_{k-1}\big) \longmapsto \omega\big(y_1, \ldots, y_{k-1}, y\big) \qquad \text{for all } \big(y_1, \ldots, y_{k-1}\big) \in N^{k-1}. \tag{4.5}$$

I is an injective linear application of $\mathrm{Sym}_k(N, E)$ into $\mathscr{L}\big(N, \mathrm{Sym}_{k-1}(N, E)\big)$ and the rank of a symmetric k–linear application ω is the rank of the the linear map $I\omega$.

Remark 4.11. (Symmetric k–linear Applications with Rank One). When $p \in N^*$ and $e \in E$, we denote by $e \cdot \underbrace{p \otimes \ldots \otimes p}_{k \text{ times}}$ the element of $\mathrm{Sym}_k(N, E)$ given by

$$\mathbf{y} \mapsto \Big[\prod_{h=1}^{k} p(y_h)\Big] e \qquad \text{for all } \mathbf{y} \in N^k.$$

When $p \neq 0$ and $e \neq 0$, the rank of this symmetric k–linear application is 1. On the contrary, every rank one element of $\mathrm{Sym}_k(N, E)$ may be written in this form for suitable p and e.

Indeed, suppose that ω is a rank one element of $\mathrm{Sym}_k(N, E)$. By Remark 1.7 there exist $\eta \in N$ with $|\eta| = 1$ and $\omega' \in \mathrm{Sym}_{k-1}(N, E)$ such that $[I\omega]y = \langle \eta; y\rangle\, \omega'$ for every $y \in N$ and then

$$\omega\,\mathbf{y} = \omega'\big(y_1, \ldots, y_{k-1}\big) \cdot \langle \eta; y_k\rangle \qquad \text{for every } \mathbf{y} \in N^k. \tag{4.6}$$

Let A be the open set of all $\mathbf{y} \in N^k$ such that $\langle \eta; y_h\rangle \neq 0$ for $h = 1, \ldots, k$ and let $\psi : A \to E$ be the function given by

$$\psi(\mathbf{y}) = \frac{\omega\mathbf{y}}{\prod_{h=1}^{k} \langle \eta; y_h\rangle} \qquad \text{for all } \mathbf{y} \in A.$$

Taking into account equality (4.6), we obtain

$$\psi(\mathbf{y}) = \frac{\omega'\big(y_1, \ldots, y_{k-1}\big)}{\prod_{h=1}^{k-1} \langle \eta; y_h\rangle} \qquad \text{for all } \mathbf{y} \in A$$

and then ψ is a function on A which does not depend on the k-th variable and admits a continuous extension to the set of all $\mathbf{y} \in N^k$ such that $\langle \eta; y_h\rangle \neq 0$ for $h = 1, \ldots, k-1$.

Taking into account that ψ is a symmetric function, we obtain that it is locally constant on A and admits a continuous extension to the set $N^k \setminus C$ where C is the closed set of all $\mathbf{y}$ such that $\langle \eta; y_h \rangle = 0$ for at least two different integers h.

Since the dimension of N^k is kn and C is a finite union of manifolds with dimension $nk-2$, $N^k \setminus C$ is a connected open set and then ψ is constant and in particular $\psi(\mathbf{y}) = \psi(\eta, \dots, \eta) = \omega(\eta, \dots, \eta)$ for every $\mathbf{y} \in A$. Since A is dense in N^k, we have that

$$\omega \mathbf{y} = \omega(\eta, \dots, \eta) \prod_{h=1}^{k} \langle \eta; y_h \rangle \quad \text{for every } \mathbf{y} \in N^k \tag{4.7}$$

and this means that $\omega = e \cdot \underbrace{\eta^* \otimes \ldots \otimes \eta^*}_{k \text{ times}}$ where $e = \omega(\eta, \dots, \eta)$.

Definition 4.12. (Derivatives of k-th Order). Let Ω be an open subset of N. When u is a function of Ω into E of class C^k and x is a point of Ω, $D^k u(x)$ denotes the element of $\mathrm{Sym}_k(N,E)$ given by

$$\big[D^k u(x)\big] : \mathbf{y} \longmapsto \frac{\partial^k u}{\partial y_1 \dots \partial y_k}(x) \quad \text{for all } \mathbf{y} \in N^k.$$

The usual recursive formula $D^k u = D\big(D^{k-1} u\big)$ becomes

$$I\big[D^k u(x)\big] = \big[D(D^{k-1}u)\big](x) \quad \text{for every } x.$$

We say that a function $u \in L^1(\mathscr{H}^n \llcorner \Omega, E)$ belongs to $BV^k(\Omega, E)$ when its k-th derivative is (represented by) a measure in $\mathscr{M}\big(\Omega, \mathrm{Sym}_k(N,E)\big)$ and this means that

$$\begin{aligned} \int_\Omega \big[D^k \phi(x) \cdot \mathbf{y}\big]\, u(x)\, d\mathscr{H}^n(x) =& (-1)^k \int_\Omega \phi(x)\, d\big[D^k u \cdot \mathbf{y}\big](x) \\ & \text{for all } \mathbf{y} \in N^k \text{ and all } \phi \in C^\infty_C(\Omega), \end{aligned} \tag{4.8}$$

where the measure $\big[D^k u \cdot \mathbf{y}\big]$ is defined by $\big[D^k u \cdot \mathbf{y}\big](B) = \big[D^k u(B)\big] \cdot \mathbf{y}$ for all Borel sets B. It may be proved that when u is a function in $BV^k(\Omega, E)$, $D^h u$ is (represented by) a function in $L^1_{\mathrm{loc}}\big(\Omega, \mathrm{Sym}_h(N,E)\big)$ for every h with $1 \le h \le k_1$.

The following generalizations of Theorem 4.5 holds.

Theorem 4.13. *Let k be a positive integer, let μ be a rectifiable measure on N and η an orientation of μ. Let E be a finite dimensional Banach space and u a function in $BV^k(N,E)$. As usual we write $D^k u = f \cdot \mu \llcorner N + \theta$ where $f \in L^1\big(\mu, \mathrm{Sym}_k(N,E)\big)$, $\theta \in \mathscr{M}\big(N, \mathrm{Sym}_k(N,E)\big)$ and $\theta \perp \mu$.*

Then, for μ almost all x, $f(x)$ is an E valued symmetric k–linear application on N of the form $f(x) = e(x)\, \underbrace{\eta(x)^* \otimes \ldots \otimes \eta(x)^*}_{k \text{ times}}$, i.e.

$$\big[f(x)\big] : \mathbf{y} \longmapsto e(x) \prod_{h=1}^{k} \langle \eta(x); y_h \rangle \quad \textit{for every } \mathbf{y} \in N^k, \tag{4.9}$$

where e is the function in $L^1(\mu, E)$ *given by*

$$e(x) = f(x) \cdot \big(\underbrace{\eta(x), \dots, \eta(x)}_{k \text{ times}}\big) \quad \text{for } \mu \text{ a.a. } x \in B \tag{4.10}$$

(we recall that for every x, $f(x) \in \mathrm{Sym}_k(N, E)$, $\eta(x) \in N$, *and then* $f(x) \cdot \big(\eta(x), \dots, \eta(x)\big)$ *is an element of* E*).*

Proof. Notice that $v = D^{k-1}u$ is a function in $BV_{\mathrm{loc}}\big(N, \mathrm{Sym}_{k-1}(N, E)\big)$ and $\big[Dv(B)\big] = I\big[D^k u(B)\big]$ for all Borel sets B. Then it is enough to apply Theorem 4.5 to the function $v = D^{k-1}u$ and take into account Remark 4.11. □

As for Theorem 4.5 we have the following straightforward corollary.

COROLLARY 4.14. (Rank One Property of Higher Order Derivatives). *Let* μ *be a positive measure on* N *which is singular with respect to* $\mathscr{H}^n \llcorner N$ *and let* k *be a positive integer. Let* E *be a finite dimensional Banach space and let* u *be a function in* $BV^k(N, E)$*. Then*

$$\big[dD^k u/d\mu\big](x) \quad \text{has rank 1 or 0 for } \mu \text{ a.a. } x \in N.$$

In particular, when λ *is the singular part* $D^k u$ *with respect to* $\mathscr{H}^n \llcorner N$*, we obtain that* $\big[d\lambda/d|\lambda|\big](x)$ *has rank 1 for* $|\lambda|$ *a.a.* $x \in N$*, i.e. the singular part of a* k*-th order derivative has rank one.*

Remark 4.15. Of course Theorem 4.13 and Corollary 4.14 hold, with suitable modifications, even if we take u in $BV^k_{\mathrm{loc}}(\Omega, E)$, where Ω is an open subset of N, instead of $BV^k(N, E)$.

Remark 4.16. (The Case $N = \mathbb{R}^n$, $E = \mathbb{R}$ and $k = 2$). When $N = \mathbb{R}^n$ and $E = \mathbb{R}$ and $k = 2$, $\mathrm{Sym}_2(\mathbb{R}^n, \mathbb{R})$ is usually identified with the space $S(n)$ of all symmetric $n \times n$ matrices. In particular, when Ω is an open subset of $\mathbb{R}^n$ and u is a function in $BV^2(\Omega, \mathbb{R})$, Du is the measure in $\mathscr{M}\big(\Omega, S(n)\big)$ given by $(Du)_{i,j} = \partial^2 u/\partial x_i \partial x_j$ for every i and j with $1 \le i, j \le n$.

In this case Theorem 4.13 may be written in the following form: when μ is a rectifiable measure on $\mathbb{R}^n$, η is an orientation of μ and u is a function in $BV^2(\Omega, \mathbb{R})$, we may write $Du = f \cdot \mu + \theta$ where f is a function which takes values in $S(n)$ and $\theta \perp \mu$ and for μ almost all x, $f(x) = c(x) \cdot \eta(x) \otimes \eta(x)$ and this means that $f_{i,j}(x) = c(x)\, \eta_i(x)\, \eta_j(x)$ for all i, j, where $c(x) = \sum_{i,j} f_{i,j}(x)\, \eta_i(x)\, \eta_j(x)$. In particular we have that the rank of the matrix $f(x)$ is 1 or 0 for μ almost all x.

A similar result has just been proved in some particular cases by Aviles and Giga in [6]: suppose that μ is a positive singular measure on the open set $\Omega \subset \mathbb{R}^n$ and that there exists $q < n$ so that, for μ almost every x,

$$0 < \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^q} < \infty.$$

Then, when u is a function in $BV^2(\Omega, \mathbb{R})$, the rank of the matrix $\big[dD^2 u/d\mu\big](x)$ is 1 or 0 for μ almost all x.

The basic idea of their method is to study the blow–up of the function u in every point x and the density hypothesis on μ seems to be essential in the proof of the result.

Remark 4.17. (The Case $N = \mathbb{R}^n$ and $E = \mathbb{R}$). Suppose that $N = \mathbb{R}^n$ and $E = \mathbb{R}$. For every $\omega \in \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R})$, the function $[J\omega] : y \mapsto \omega(y, \dots, y)$ is a function on N which is an homogeneous polynomial with degree k and J turns out to be a one-to-one linear application of $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R})$ into the linear space $P(n, k)$ of all homogeneous polynomials of n variables with degree k and then $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R})$ is sometimes identified with $P(n, k)$.

Taking into account Remark 4.11 we have that the rank of $p \in P(n, k)$ is 1 (that is, $J^{-1}p$ is a rank one element of $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R})$) if and only if there exists $c \in \mathbb{R}$ and $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ such that $p(y) = c\langle \eta; y\rangle^k = c\big(\sum_i \eta_i y_i\big)^k$ for every $y \in \mathbb{R}^n$.

In particular we have that Theorem 4.13 may be stated as follows: when μ is a rectifiable measure on $\mathbb{R}^n$, η is an orientation of μ and u is a function in $BV^k(\Omega, \mathbb{R})$, we may write $Du = f \cdot \mu + \theta$ where f is a function which takes values in $P(n, k)$ and $\theta \perp \mu$, and for μ almost all x, $f(x)$ is a polynomial of the form $y \mapsto c(x)\big(\sum_i \eta_i(x)\, y_i\big)^k$.

Remark 4.18. (Open Questions). Eventually, we want to state two open questions which seem to be interesting.

Statement (iv) of Proposition 4.4 says that every measure which is concentrated in a purely unrectifiable set is purely unrectifiable. Then it is natural to consider the opposite problem: when μ is a purely unrectifiable measure, is it concentrated on a purely unrectifiable set? In other words, when μ is a singular measure such that $\mu(T) = 0$ for every purely unrectifiable set T, is μ rectifiable?

The second problem is the following: when k is a positive integer greater than 1, we may consider the class of all positive singular measure μ such that there exists a function $u \in BV^k(N)$ and $\mu \ll D^k u$ (cf. Proposition 4.2). Of course, any measure in this class is rectifiable and we conjecture that every rectifiable measure belongs to this class but we are not able to prove it.

This problem is somehow connected with the following: when S is an $(n-1)$–dimensional submanifold of N of class C^1, is there a function $u \in BV^k(N)$ such that $\mathscr{H}^{n-1} \llcorner S \ll D^k u$? The answer is positive when S is a manifold of class C^k but it is not clear whether the same holds when S is a manifold of class C^h with $1 \le h < k$.

5. Proof of Technical Results

Statements (i), (ii), (iii) and (iv) of Proposition 1.3 are trivial and we omit to prove them.

Proof of Proposition 1.3(v). Let Y be a closed convex cone in E of the form $Y = X(e, a)$ (cf. (1.1)) and suppose that $Y \cap X = \{0\}$. Then there exists a real number $b < 1$ such that

$$\langle y; x\rangle \le b\,|x|\,|y| \quad \text{for every } y \in Y,\ x \in X. \tag{5.1}$$

For every $t \in M$, let S_t be the set of all points x such that

$$\left[\frac{d\psi_t}{d|\psi_t|}\right](x) \in Y \tag{5.2}$$

and set $\psi_t^1 = \psi_t \llcorner S_t$, $\psi_t^2 = \psi_t \llcorner \Omega \setminus S_t$, $\psi^1 = \int \psi_t^1 d\lambda(t)$ and $\psi^2 = \int \psi_t^2 d\lambda(t)$ (see Definition 1.2, it may be verified that both $t \mapsto \psi_t^1$ and $t \mapsto \psi_t^2$ are λ measurable function of M into $\mathscr{M}(\Omega, E)$).

Then $\psi_t^1 \in \mathscr{M}(\Omega, Y)$ for every t by (5.2) and Proposition 1.3(i), $\psi^1 \in \mathscr{M}(\Omega, Y)$ by Proposition 1.3(iv) and $\psi = \psi^1 + \psi^2$. Hence, for every Borel set B such that $|\psi(B)| > 0$ we have $\psi(B) = \psi^1(B) + \psi^2(B)$ and taking into account (5.1),

$$\bigl|\psi(B)\bigr| = \frac{\bigl\langle \psi(B); \psi(B)\bigr\rangle}{\bigl|\psi(B)\bigr|} = \frac{\bigl\langle \psi^1(B); \psi(B)\bigr\rangle + \bigl\langle \psi^2(B); \psi(B)\bigr\rangle}{\bigl|\psi(B)\bigr|} \leq b\bigl|\psi^1(B)\bigr| + \bigl|\psi^2(B)\bigr|.$$

It follows immediately from the definition of total variation of a measure that $|\psi| \leq b|\psi^1| + |\psi^2|$ and in particular

$$\|\psi\| \leq b\|\psi^1\| + \|\psi^2\|. \tag{5.3}$$

Moreover, by hypothesis we have that

$$\|\psi\| = \int_M \|\psi_t\| \, d\lambda(t) = \int_M \bigl(\|\psi_t^1\| + \|\psi_t^2\|\bigr) \, d\lambda(t) \geq \|\psi^1\| + \|\psi^2\|. \tag{5.4}$$

(5.3) and (5.4) yield $b\|\psi^1\| \geq \|\psi^1\|$ and then $\|\psi^1\| = 0$ because $b < 1$. Since $Y = X(e, a)$ and $\psi_t^1 \in \mathscr{M}(\Omega, Y)$ for every t,

$$0 = |\psi^1| \geq \langle e; \psi^1\rangle = \int_M \langle e; \psi_t^1\rangle \, d\lambda(t) \geq a \int_M |\psi_t^1| \, d\lambda(t) \tag{5.5}$$

(when $\psi \in \mathscr{M}(\Omega, E)$ and $e \in E$, $\langle e; \psi\rangle$ is the real measure given by $\langle e; \psi\rangle(B) = \langle e; \psi(B)\rangle$ for all Borel sets B). Then $\psi_t^1 = 0$ for λ a.a. t and this means that for λ a.a. t, $\bigl[d\psi_t/d|\psi_t|\bigr](x) \notin Y$ for $|\psi_t|$ a.a. x. Since this fact holds for every closed convex cone Y of the form $X^*(e, a)$ such that $Y \cap X = \{0\}$ and $N^* \setminus X$ may be covered by countably many cones of this kind, Proposition 1.3(v) is proved. □

Proposition 1.5 may be found in [9], nos. 70 to 74. Proposition 1.9 is a straightforward generalization of well–known results about the traces of BV (cf. [10], section 5.4, or [14], Remark 2.13, or [17], section 5.10). Proposition 1.10 follows from Proposition 1.5 (see for instance [2], Theorem 3.3). Statement (i) of Theorem 1.12 is standard while statement (ii) (cf. [2], formulas (1.7) and (1.8)) is a straightforward generalization of the usual coarea formula for BV functions (see for instance [10], section 5.5, or [14], Theorem 1.23, or [17], Theorem 5.4.4).

In order to prove the statements in Remark 1.14, we need a preliminary lemma.

Lemma 5.1. *Let μ be a finite positive measure in N and let B be a subset of N. Suppose that B has μ density one in everyone of its points, and this means that*

$$\lim_{r\to 0} \frac{\mu\bigl(B(t,r) \cap B_*\bigr)}{\mu\bigl(B(t,r)\bigr)} = 1 \qquad \textit{for every } t \in B, \tag{5.6}$$

where B_* be a μ maximal Borel set included in B (i.e. $\mu\big(C \setminus B_*\big) = 0$ for every Borel set $C \subset B$). Then B is μ measurable.

Proof. With no loss in generality we may suppose that the support of μ is N. For every $r > 0$ set

$$u_r(t) = \frac{\mu\big(B(t,r) \cap B_*\big)}{\mu\big(B(t,r)\big)} \quad \text{for every } t \in N.$$

Each u_r is a Borel function of N and well–known theorems about densities ensure that u_r converge to the characteristic function of the set B_* for μ almost all x. In particular, if B_1 is the Borel set of all x such that $\lim u_r(x) = 1$, we have that $\mu(B_1 \setminus B_*) = 0$ and taking into account (5.6), $B_1 \supset B \supset B_*$. Hence B is μ measurable. □

Proof of statements in Remark 1.14. For the equivalence of statements (a), (b) and (c) see for instance [8], section II.1 and chapter III). The fact that every μ approximately lower semicontinuous function of N into $\mathscr{F}(X)$ is μ measurable immediately follows from (1.11), Lemma 5.1 and statement (a) of Remark 1.14. □

Theorem 1.15 is a straightforward corollary of Theorem III.8 in [8]. The proof of Propositions 2.6 is standard and we omit it.

Proof of Proposition 2.11. We want to show that $x \mapsto E(\mu, x)$ is a μ approximately lower semicontinuous function of N into $\mathscr{F}(N^*)$ and this means that for every open set $A \subset N^*$, the set $S(A)$ of all points $x \in N$ such that $E(\mu,x) \cap A \neq 0$ has μ density 1 in everyone of its points (see (1.11)).

If $0 \in A$, we have that $0 \in E(\mu,x) \cap A$ for every x and then $S(A) = N$. Hence we may suppose that $0 \notin A$. Let x be a fixed point of $S(A)$. Then there exists $v \in A$ and $u \in BV(N)$ such that Du is tangent to $v \cdot \mu$ in x (see Definition 2.3). Write $Du = f \cdot \mu + \theta$ with $\theta \perp \mu$ and let B be the Borel set of all points t such that $f(t) \in A$. Remark 2.2 shows that

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \big|f(t) - v\big| \, d\mu(t) = 0 \tag{5.7}$$

and that Du is tangent to $f(t) \cdot \mu$ for μ almost all t with $f(t) \neq 0$ and this means that $f(t)$ belongs to $E(\mu,x) \cap A$ for μ almost all $x \in B$ and then $B \subset S(A)$. Take $\varepsilon > 0$ such that $B(v,\varepsilon) \subset A$, then $\varepsilon^{-1}\big|f(t) - v\big| \geq 1$ for every $t \in N \setminus B$ and (5.7) yields

$$\limsup_{r \to 0} \frac{\mu\big(B(x,r) \setminus B\big)}{\mu\big(B(x,r)\big)} \leq \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \varepsilon^{-1}\big|f(t) - v\big| \, d\mu(t) = 0$$

and this means that B has μ density 1 in x and then $S(A)$ has μ density 1 in x. □

To prove Theorem 2.12 we need some preliminary lemmas.

Remark 5.2. Let Ω be an open subset of N. We denote by $BV\big(\overline{\Omega}\big)$ the subspace of all functions $u \in BV(N)$ with support included in $\overline{\Omega}$ (i.e. all functions u which take value 0 $\mathscr{H}^n$ a.e. in $\mathbb{N} \setminus \overline{\Omega}$). Poincaré inequalities (see [10], section 5.6.1, or

[17], section 5.11) yield a constant C_1, which depends on the dimension of N only, such that

$$\|u\|_1 \le C_1 r \|Du\| \tag{5.8}$$

whenever $u \in BV(N)$ and the support of u is included in a ball with radius r. Hence, when Ω is bounded, $\|Du\|$ is a norm in $BV\big(\overline{\Omega}\big)$ which is equivalent to the usual BV norm $\|u\|_1 + \|Du\|$.

Let B be an open ball of N and let η be the inner normal of ∂B. For every real function u of B, let Tu be the real function of N defined by $\big[Tu\big](x) = u(x)$ when $x \in B$ and $\big[Tu\big](x) = 0$ when $x \notin B$. $Tu \in BV(N)$ whenever $u \in BV(B)$, T is a continuous linear operator of $BV(B)$ into $BV(N)$ (more precisely, into $BV\big(\overline{B}\big)$), and for every $u \in BV(B)$

$$D[Tu] = Du + u^+ \eta \cdot \mathscr{H}^{n-1} \llcorner \partial B \tag{5.9}$$

where u^+ is the (inner) trace of u on ∂B (see Proposition 1.9). As T is continuous, Poincaré inequalities yield a constant C_2 such that

$$\big\|D[Tu]\big\| \le C_2 \|Du\| \quad \text{for all } u \in BV(B) \text{ with mean value 0.} \tag{5.10}$$

A simple homogeneity argument shows that the best constant in (5.10) does not depends on the choice of B and this means that C_2 may be taken in (5.10) which depends on the dimension of N only. In particular, when u is a function in $BV(N)$, (5.9) and (5.10) yield

$$D[u \llcorner B] \;= Du \llcorner B + t\,\eta \cdot \mathscr{H}^{n-1} \llcorner \partial B \tag{5.11}$$

$$\big\|D[u \llcorner B]\big\| \le C_2 |Du|(B) \quad \text{for all } u \in BV(N) \text{ with mean value 0 on } B. \tag{5.12}$$

Lemma 5.6. *Suppose that Ω is a bounded open subset of N. Let ϕ be a positive real measure on N and let $\psi \in \mathscr{D}(N)$ be a measure whose support is included in Ω. Then there exist $u \in BV\big(\overline{\Omega}\big)$ and $\lambda \in \mathscr{M}(N, N^*)$ such that*

(i) *$Du - \lambda \perp \lambda$ and $Du - \lambda \perp \phi$,*

(ii) *$\|\lambda - \psi\| \le \frac{1}{2}\|\psi\|$,*

(iii) *$\|Du\| \le 2C_2\|\psi\|$ where C_2 is the same constant as in* (5.12).

Proof. We denote by f the function $\big[d\psi/d|\psi|\big]$.

Let E be the (Borel) set of all points $x \in \operatorname{supp}\psi$ such that $f(x)\cdot|\psi|$ is tangent to ψ in x and $f(x)$ belongs to $E\big(|\psi|, x\big)$. Then there exists a function $u_x \in BV(N)$ the derivative of which is tangent to $f(x)\cdot|\psi|$, and then to ψ, in x. Using the Radon–Nikodym theorem, the fact that $f(x) \in E\big(|\psi|, x\big)$ for $|\psi|$ almost all x, and Proposition 2.6, we obtain that $|\psi|\big(\Omega \setminus E\big) = 0$.

Let $\mathscr{F}$ be the collection of all balls $B(x,r) \subset \Omega$ such that $x \in E$,

$$\big|Du_x - \psi\big|(B(x,r)) \le \frac{1}{2}|\psi|(B(x,r)) \tag{5.13}$$

and $\phi\big(\partial B(x,r)\big) = 0$. Verify that for all $x \in E$ there exist balls $B(x,r)$ in $\mathscr{F}$ with arbitrary small radius r and then we may apply a well–known corollary of

Besicovitch lemma (see [10], section 1.5, or [16], Lemma 4.6, or [17], section 1.3) to obtain pairwise disjoint balls $B_n = B(x_n, r_n) \in \mathscr{F}$ for $n = 1, 2, \dots$ such that

$$|\psi|\Big(\Omega \setminus \bigcup_n B_n\Big) = 0. \tag{5.14}$$

For all n, set $u_n = u_{x_n}$, let a_n be then mean value of u_n in B_n, η_n the inner normal of the sphere ∂B_n, t_n the inner trace of u_n on ∂B_n.

Taking into account (5.12), (5.13), (5.14) and recalling that the balls B_n are pairwise disjoint,

$$\sum_1^\infty \big\| D\big[(u_n - a_n) \llcorner B_n\big]\big\| \le \sum_1^\infty C_2 |Du_n|(B_n) \le \sum_1^\infty 2C_2 |\psi|(B_n) = 2C_2\|\psi\|. \tag{5.15}$$

Finally set $u = \sum (u_n - a_n) \llcorner B_n$ and notice that the series converges in the norm of $BV\big(\overline{\Omega}\big)$ by (5.15) and u is well–defined. Then (5.11) yields

$$Du = \sum_1^\infty \Big[Du_n \llcorner B_n\Big] + \sum_1^\infty \Big[(t_n - a_n)\eta_n \cdot \mathscr{H}^{n-1} \llcorner \partial B_n\Big]. \tag{5.16}$$

Notice that both series converge in norm to mutually singular measures. Then $\lambda = \sum \big[Du_n \llcorner B_n\big]$ is well–defined, $Du - \lambda \perp \lambda$ by construction, $Du - \lambda \perp \phi$ because $Du - \lambda$ is null out of the union of all ∂B_n and we have chosen the balls B_n so that $\phi\big(\partial B_n\big) = 0$ for all n. Hence (i) is proved and (iii) follows from (5.15). Eventually (5.13) and (5.14) yield

$$\|\lambda - \psi\| = \sum_1^\infty |Du_n - \psi|(B_n) \le \sum_1^\infty \frac{1}{2}|\psi|(B_n) = \frac{1}{2}\|\psi\|$$

and (ii) is proved. □

Remark 5.4. Regarding Lemma 5.3, notice that $Du - \lambda \perp \lambda$ and 2.6(v) yields that both λ and $Du - \lambda$ belong to $\mathscr{D}(N)$. Since $\psi \in \mathscr{D}(N)$ by hypothesis, $\psi - \lambda$ belongs to $\mathscr{D}(N)$ by Proposition 2.6(vi).

Lemma 5.5. *Suppose that Ω is a bounded open subset of N. Let ϕ be a positive real measure on N and let ψ be a measure in $\mathscr{D}(N)$ with support included in Ω. Then there exists a function $u \in BV\big(\overline{\Omega}\big)$ such that $Du - \psi \perp \phi$ and $\|Du\| \le 4C_2\|\psi\|$ where C_2 is the same constant as in* (5.12).

Proof. Set $\psi_0 = \psi$, $u_0 = 0$ and $\lambda_0 = 0$. Taking into account Lemma 5.3 and Remark 5.4, for all integers $n > 0$ we may define by induction on n functions $u_n \in BV\big(\overline{\Omega}\big)$ and measures $\psi_n, \lambda_n \in \mathscr{M}(N, N^*)$ such that

(a) $\psi_n = \psi_{n-1} - \lambda_{n-1}$,

(b) $\psi_n \in \mathscr{D}(N)$,

(c) $Du_n - \lambda_n \perp \lambda_n$, $Du_n - \lambda_n \perp \phi$ and $Du_n - \lambda_n, \lambda_n \in \mathscr{D}(N)$,

(d) $\|\lambda_n - \psi_n\| \le \frac{1}{2}\|\psi_n\|$,

(e) $\|Du_n\| \le 2C_2\|\psi_n\|$.

Notice that (a) and (d) yield $\psi_1 = \psi_0 = \psi$ and, for all $n > 1$, $\|\psi_n\| = \|\psi_{n-1} - \lambda_{n-1}\| \leq \frac{1}{2}\|\psi_{n-1}\|$. Hence, for all $n > 0$

$$\|\psi_n\| \leq 2^{1-n}\|\psi\| \tag{5.17}$$

and taking into account (e)

$$\sum_1^\infty \|Du_n\| \leq 2C_2 \sum_1^\infty \|\psi_n\| \leq 2C_2 \sum_1^\infty 2^{1-n}\|\psi\| = 4C_2\|\psi\|. \tag{5.18}$$

Finally set $u = \sum u_n$. By (5.18) the series $\sum u_n$ converges in the norm of $BV(\overline{\Omega})$ and $\|Du\| \leq 4C_2\|\psi\|$. Moreover (c) yields

$$\sum \|\lambda_n\| \leq \sum \|Du_n\| < \infty,$$

hence $\lambda = \sum \lambda_n$ is well–defined and $Du - \lambda = \sum Du_n - \lambda_n \perp \phi$.

The proof is complete if we show that $\psi = \lambda$. For all integers $m > 1$, (a) yields $\psi - \sum_1^{m-1} \lambda_n = \psi_m$ and then

$$\|\psi - \lambda\| \leq \Big\|\psi - \sum_1^{m-1} \lambda_n\Big\| + \sum_m^\infty \|\lambda_n\| = \|\psi_m\| + \sum_m^\infty \|\lambda_n\|.$$

The sequence $m \mapsto \|\psi_m\|$ converges to 0 by (5.17), the sequence $m \mapsto \sum_m^\infty \|\lambda_n\|$ converges to 0 because $\sum_n \|\lambda_n\| < \infty$ and then $\psi = \lambda$. □

Proof of Theorem 2.12. Let $\{E_n : n = 1, 2, \ldots\}$ be a sequence of pairwise disjoint Borel sets with diameter smaller than 1 which cover N. Each E_n is included in an open ball B_n with radius smaller than 1.

For every integer n set $f_n(x) = f(x)$ when $x \in E_n$ and $f_n(x) = 0$ when $x \notin E_n$ and apply Lemma 5.5 with $\psi_n = f_n \cdot \mu$ and $\phi = \mu$ to get $u_n \in BV(\overline{B}_n)$ such that $Du_n - \psi_n \perp \phi$ and $\|Du_n\| \leq 4C_2\|\psi_n\|$.

Set $\theta_n = Du_n - \psi_n$, notice that $Du_n = f_n \cdot \mu + \theta_n$, $\theta_n \perp \mu$ and, taking into account (5.8),

$$\|u_n\|_1 + \|Du_n\| \leq 4C_2(1 + C_1)\|f_n\|_{L^1(\mu)}.$$

Finally, set $u = \sum u_n$ and $\theta = \sum \theta_n$. Both u and θ are well-defined and satisfy the assertion of Theorem 2.12 with $C = 4C_2(1 + C_1)$. □

Regarding Theorem 3.5, statements (i) and (ii) are trivial and we omit to prove them. In order to prove statement (iii) we need some preliminary lemmas.

Lemma 5.6. *Let $X^*(e,a)$ be a closed convex cone in N^*. When u is a function in $L^\infty(\mathscr{H}^n \llcorner N)$ such that $u \in BV_{\mathrm{loc}}(N)$ and $Du \in \mathscr{M}_{\mathrm{loc}}(N, X^*(e,a))$, then*

$$(2C/a)\,\|u\|_\infty r^{n-1} \geq |Du|\big(B(0,r)\big) \quad \textit{for every } r > 0, \tag{5.19}$$

where C is the measure $\mathscr{H}^{n-1}$ of the unit ball in $\mathbb{R}^{n-1}$.

Proof. The proof is divided in two cases.

<u>*Case 1*</u>: u is a function of class C^∞. In this case $Du(x) \in X^*(e,a)$ for every $x \in N$ and

$$2\|u\|_\infty \geq \lim_{t\to+\infty} u(x+te) - \lim_{t\to-\infty} u(x+te) = \int_{\mathbb{R}} \frac{\partial u}{\partial e}(x+te)\, d\mathscr{H}^1(t). \tag{5.20}$$

Let M be the orthogonal complement of e, π_M the projection of N into M and $B = \{x \in N : |\pi_M(x)| < r\}$. Then $B \supset B(0,r)$ and taking into account (5.20) and the fact that $[\partial u/\partial e] \geq a|Du|$ in every point,

$$2\|u\|_\infty C r^{n-1} \geq \int_B \frac{\partial u}{\partial e}\, d\mathscr{H}^n \geq \int_{B(0,r)} \frac{\partial u}{\partial e}\, d\mathscr{H}^n \geq a \int_{B(0,r)} |Du|\, d\mathscr{H}^n. \tag{5.21}$$

<u>*Case 2*</u>: u is a function in $BV_{\rm loc}(N)$. For every $\varepsilon > 0$, let ρ_ε denote as usual some positive mollifiers of class C^∞ on N and set $u_\varepsilon = u * \rho_\varepsilon$. For every $\varepsilon > 0$, u_ε is a function of class C^∞, $\|u_\varepsilon\|_\infty \leq \|u\|_\infty$, $Du_\varepsilon(x) \in X^*(e,a)$ for every x and $Du_\varepsilon \cdot \mathscr{H}^n \llcorner N$ converges to Du in the weak* topology of $\mathscr{M}_{\rm loc}(N,N^*)$ as $\varepsilon \to 0$. Then, taking into account (5.21),

$$\begin{aligned}(2C/a)\,\|u\|_\infty r^{n-1} &\geq \liminf_{\varepsilon\to 0}\,(2C/a)\,\|u_\varepsilon\|_\infty r^{n-1}\\ &\geq \liminf_{\varepsilon\to 0} \int_{B(0,r)} |Du_\varepsilon| d\mathscr{H}^n \geq |Du|(B(0,r)). \qquad \square\end{aligned}$$

LEMMA 5.7. *Let $X^*(e,a)$ be a closed convex cone in N^* and let S be the set of all functions $u \in L^\infty(\mathscr{H}^n \llcorner N)$ such that $u \in BV_{\rm loc}(N)$, $\|u\|_\infty \leq 1$ and $Du \in \mathscr{M}_{\rm loc}(N, X^*(e,a))$ (cf. Definition 3.4).*

If S is endowed with the topology induced by the weak topology of $L^\infty(\mathscr{H}^n \llcorner N)$, then it is compact and separable, and moreover the derivative is a continuous linear application of S into $\mathscr{M}_{\rm loc}(N,N^*)$ (endowed with the weak* topology).*

Proof. Since S is bounded and L^∞ is the dual of a separable Banach space, it is enough to prove that when $\{u_n\}$ is a sequence of functions in S which weakly* converges to $u \in L^\infty$, then u belongs to S and Du_n weakly* converge to Du. Notice that $\|u\|_\infty \leq 1$ and taking into account (5.19)

$$(2C/a) r^{n-1} \geq |Du_n|(B(x,r)) \quad \text{for every } n \text{ and every } r > 0.$$

Hence u belongs to $BV_{\rm loc}(N)$, Du_n weakly* converges to Du and Du belongs to $\mathscr{M}_{\rm loc}(N, X^*(e,a))$ because $\mathscr{M}_{\rm loc}(N, X^*(e,a))$ is weak* closed (cf. statement (ii) of Proposition 1.3). $\square$

LEMMA 5.8. *Let Y_1 be the class of all positive finite measures on N and let $Y_2 = \mathscr{M}_{\rm loc}(N,N^*)$, both endowed with the w* topology. Let $T : Y_1 \times Y_2 \to [0,\infty[$ be the function which associate to each pair (μ,ψ) the total variation of the part of μ which is absolutely continuous with respect to ψ, i.e.*

$$T(\mu,\psi) = \int_N \frac{d\mu}{d|\psi|} d|\psi|.$$

Then T is a Borel function of $Y_1 \times Y_2$. Notice that in general $T(\mu, \psi) \le \|\mu\|$ and equality holds if and only μ is absolutely continuous with respect to ψ.

Proof. For every $\mu \in Y_1$ and every $\psi \in Y_2$, let $\mu \wedge |\psi|$ be the positive measure given by

$$\big[\mu \wedge |\psi|\big](B) = \int_B \Big[1 \wedge \frac{d\mu}{d|\psi|}(x)\Big]\, d|\psi|(x) \quad \text{for every Borel set } B. \tag{5.22}$$

One easily checks that $\mu \wedge |\psi| \le \mu, \psi$ and

$$\big\|\mu \wedge |\psi|\big\| = \inf\big\{\mu(B) + |\psi|(N \setminus B) : \ B \text{ is a Borel subset of } N\big\}. \tag{5.23}$$

Let $\mathscr{F}_1$ be the class of all functions f in $C_C(N)$ such that $0 \le f \le 1$ and $\mathscr{F}_2 = C_C(N, N)$, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be countable dense subsets of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Then (5.23) yields

$$\begin{aligned}\big\|\mu \wedge |\psi|\big\| &= \inf_{f \in \mathscr{F}_1} \Big[\int f\, d\mu + \int (1-f)\, d|\psi|\Big] \\ &= \inf_{f \in \mathscr{F}_1} \Big[\int f\, d\mu + \sup_{\substack{g \in \mathscr{F}_2 \\ |g| \le 1-f}} \int g\, d|\psi|\Big] = \inf_{f \in \mathscr{G}_1} \sup_{\substack{g \in \mathscr{G}_2 \\ |g| \le 1-f}} \Big[\int f\, d\mu + \int g\, d\psi\Big]\end{aligned} \tag{5.24}$$

Since both $\mu \mapsto \int f\, d\mu$ and $\psi \mapsto \int g\, d\psi$ are continuous linear functional on Y_1 and Y_2 respectively, (5.24) yields that $(\mu, \psi) \mapsto \big\|\mu \wedge |\psi|\big\|$ is a Borel function of $Y_1 \times Y_2$ (and then also $(\mu, \psi) \mapsto \big\|\mu \wedge |n \cdot \psi|\big\|$ is a Borel function for every integer n). Taking into account (5.22) we obtain

$$T(\mu, \psi) = \int_N \frac{d\mu}{d|\psi|} d|\psi| = \sup_n \int_N \Big[n \wedge \frac{d\mu}{d|\psi|}\Big] d|\psi| = \sup_n \big\|\mu \wedge |n \cdot \psi|\big\|$$

and then T is a Borel function of $Y_1 \times Y_2$. □

Proof of Theorem 3.5(iii). The proof of this statement is divided in two steps.

Step 1: Let S be taken as in Lemma 5.7. For λ almost every $t \in M$, μ_t belongs to $\mathscr{E}(N, X)$ and then Definition 3.4 yields a function $u_t \in S$ so that $\mu_t \ll Du_t$. I want to show that for λ almost all t, functions $u_t \in S$ may be chosen so that $\mu_t \ll Du_t$ and $t \mapsto u_t$ is a Borel function of M into S.

For every $t \in M$, let F_t be the set of all $u \in S$ such that $\mu_t \ll Du$, F_t is assumed not empty for λ almost all t. Let F the set of all pairs $(t, u) \in M \times S$ such that $t \in M$ and $u \in F_t$, let T be given as in Lemma 5.8 and notice that (t, u) belongs to F if and only if $T(\mu_t, Du) = \|\mu_t\|$.

Taking into account Lemmas 5.7 and 5.8 and the fact that $t \mapsto \mu_t$ may be taken Borel measurable, we have that

$$(t, u) \longmapsto T(\mu_t, Du) - \|\mu_t\|$$

is a Borel function of $M \times S$ and then F is a Borel subset of $M \times S$.

Since F_t is not empty for λ almost every t, M is locally compact and separable and S is a compact and separable, we may apply Aumann's measurable selection theorem (see [8], Theorem III.22) to find a Borel function $t \mapsto u_t$ such that $u_t \in F_t$ for λ almost all t.

Step 2: let c be a positive function in $L^1(\lambda)$ such that $\int c\, d\lambda = 1$, taking into account Lemma 5.6 and Proposition 1.3(iv), it may be verified that function u given by

$$u(x) = \int_M c(t) u_t(x)\, d\lambda(t) \quad \text{for } \mathscr{H}^n \text{ a.a. } x \in N \tag{5.25}$$

is well–defined, belongs to S and $Du = \int_M \big[c(t)\cdot Du_t\big]\, d\lambda(t)$ and then (cf. formula (5.5))

$$|Du| \geq \frac{\partial u}{\partial e} = \int \Big[c(t)\cdot \frac{\partial u_t}{\partial e}\Big]\, d\lambda(t) \geq a \int \big[c(t)\cdot |Du_t|\big]\, d\lambda(t) \tag{5.26}$$

In particular, for every Borel set B, $|Du|(B) = 0$ yields $|Du_t|(B) = 0$ for λ a.a. t because $c(t) > 0$ for λ a.a t, hence $\mu_t(B) = 0$ because $\mu_t \ll |Du_t|$ and eventually $\mu(B) = 0$. This shows that $\mu \ll Du$ and then $\mu \in \mathscr{E}\big(N, X^*(e,a)\big)$. □

LEMMA 5.9. *Let (S,η) be an oriented rectifiable set in N such that η belongs to the interior of the closed convex cone $X(e,a)$ for $\mathscr{H}^{n-1}$ a.a. $x \in S$. Then we may find countably many open sets A_n with Lipschitz boundary and inner normal η_n so that $\eta_n(x) \in X(e,a)$ for all n and $\mathscr{H}^{n-1}$ a.a. $x \in \partial A_n$ and the sets ∂A_n cover $\mathscr{H}^{n-1}$ almost all of S.*

Proof. To begin with, notice that it is enough to prove this lemma when S is a subset of a manifold M of class C^1 in N.

In this case we have that $\eta(x)$ is orthogonal to the tangent space of M in x for $\mathscr{H}^{n-1}$ a.a. $x \in S$ (see Definition 1.6).

Let x be a fixed point of S such that $\eta(x)$ is orthogonal to the tangent space of M in x and belongs to the interior of $X(e,a)$. Then there exists an open neighborhood B of x and a continuous orientation ν of $M \cap B$ such that ν belongs to $X(e,a)$ in every point of B. Hence η agrees with ν in $\mathscr{H}^{n-1}$ almost every point of $S \cap B$.

Then we may find $r > 0$ and an open set $D \subset B(x,r)$ so that $B(x,r) \subset B$ and $\partial D \cap B(x,r) = M \cap B(x,r)$. Then ν is the inner normal of D for every point of $B(x,r)$. Now we may choose a point t in $B(x,r) \setminus D$ so that x belongs to the open cone with vertex t given by

$$C = \Big\{y : \langle y-t; e\rangle \geq \sqrt{1-a^2}\, |y-t|\Big\}$$

and $\partial D \cap C$ is relatively compact in $B(x,r)$. Notice that C is an open set with Lipschitz boundary and the inner normal of C always belongs to $X(e,a)$. Finally we take $A = \big\{y \in C : y \notin B(x,r) \text{ or } y \in D\big\}$.

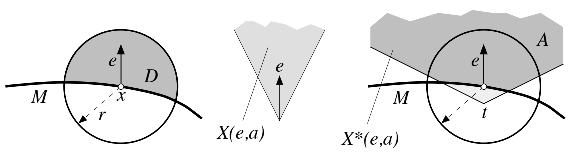

FIGURE 1

It may be verified that A is an open set with Lipschitz boundary and its inner normal belongs to $X(e,a)$ for $\mathscr{H}^{n-1}$ a.a. points of ∂A. Moreover ∂A covers a neighborhood of x in S and Lemma 5.9 is proved because S may be covered by countably many neighborhoods of this kind. □

Proof of Theorem 3.5(iv). Since $\eta(x)$ belongs to the interior of $X(e,a)$ for $\mathscr{H}^{n-1}$ a.a. $x \in S$, by Lemma 5.9 we may find countably many open sets A_n with Lipschitz boundary and inner normals η_n such that $\eta_n(x) \in X(e,a)$ for all n and $\mathscr{H}^{n-1}$ a.a. $x \in \partial A_n$, so that the sets ∂A_n cover $\mathscr{H}^{n-1}$ almost all of S. For all n, let u_n be the characteristic function of the set A_n. It is well known that each u_n belongs to $BV_{\mathrm{loc}}(N)$ and

$$Du_n = \eta_n^* \cdot \mathscr{H}^{n-1} \llcorner \partial A_n \tag{5.27}$$

(see for instance [14], Remark 2.13, or [17], Remark 5.8.3) and then $Du_n \in \mathscr{E}\big(N, X^*(e,a)\big)$. For every positive integer m, set

$$S_m = \big(S \cap \partial A_m\big) \setminus \Big(\bigcup_{n<m} \partial A_n\Big).$$

Then $\mathscr{H}^{n-1} \llcorner S_m \in \mathscr{E}\big(N, X^*(e,a)\big)$ for all m because it is absolutely continuous with respect to Du_n (see Definition 3.4) and $\mathscr{H}^{n-1} \llcorner S$ belongs to $\mathscr{E}\big(N, X^*(e,a)\big)$ because $\mathscr{H}^{n-1} \llcorner S = \sum_m \mathscr{H}^{n-1} \llcorner S_m$ (apply statement (i) of Theorem 3.5). □

Proof of Theorem 3.5(v). For every integer n, let X_n be the cone $X^*(e, a-1/n)$, B_n the set of all points x such that $E(\mu,x) \cap X_n \neq \{0\}$ and $E(\mu,x) \cap X_{n-1} = \{0\}$, and set $\mu_n = \mu \llcorner B_n$. Then the sets B_n are pairwise disjoint μ measurable sets which cover μ a.a. of N and then $\mu = \sum \mu_n$.

Let n be a fixed integer.

Set $C = \big\{v \in N^* : |v| = 1\big\}$. Since $E(\mu_n,x) \cap X_n = E(\mu,x) \cap X_n \neq \{0\}$ for μ_n a.a. x (cf. statement (iii) of Proposition 2.6), $E(\mu_n,x) \cap X_n \cap C \neq \emptyset$ for μ almost all x and by Proposition 2.11 and Remark 1.14 $x \mapsto E(\mu_n,x) \cap X_n \cap C$ is μ measurable.

Hence, by Theorem 1.15 we may find a μ measurable function $f : N \to N^*$ such that $f(x) \in E(\mu_n,x) \cap X_n$ and $|f(x)| = 1$ for μ_n a.a. x. By Theorem 2.12 there exists a function $u \in BV(N)$ such that $Du = f \cdot \mu_n + \theta$ where $\theta \in \mathscr{M}(N,N^*)$ and $\theta \perp \mu_n$ and then there exists a set B such that $\mu_n(N \setminus B) = 0$ and $|\theta|(B) = 0$.

Hence $f \cdot \mu_n = Du \llcorner B$ and by the coarea formula (equality (1.9) and (1.10)) we have that

$$f \cdot \mu_n = Du \llcorner B = \int_{\mathbb{R}} \big(Du_t \llcorner B\big)\, d\mathscr{H}^1(t) = \int_{\mathbb{R}} \eta_t^* \cdot \big(\mathscr{H}^{n-1} \llcorner S_t \cap B\big)\, d\mathscr{H}^1(t) \tag{5.28}$$

$$\mu_n = |Du \llcorner B| = \int_{\mathbb{R}} \big|Du_t \llcorner B\big|\, d\mathscr{H}^1(t) = \int_{\mathbb{R}} \big(\mathscr{H}^{n-1} \llcorner S_t \cap B\big)\, d\mathscr{H}^1(t) \tag{5.29}$$

Since $f \cdot \mu_n \in \mathscr{M}(N, X_n)$ and (5.28) and (5.29) hold, $\eta_t^* \cdot \mathscr{H}^{n-1} \llcorner S_t \cap B \in \mathscr{M}\big(N, X_n\big)$ for $\mathscr{H}^1$ a.a. t by Proposition 1.3(v). In particular, $\eta_t^*(x)$ belongs to the interior of $X^*(e,a)$ for $\mathscr{H}^1$ a.a. t and $\mathscr{H}^{n-1}$ a.a. $x \in S_t \cap B$. By Theorem 3.5(iv), $\mathscr{H}^{n-1} \llcorner S_t \cap B \in \mathscr{E}\big(N, X^*(e,a)\big)$ for $\mathscr{H}^1$ a.a. t and then $\mu_n \in \mathscr{E}\big(N, X^*(e,a)\big)$ by Theorem 3.5(iii) and (5.29). Hence $\mu \in \mathscr{E}\big(N, X^*(e,a)\big)$ because $\mu = \sum \mu_n$ and $\mathscr{E}\big(N, X^*(e,a)\big)$ is a strongly closed subset of $\mathscr{M}(N)$ (Theorem 3.5(i)). □

Proof of Proposition 3.6. By Theorem 1.15 there exist functions $f_n \in L^1(\mu, N^*)$ for $n = 1, 2, \dots$ such that $E(\mu, x)$ is the closure of the set $\{f_n(x) : n = 1, 2, \dots\}$ for μ almost all x.

By Theorem 2.12 there exist functions $u_n \in BV(N)$ such that $[dDu_n/d\mu](x) = f_n(x)$ for μ almost all x and all n and then, for μ almost all x, $E(\mu, x)$ is the closure of the set of all $[dDu_n/d\mu](x)$. Now it is enough to apply Proposition 1.10. $\square$

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