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A Lusin Type Theorem for Gradients

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We prove that for every Borel vector field f, there exists a function u of class $\mathscr{C}^1$ whose gradient Du agrees with f outside a set of arbitrary small measure.

INTRODUCTION

It is well-known that given any vector field f of class $\mathscr{C}^1$ on a simply connected open set $\Omega \subset \mathbb{R}^N$, there exists a function whose gradient is f if and only if $\operatorname{curl} f = 0$, where $\operatorname{curl} f$ is the function of Ω into $\mathbb{R}^{N\times N}$ defined by

$$(\operatorname{curl} f)_{j,i} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \qquad \text{for all } j, i = 1, \dots, N.$$

By using convolutions, the analogous result may be easily proved when f is a distribution and $\operatorname{curl} f = 0$ in the distributional sense.

In this paper we prove that if f is a Borel vector field on Ω and ε is a positive real number, then there exists a function u of class $\mathscr{C}^1$ such that f agrees with Du outside an open set A with measure less than ε. Notice that this holds even if f is a field such that $\operatorname{curl} f \neq 0$ everywhere; it may easily be proved that in this case the set A must be dense in Ω.

Our main result is the following.

THEOREM 1. *Let Ω be a open subset of $\mathbb{R}^N$ $(N > 1)$ with finite measure, and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Then, for every $\varepsilon > 0$, there exist an open set $A \subset \Omega$ and a function $u \in \mathscr{C}^1_0(\Omega)$ such that*

$$|A| \le \varepsilon |\Omega| \tag{1a}$$

$$f = Du \qquad \text{in } \Omega \setminus A, \tag{1b}$$

$$\|Du\|_p \le C\, \varepsilon^{1/p-1} \|f\|_p \qquad \text{for all } p \in [1, \infty], \tag{1c}$$

where C is a constant which depends on N only.

We add some remarks and further results.

Remark 2. Notice that when $p = 1$ the condition $|\Omega| < \infty$ may be dropped and Theorem 1 may be stated as follows:

Let Ω be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Then, for every $\varepsilon > 0$, there exists a function $u \in \mathscr{C}^1_0(\Omega)$ such that $f = Du$ outside an open set with measure less than ε and $\|Du\|_1 \leq C\|f\|_1$ (C is the same constant of Theorem 1).

If the function u in the statement of Theorem 1 is allowed to be taken in the space BV, (1a), (1b) and (1c) may be strenghtened as follows.

THEOREM 3. *Let Ω be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a function in L^1. Then there exists a function $u \in BV(\mathbb{R}^N)$ and a Borel function $g : \Omega \to \mathbb{R}^N$ such that*

$$Du = f \cdot \mathscr{L}^N + g \cdot \mathscr{H}^{N-1}, \tag{2a}$$

$$\int |g| \, d\mathscr{H}^{N-1} \leq C\|f\|_1, \tag{2b}$$

where $\mathscr{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$, $\mathscr{H}^{N-1}$ is the $(N-1)$ dimensional Hausdorff measure, and C is a constant which depends on N only.

Remark 4. In Theorem 1, (1c) gives an upper bound of the L^p norm of the gradient of u which essentially depends on the measure of the set A. We may ask whether this is the best estimate we can get in general, that is, whether for some p formula (1c) may be replaced with

$$\|Du\|_p \leq \phi(\varepsilon) \, \|f\|_p,$$

where ϕ is a function such that $\lim_{\varepsilon \to 0} \phi(\varepsilon) \, \varepsilon^{1-1/p} = 0$.

The answer is "no" as the following proposition shows.

PROPOSITION 5. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega)$ and let $A_n = \big\{x \in \Omega : f(x) \neq Du_n(x)\big\}$. If we have that*

$$\lim_{n \to \infty} |A_n| = 0, \quad \textit{and} \quad \liminf_{n \to \infty} |A_n|^{1-1/p} \|Du_n\|_p = 0, \tag{3}$$

then $\operatorname{curl} f = 0$ *as a distribution on* Ω.

The proposition above shows that if $\operatorname{curl} f \neq 0$ as a distribution on Ω (for example, take $N = 2$ and $f(x, y) = (y, 0)$), then no sequence $\{u_n\} \subset W^{1,p}(\Omega)$ can satisfy (3).

Theorem 1 can be applied to study integral functionals on Sobolev space of the form (cf. [2])

$$F(u,A)=\int_A g\big(x,Du(x)\big)\,dx$$

where Ω is an open subset of $\mathbb{R}^N$, $g:\Omega\times\mathbb{R}^N\to[-\infty,\infty]$ is a Borel function, A varies among all open subsets of Ω and u varies in the space $W^{1,p}(\Omega)$. We may ask in which sense the function g which represents F is determined.

Corollary 6. *Let Ω be an open subset of $\mathbb{R}^N$ and let h and g be two Borel functions of $\Omega\times\mathbb{R}^N$ into $[-\infty,\infty]$ such that for every $u\in C^1_c(\Omega)$*

$$h\big(x,Du(x)\big)=g\big(x,Du(x)\big)\qquad \text{a.e. in } \Omega, \tag{4}$$

that is, h and g represent the same integral functional. Then there exists a negligible Borel set $N\subset\Omega$ such that $h(x,s)=g(x,s)$ for all $x\in\Omega\setminus N$ and $s\in\mathbb{R}^N$.

Proof of the results

To begin with, we prove the following auxiliary lemma.

Lemma 7. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure, let $f:\Omega\to\mathbb{R}^N$ be a continuous function and let η and ε be positive real numbers. Then there exist a compact set $K\subset\Omega$ and a function $u\in\mathscr{C}^1_c(\Omega)$ such that*

$$|\Omega\setminus K|\le\varepsilon|\Omega| \tag{5a}$$

$$|f-Du|\le\eta \qquad \text{on } K, \tag{5b}$$

$$\|Du\|_p\le C'\varepsilon^{1/p-1}\|f\|_p \qquad \text{for all } p\in[1,\infty], \tag{5c}$$

where C' is a constant which depends on N only.

Proof. Of course we may suppose $\varepsilon<1$. Let K' be a compact subset of Ω such that $|\Omega\setminus K'|<|\Omega|\,\varepsilon/2$; there exists a positive δ such that, for all $x\in K'$, $y\in\Omega$

$$|x-y|<\delta \;\Rightarrow\; |f(x)-f(y)|<\eta \quad \text{and} \quad Q(x,4\delta)\subset\Omega \tag{6}$$

where $Q(x,4\delta)$ is the cube with center x and side 4δ.

Let $\{T_i\}_{i\in I}$ be the (finite) family of all closed cubes T whose sides' length is δ, whose centers y_i belong to lattice $(\delta\mathbb{Z})^N$ and which intersect K: by the choice of δ, each T_i is included in Ω. For all $i\in I$, let Q_i be the closed cube with the same center of T_i and side $(1-\varepsilon/(2N))\,\delta$; let a_i be the mean value of f on T_i and let ϕ_i be a function of class $\mathscr{C}^1$ such that $\phi_i \equiv 1$ in Q_i, $\phi_i \equiv 0$ outside T_i and

$$\|D\phi_i\|_\infty \le \frac{8N}{\delta\varepsilon}. \tag{7}$$

For all $x \in \mathbb{R}^N$ set

$$u(x) = \sum_i \phi_i(x)\ < a_i, x - y_i > . \tag{8}$$

It is easy to see that u is a function of class $\mathscr{C}^1$ whose support is included in $\bigcup_i T_i \subset \Omega$ and whose gradient is a_i within each cube Q_i. Finally we set $K = \bigcup_i Q_i$. We have to prove that u and K satisfy (5a), (5b) and (5c).

(5a): By the choice of each Q_i we have that

$$|T_i \setminus Q_i| \le \Big[1 - \Big(1 - \frac{\varepsilon}{2N}\Big)^N\Big]\ |T_i| \le \frac{\varepsilon}{2}|T_i| \tag{9}$$

and then, as each T_i is a subset of Ω by (6),

$$|\Omega \setminus K| \le |\Omega \setminus K'| + \sum_i |T_i \setminus Q_i| \le \varepsilon|\Omega|.$$

(5b): By (8), Du is equal to the mean value of f on T_i within each Q_i and then $|Du(x) - f(x)| \le \eta$ within each Q_i by (6).

(5c): By (8) we have that

$$Du(x) = \sum_i D\phi_i(x) < a_i, x - y_i > + \sum_i a_i\, \phi_i(x);$$

and then, for all $p \in [1, \infty[$, taking into account (6), (7) and recalling that $D\phi_i = 0$ outside $T_i \setminus Q_i$ and that a_i is the mean value of f on T_i,

$$\begin{aligned}
\|Du\|_p &\le \Big[\sum_i \Big(\|D\phi_i\|_\infty |a_i| \sqrt{N}\delta\Big)^p |T_i \setminus Q_i|\Big]^{1/p} + \Big[\sum_i |a_i|^p |T_i|\Big]^{1/p} \\
&\le \Big[\sum_i \Big(8N^{3/2}|a_i|\varepsilon^{-1}\Big)^p \varepsilon |T_i|\Big]^{1/p} + \Big[\sum_i |a_i|^p |T_i|\Big]^{1/p} \\
&\le \Big(8N^{3/2}\varepsilon^{1/p-1} + 1\Big)\Big[\sum_i \Big|\frac{1}{|T_i|}\int_{T_i} f\, dx\Big|^p |T_i|\Big]^{1/p} \\
&\le \Big(8N^{3/2}\varepsilon^{1/p-1} + 1\Big)\ \Big[\int_\Omega |f|^p dx\Big]^{1/p}.
\end{aligned}$$

As the same inequality hold when $p=\infty$ and $\varepsilon<1$, Lemma 7 is proved. □

Proof of Theorem 1. Of course we may suppose $\varepsilon<1$ and that f is not almost everywhere 0.

First Case. f is a continuous bounded function.

Let $\{\eta_n\}$ be a sequence of positive real numbers; by induction on n we build a sequence $\{u_n, K_n, f_n\}$ as follows: set $u_0=0$, $K_0=\emptyset$ and $f_0=f$. Let $n>0$ and let u_{n-1}, K_{n-1} and f_{n-1} be chosen. Apply Lemma 7 to obtain a compact set $K_n\subset\Omega$ and a function $u_n\in\mathscr{C}^1_c(\Omega)$ such that

$$|\Omega\setminus K_n|\le|\Omega|2^{-n}\varepsilon \tag{10a}$$

$$|f_{n-1}-Du_n|\le\eta_n \quad \text{on } K_n, \tag{10b}$$

$$\|Du_n\|_p\le C'(2^{-n}\varepsilon)^{1/p-1}\|f_{n-1}\|_p \quad \text{for all } p\in[1,\infty]. \tag{10c}$$

Define $f_n(x)=f_{n-1}(x)-Du_n(x)$ for all $x\in K_n$ and apply Titze's lemma to extend f_n to the whole of Ω so that

$$\sup_{x\in\Omega}|f_n(x)|=\sup_{x\in K_n}|f_n(x)|\le\eta_n. \tag{11}$$

We set $A=\Omega\setminus\bigcap_n K_n$, $u=\sum_n u_n$ and then choose a sequence $\{\eta_n\}$ so that these definitions make sense and satisfy (1a), (1b) and (1c). By (10a) we obtain

$$|A|\le\sum_1^\infty|\Omega\setminus K_n|\le\sum_1^\infty|\Omega|2^{-n}\varepsilon=|\Omega|\varepsilon$$

and (1a) holds. For all $p\in[1,\infty]$, (10c) and (11) yield

$$\begin{aligned}\sum_1^\infty\|Du_n\|_p&\le\sum_1^\infty C'\varepsilon^{1/p-1}2^n\|f_{n-1}\|_p\\&\le 2C'\varepsilon^{1/p-1}\left[\|f_0\|_p+\sum_1^\infty 2^n\|f_n\|_\infty|\Omega|^{1/p}\right]\\&\le 2C'\varepsilon^{1/p-1}\|f\|_p\left[1+\frac{|\Omega|^{1/p}}{\|f\|_p}\sum_1^\infty 2^n\eta_n\right]\quad.\end{aligned}$$

As f is bounded and not almost everywhere 0, an easy computation shows that the function $p\mapsto|\Omega|^{1/p}\big/\|f\|_p$ is continuous and positive on $[1,\infty]$, hence it has a positive upper bound a and we may choose all η_n small enough to have that $\sum_1^\infty 2^n\eta_{n-1}\le 1/a$ and then

$$\sum_1^\infty\|Du_n\|_p\le 4\,C'\varepsilon^{1/p-1}\|f\|_p.$$

Poincaré's inequality (cf. [1, Chap. 9]) shows that the series $\sum_n u_n$ converges in the $\mathscr{C}^1_0(\Omega)$ norm to a function u that satisfies (1c) with $C = 4\,C'$. By the definition of f_n we have that, for all x in $\Omega \setminus A$ and for all integers m, $f(x) - \sum_1^m Du_n(x) = f_m(x)$ and then by (10b)

$$|f(x) - Du(x)| \leq |f_m(x)| \ + \sum_{m+1}^{\infty} |Du_n(x)| \leq \eta_m + \sum_{m+1}^{\infty} |Du_n(x)|.$$

Hence (1b) immediately follows because the sequences η_m and $\sum_m^\infty \|Du_n\|_\infty$ converge to 0.

Second Case. f is a Borel function.

Let $\varepsilon > 0$ be fixed. There exists a positive r such that $|B| < \varepsilon/4$, where $B = \{x : |f(x)| > r\}$. By Lusin's theorem there exists a continuous function $f_1 : \Omega \to \mathbb{R}^N$ which agrees with f outside a Borel set C with $|C| < |B|$. Set

$$f_2(x) = \begin{cases} f_1(x) & \text{if } |f_1(x)| \leq r, \\ r\, f_1(x)/|f_1(x)| & \text{if } |f_1(x)| > r. \end{cases}$$

The function f_2 is bounded and continuous, agrees with f outside $C \cup B$ and since $|C \cup B| < \varepsilon/2$, there exists an open set A_1 such that $|A_1| < \varepsilon/2$ and f_2 agrees with f outside A_1. Moreover, for all $p \in [1, \infty[$,

$$\begin{aligned} \int_\Omega |f_2|^p dx &\leq \int_{\Omega\setminus(B\cup C)} |f|^p dx \ + \int_{B\cup C} r^p dx \\ &\leq \int_{\Omega\setminus(B\cup C)} |f|^p dx \ + \ 2\int_B |f|^p dx \leq 2\int_\Omega |f|^p dx \quad , \end{aligned}$$

that is, $\|f_2\|_p \leq 2\,\|f\|_p$ for all p (infact that the same inequality holds for $p = \infty$).

As f_2 is bounded and continuous we may apply Theorem 1 to obtain an open set A_2 with $|A_2| \leq \varepsilon/2$ and a function $u \in \mathscr{C}^1_c(\Omega)$ such that $Du = f_2$ outside A_2 and $\|Du\|_p \leq 4\,C'(\varepsilon/2)^{1/p-1}\|f_2\|_p$ for all $p \in [1, \infty]$.

Hence $Du = f$ outside the set $A_1 \cup A_2$, $|A_1 \cup A_2| \leq \varepsilon$, and for all $p \in [1, \infty]$,

$$\|Du\|_p \leq 4\,C'(\varepsilon/2)^{1/p-1}\|f_2\|_p \leq 16\,C'\varepsilon^{1/p-1}\|f\|_p.$$

Then Theorem 1 holds with $A = A_1 \cup A_2$. □

The proof of Theorem 3 is quite similar to the one of Theorem 1; with no loss in generality we may suppose that $\Omega = \mathbb{R}^N$.

To begin with, we prove an auxiliary lemma that will be used instead of Lemma 7.

LEMMA 8. *Let $f \in L^1(\mathbb{R}^N, \mathbb{R}^N)$ and let $\eta > 0$. Then there exist a function $u \in BV(\mathbb{R}^N)$ and two Borel functions g^a and g^s such that $Du = g^a \cdot \mathscr{L}^N + g^s \cdot \mathscr{H}^{N-1}$ and*

$$\|u\|_1 \leq \|f\|_1 \tag{12a}$$

$$\|f - g^a\|_1 \leq \eta \tag{12b}$$

$$\int |g^s| d\mathscr{H}^{N-1} \leq C'\|f\|_1. \tag{12c}$$

where C' is a constant which depends on N only.

Proof. Let δ be a fixed positive number. Let $\{T_i\}_{i \in I}$ be the family of all open cubes whose sides' length is δ and whose centers y_i belong to lattice $(\delta\mathbb{Z})^N$. For all $i \in I$ let a_i be the mean value of f on T_i , let χ_i be the characteristic function of the set T_i , let ν_i be the inner normal of ∂T_i (namely, if x is a smooth point for ∂T_i then $\nu_i(x)$ is the inner normal of ∂T_i in x, otherwise $\nu_i(x)$ is 0). For all $x \in \mathbb{R}^N$ set

$$u_\delta(x) = \sum_i < a_i, x - y_i > \chi_i(x)$$

An easy computation shows that u_δ belongs to BV and $Du_\delta = g^a_\delta \cdot \mathscr{L}^N + g^s_\delta \cdot \mathscr{H}^{N-1}$ where $g^a_\delta(x) = \sum_i a_i \chi_i(x)$ and $g^s_\delta(x) = \sum_i < a_i, x - y_i > \nu_i(x)$. Then

$$\|u_\delta\|_1 \leq \sum_i \sqrt{N}\delta\, |a_i| \cdot |T_i| \leq \sqrt{N}\delta \|f\|_1$$

$$\|g^a_\delta\|_1 \leq \sum_i |a_i| \cdot |T_i| \leq \|f\|_1$$

$$\int |g^s_\delta| d\mathscr{H}^{N-1} \leq \sum_i \sqrt{N}\delta |a_i| \mathscr{H}^{N-1}\big(\partial T_i\big) \leq \sum_i |a_i| 2N^{3/2} |T_i| \leq 2N^{3/2}\|f\|_1.$$

Now it is enough to show that δ may be chosen so that (12a), (12b) and (12c) hold. Hence the proof is complete if we show that

$$\lim_{\delta \to 0} \|f - g^a_\delta\|_1 = 0. \tag{13}$$

Let $\Gamma_\delta : L^1 \to L^1$ be the linear operator taking each f into g^a_δ. By construction we have that $\|\Gamma_\delta\| \leq 1$ for all δ and an easy computation shows that

$\lim_{\delta\to 0} \|\Gamma_\delta f - f\|_1 = 0$ whenever $f \in C_c$. Hence (13) follows because C_c is dense in L^1. □

Proof of Theorem 3. As in the proof of Theorem 1 we build by induction on n a sequence $\{u_n, f_n\}$ as follows.

Set $u_0 = 0$ and $f_0 = f$. Let $n > 0$ and suppose that u_{n-1} and f_{n-1} has been chosen. Apply Lemma 8 to obtain a function $u_n \in BV$ such that $Du_n = g_n^a \cdot \mathscr{L}^N + g_n^s \cdot \mathscr{H}^{N-1}$ and

$$\|u_n\|_1 \le \|f_{n-1}\|_1, \quad \|g_n^a - f_{n-1}\|_1 \le 2^{-n}\|f\|_1, \quad \text{and}$$

$$\int |g_n^s| d\mathscr{H}^{N-1} \le C'\|f_{n-1}\|_1.$$

Set $f_n = f_{n-1} - g_n^a$.

Hence the series $\sum_n u_n$ converges in BV norm to a function u and $Du = g^a \cdot \mathscr{L}^N + g^s \cdot \mathscr{H}^{N-1}$ with $g^a = \sum_n g_n^a$, $g^s = \sum_n g_n^s$. Arguing as in the proof of Theorem 1 we get $\|u\|_1 \le 2\|f\|_1$, $g^a = f$ almost everywhere and $\int |g^s| d\mathscr{H}^{N-1} \le 2C'\|f\|_1$. □

Proof of Proposition 5. Possibly passing to a subsequence we may assume

$$\lim_{n\to\infty} |A_n|^{1-1/p} \|Du_n\|_p = 0. \tag{14}$$

For all n set

$$g_n(x) = \begin{cases} |Du_n(x)| & \text{if } x \in A_n, \\ 0 & \text{if } x \notin A_n. \end{cases}$$

Then $|Du_n| \le |f| + g_n$ everywhere by definition of A_n and $\|g_n\|_1 \le |A_n|^{1-1/p}\|Du_n\|_p$ by Schwartz-Hölder inequality. Now (14) implies that $\|g_n\|_1$ converges to 0; Hence $\{Du_n\}$ is a sequence of uniformly integrable functions and Dunford-Pettis theorem (cf. [4, Theorem II.25]) ensures that it has at least one limit point in $w - L^1(\Omega, \mathbb{R}^N)$. This limit point must be f, that is, Du_n converges to f in the weak topology of L^1.

Then $\operatorname{curl} f = \lim_n \operatorname{curl} Du_n$ in the sense of distributions and the conclusion follows immediately because $\operatorname{curl} Du = 0$ for any distribution $\mathscr{D}'(\Omega)$ (cf. [5, Chap. 6]). □

Proof of Corollary 6. Set $B = \{(x,s) : h(x,s) \ne g(x,s)\}$ and let π be the projection of $\Omega \times \mathbb{R}^N$ on Ω. By the Aumann measurable selection theorem (cf. [3, Theorems III.22 and III.23]) we have

(i) $\pi(B)$ is Lebesgue measurable

(ii) there exists a Lebesgue measurable function $f : \pi(B) \to \mathbb{R}^N$ whose graph is a subset of B.

As $\pi(B)$ is Lebesgue measurable, it is enough to show that $|\pi(B)| = 0$. By contradiction, suppose that $|\pi(B)| > 0$; then, by (ii) and Theorem 1 there exists a function $u \in \mathscr{C}^1(\mathbb{R}^N)$ such that $f = Du$ in a compact set C of positive measure. Therefore

$$h\big(x, Du(x)\big) \neq g\big(x, Du(x)\big) \quad \text{for every } x \in C \ ,$$

and this contradicts the assumption (4). □

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Refernces

1. H. Brezis, "Analyse fonctionnelle et applications", Masson, Paris, 1973.
2. G. Buttazzo and G. Dal Maso, Integral representation and relaxation of local functionals, *Nonlinear Anal.* **9** (1985), 515-532.
3. C. Castaing and M. Valadier, "Convex analysis and measurable multifunctions", Lecture Notes in Mathematics, 580. Springer-Verlag, Berlin, 1977.
4. C. Dellacherie and P.A. Meyer, "Probabilities and potential", Mathematics Studies, 29. North Holland, Paris, 1978.
5. W. Rudin, "Functional analysis", McGraw-Hill, 1973.