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On the structure of singular sets of convex functions

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Abstract. When f is a convex function of $\mathbb{R}^h$, and k is an integer with $0 < k < h$, then the set $\Sigma^k(f) := \{x : \dim(\partial f(x)) \geq k\}$ may be covered by countably many manifolds of dimension $h-k$ and class C^2 except an $\mathscr{H}^{h-k}$ negligible subset.

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Introduction

The aim of this paper is to study the structure of the singular set (points of non-differentiability) of a convex function. When f is a convex function of $\mathbb{R}^h$ and k is an integer such that $0 \leq k \leq h$, we define

$$\Sigma^k(f) := \{x : \dim(\partial f(x)) \geq k\}$$

where ∂f is the subdifferential of f (see Definitions 1.2 and 1.5). It is well-known that f is differentiable out of the union of all $\Sigma^k(f)$ with $k > 0$ (cf. Proposition 1.6) and that $\Sigma^h(f)$ is a countable set and then it seems a natural problem to study the dimension and the structure of the sets $\Sigma^k(f)$ when $0 < k < h$. In the recent paper [2] it was proved that $\Sigma^k(f)$ is a $(\mathscr{H}^{h-k}, h-k)$ rectifiable set of class C^1 (following [5] we say that a Borel set $S \subset \mathbb{R}^h$ is a $(\mathscr{H}^n, n)$ rectifiable set of class C^m when $\mathscr{H}^n$ almost all of S may be covered by a countable family of n-dimensional submanifolds of $\mathbb{R}^h$ of class C^m, and $\mathscr{H}^n$ denotes as usual the n-dimensional Hausdorff measure in $\mathbb{R}^h$, see for instance definition 1.1).

The main theorem of this paper improves that result.

Theorem 1. *Let $f : \mathbb{R}^h \to \mathbb{R}$ be a convex function and let k be an integer such that $0 < k < h$. Then $\Sigma^k(f)$ is a $(\mathscr{H}^{h-k}, h-k)$ rectifiable set of class C^2 (cf. Definitions 1.1 and 1.5) and this means that $\mathscr{H}^{h-k}$ almost all of it may be covered by countably many $(h-k)$-dimensional submanifolds of class C^2.*

We recall that when Ω is a convex open subset of $\mathbb{R}^h$ and f a real convex function on Ω, then for every compact set $K \subset \Omega$ there exists a real convex function f' on $\mathbb{R}^h$

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which agrees with f on K. Hence the statement of Theorem 1 may be extended to every real convex function defined on a convex open subset of $\mathbb{R}^h$.

The next theorem shows that the class of regularity given in Theorem 1 cannot be improved.

Theorem 2. *Let k be a fixed integer such that $0 < k < h$ and let S be a subset of $\mathbb{R}^h$ which may be covered by countably many $(h-k)$-dimensional submanifolds of class C^2 included in $\mathbb{R}^h$. Then there exists a convex function f such that $S \subset \Sigma^k(f)$.*

Theorem 1 may be extended to the singularities of convex surfaces (boundary of h-dimensional convex subsets of $\mathbb{R}^h$). When C is an h-dimensional closed convex subset of $\mathbb{R}^h$ and $x \in \partial C$, we may consider the set $\mathscr{N}(C,x)$ of all outer normals of C in x and then we take $\Sigma^k(C) := \{x \in \partial C : \dim(\mathscr{N}(C,x)) \geq k\}$ for all integers k with $0 \leq k < h$ (see Definition 1.7). Then the boundary of C is a Lipschitz manifold of co-dimension 1 and it is differentiable in every point x out of the union of all $\Sigma^k(C)$ with $k > 0$. The structure of the set $\Sigma^k(C)$ with $k > 0$ is described by the following straightforward corollary of Theorem 1.

Theorem 3. *Let C be an h-dimensional closed convex subset of $\mathbb{R}^h$ and let k be an integer with $0 \leq k < h$. Then $\Sigma^k(C)$ is a $(\mathscr{H}^{h-k-1}, h-k-1)$ rectifiable subset of $\mathbb{R}^h$ of class C^2 (cf. Definitions 1.1 and 1.7). In particular, if we take $k = 0$ we obtain that the boundary of C is a rectifiable set of class C^2 and co-dimension 1.*

Theorem 1 and Theorem 3 may be very useful when studying the generalized curvatures of convex surfaces in $\mathbb{R}^h$ (see [4]) and the structure of weakly defined minors of second order derivatives of convex functions (see [3] and [1]).

The proof of the C^1 rectifiability in [2] is based upon a particular rectifiability criterion and it may be extended to the singular sets of semi-convex functions as well. Different proofs of this theorem (for convex functions only) are given in [1] and [4]. The former paper deals in general with singular sets (points of discontinuity) of monotone functions (cf. Proposition 1.4) and the rectifiability result is an almost straightforward corollary of some simple properties of monotone functions. On the contrary the latter paper approaches the problem from a GMT viewpoint and a rectifiability theorem is stated for the singularities of a class of surfaces which includes convex surfaces (and then graphs of convex functions) as a particular case. Moreover a geometric proof (due to B. White) of the C^2 rectifiability of singularities of convex surfaces in $\mathbb{R}^3$ is given.

In our paper we follow a quite different approach: we prove that the singular set $\Sigma^k(f)$ may be covered by countably many graphs of locally Lipschitz functions which belong to BV^2_{loc} (i.e. sets which may be written in the form $\Phi(G)$ where Φ is a linear isometry of $\mathbb{R}^h$ and G is the graph of a locally Lipschitz function g in $BV^2_{\mathrm{loc}}(\mathbb{R}^{h-k}, \mathbb{R}^k)$) and then we show that these graphs are rectifiable sets of class C^2.

1. Definitions and Preliminary Results

In the following we shall deal with monotone and convex functions and rectifiable sets as well and so we need to recall some basic facts and definitions about these topics. For the general theory of monotone and convex functions see for instance [6] and [10].

When n is a positive integer, we denote by $\mathscr{L}_n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathscr{H}^n$ the n-dimensional Hausdorff measure in every metric space. For the general properties of Hausdorff measures we refer essentially to [11] and [12].

Definition 1.1. If m, n and h are positive integers with $n < h$ and S is a Borel subset of $\mathbb{R}^h$, we say that S is a $(\mathscr{H}^n, n)$ *rectifiable set of class* C^m if, for $i = 1, 2, \dots$, there exist n-dimensional submanifolds $M_i \subset \mathbb{R}^h$ of class C^m such that

$$\mathscr{H}^n\Big(S \setminus \bigcup_i M_i\Big) = 0.$$

in particular every $(\mathscr{H}^n, n)$ rectifiable set is the union of countably many sets with finite n-dimensional Hausdorff measure and then its Hausdorff dimension is at most n, moreover every countable union of $(\mathscr{H}^n, n)$ rectifiable sets of class C^m is a $(\mathscr{H}^n, n)$ rectifiable sets of class C^m.

When no doubts can arise, we simply say C^m *rectifiable* instead of $(\mathscr{H}^n, n)$ rectifiable of class C^m. The concept of rectifiability of class C^m has been recently introduced in [5] (unlike our definition, in that paper a $(\mathscr{H}^n, n)$ rectifiable sets are assumed to have finite n-dimensional Hausdorff measure).

If u and u' are multifunctions of $\mathbb{R}^h$ into $\mathbb{R}^h$ (i.e. functions of $\mathbb{R}^h$ which take values in the class of all subsets of $\mathbb{R}^h$), then $u + u'$ and u^{-1} are the multifunctions which take every x in the sets $\{y + y' : y \in u(x), y' \in u'(x)\}$ and $\{y : x \in u(y)\}$ respectively. We write $u \supset u'$ when $u(x) \supset u'(x)$ for every x. The domain of u is the set of all x such that $u(x)$ is not empty.

A *monotone function* of $\mathbb{R}^h$ is any multifunction u of $\mathbb{R}^h$ into $\mathbb{R}^h$ such that

$$\langle y_1 - y_2; x_1 - x_2\rangle \geq 0 \quad \text{for all } x_i \in \mathbb{R}^h,\ y_i \in u(x_i) \text{ with } i = 1, 2. \tag{1.1}$$

A monotone function is maximal if it is maximal in the class of all monotone functions with respect to inclusion ($\supset$). It is obvious that u is a (maximal) monotone function if and only if u^{-1} is.

Definition 1.2. By *convex functions* on $\mathbb{R}^h$ we mean real convex functions only, i.e. functions which are allowed to take finite values only. When f is a convex function of $\mathbb{R}^h$ and x is a point of $\mathbb{R}^h$, we define the subdifferential of f in x as the set $\partial f(x)$ of all points $y \in \mathbb{R}^h$ such that

$$f(x') \geq f(x) + \langle y; x' - x\rangle \quad \text{for all } x' \in \mathbb{R}^h \tag{1.2}$$

(cf.[6], section II.3, and [10], section 23).

Remark 1.3. When f is a real function of $\mathbb{R}^h$, the subdifferential of f in a point x is sometimes defined as the set of all $y \in \mathbb{R}^h$ such that

$$\liminf_{x' \to x} \frac{f(x') - f(x) - \langle y; x' - x\rangle}{|x' - x|} \geq 0. \tag{1.3}$$

It is not difficult to prove that when f is convex, (1.2) holds if and only if (1.3) does and then these two definitions of subdifferential are equivalent for convex functions.

Moreover we remark that a function $f : \mathbb{R}^h \to \mathbb{R}$ is a convex if and only if for every $x \in \mathbb{R}^h$ there exists $y \in \mathbb{R}^h$ such that (1.2) holds. Indeed, if f is convex then ∂f is never empty (cf. Proposition 1.4) and every function f such that ∂f is never empty is the supremum of all affine functions λ such that $f \geq \lambda$ everywhere and then it is convex (see section 23 of [10]).

About the subdifferential of a convex function, we shall need the following immediate result (cf. [10], sections 23 and 24, and [6], example 2.3.4).

Proposition 1.4. *If f is a convex function on $\mathbb{R}^h$ then ∂f is a maximal monotone function with domain $\mathbb{R}^n$ and then $\partial f(x)$ is always a non-empty closed convex set.*

Definition 1.5. When C is a non-empty convex set, the *dimension of C* is defined as the dimension of the affine hull of C (the affine space spanned by C), that is the least integer k such that there exists a k-dimensional affine space which includes C (we remark that every convex set has non-empty interior relative to its affine hull and then the dimension of a convex set is its Hausdorff dimension; see also [10], section 2). Then, when f is a convex function on $\mathbb{R}^h$, taking into account Proposition 1.4 it is clear what the dimension of $\partial f(x)$ is. For every integer k with $0 < k \leq h$ we set

$$\Sigma^k(f) := \{x : \dim(\partial f(x)) \geq k\} \quad .$$

It may be proved that $\Sigma^k(f)$ is always a Borel set (actually a countable union of closed sets).

Eventually we recall some elementary properties of convex functions and subdifferentials and then we give the definitions of outer normals and singularities of a convex surfaces.

Proposition 1.6. *Let f be convex functions on $\mathbb{R}^h$. Then*
(i) *f is differentiable in x if and only if $\partial f(x)$ consists of just one element which actually is the gradient of f in x;*
(ii) *x is a minimum point of f if and only if $0 \in \partial f(x)$;*
(iii) *when f' is a convex function of $\mathbb{R}^h$, $\partial(f + f') \supset \partial f + \partial f'$ and in particular the dimension of $\partial(f + f')(x)$ is never less than the dimension of $\partial f(x)$;*

Statement (i) follows from Theorem 25.1 of [10]; (ii) and (iii) are trivial. Notice that (iii) may be improved by showing that $\partial(f + f') = \partial f + \partial f'$ but this is a little more difficult (see for instance [6], Corollary 2.11).

Definition 1.7. By *convex surface* in $\mathbb{R}^h$ we mean the boundary of any (closed) convex subset of $\mathbb{R}^h$ with non-empty interior (i.e. with dimension h). When C is a closed subset of $\mathbb{R}^h$ with non empty interior, its boundary is an (oriented) Lipschitz surface of co-dimension 1, and for every $x \in \partial C$ we define the set of all outer normals to C in x, $\mathscr{N}(C, x)$, as the set of all unitary vectors $y \in \mathbb{R}^h$ such that

$$\langle y; x - x' \rangle \geq 0 \quad \text{for all } x' \in C. \tag{1.4}$$

Eventually, for every integer k with $0 \leq k < h$ we set

$$\Sigma^k(C) := \{x \in \partial C : \dim(\mathscr{N}(C, x)) \geq k\}.$$

Notice that $\mathscr{N}(C, x)$ is given by the the intersection of the unitary sphere of $\mathbb{R}^h$, S^{h-1}, and the convex cone D of all $y \in \mathbb{R}^h$ which satisfy (1.4) (actually the subdifferential in x of the indicator function of C, i.e. the convex function which takes every point of C in 0 and every point outside C in $+\infty$) and then it is always non-empty and its (Hausdorff) dimension is the dimension of D minus one.

It may be proved that a closed half-space M is tangent to C in x (i.e. satisfies $C \subset M$ and $x \in \partial M$, cf. [10], section 18) if and only if it may be written in the form $M = \{x' \in \mathbb{R}^h : \langle y; x - x' \rangle \geq 0\}$ for some $y \in \mathscr{N}(C, x)$ and ∂C is differentiable in x (i.e. admits an ordinary $(h-1)$-dimensional tangent space in x) if and only if $\mathscr{N}(C, x)$ consists of only one vector.

Moreover, when C is the epigraph of a convex function on $\mathbb{R}^h$, ∂C is the graph of f and $\partial f(t)$ is closely connected to $\mathscr{N}(C,x)$ where $x=(t,f(t))$; in particular $\partial f(t)$ is the set of all points $-y_1/y_2$ with (y_1,y_2) in $\mathscr{N}(C,x)$. Hence the dimension of $\partial f(t)$ and $\mathscr{N}(C,x)$ are always the same and for every k, $\Sigma^k(f)=\pi(\Sigma^k(C))$ where π is the projection of $\mathbb{R}^h\times\mathbb{R}$ onto $\mathbb{R}^h$.

2. Proof of the Results

We begin with some important remarks about the conjugate of a convex function.

Proposition 2.1. *Let $f:\mathbb{R}^k\to\mathbb{R}$ be a convex function such that $f(x)\geq|x|^2$ for all x and let f^* be its conjugate function (cf.* [10]*, section 12, and* [6]*, section II.3), i.e. the function defined by*

$$f^*(y)=\sup_{x\in\mathbb{R}^k}[\langle y;x\rangle-f(x)]\qquad \textit{for all } y\in\mathbb{R}^k.$$

Then
(i) *f^* is a real convex function such that $f^*(y)\leq|y|^2/4$ for all y;*
(ii) *$\partial f^*=(\partial f)^{-1}$, and this means that $y\in\partial f(x)$ if and only if $x\in\partial f^*(y)$.*

Proof. (i) is trivial.

Since ∂f is a maximal monotone functions and $(\partial f^*)^{-1}$ is a monotone function, it is enough to prove the inclusion $\partial f\subset(\partial f^*)^{-1}$, namely that $y\in\partial f(x)$ always implies $x\in\partial f^*(y)$. Let $x\in\mathbb{R}^k$ and $y\in\partial f(x)$ be fixed. By definition of subdifferential we obtain that

$$f(x')\geq f(x)+\langle y;x'-x\rangle\quad\text{for all } x'. \tag{2.1}$$

Then (2.1) yields $\langle y;x'\rangle-f(x')\leq\langle y;x\rangle-f(x)$ for all x'. Hence

$$f^*(y)=\sup_{x'\in\mathbb{R}^n}[\langle y;x'\rangle-f(x')]=\langle y;x\rangle-f(x)$$

and moreover

$$\begin{aligned}f^*(y')&=\sup_{x'\in\mathbb{R}^n}[\langle y';x'\rangle-f(x')]\geq\langle y';x\rangle-f(x)\\&=\langle y'-y;x\rangle+\langle y;x\rangle-f(x)=f^*(y)+\langle x;y'-y\rangle\end{aligned}$$

and this means that $x\in\partial f^*(y)$. $\square$

Lemma 2.2. *Let $f:\mathbb{R}^k\to\mathbb{R}$ be a convex function such that $f(x')\geq|x'|^2$ for all x', and suppose that x is a point of $\mathbb{R}^k$ such that $\partial f(x)$ includes the open set A. Then $\partial f^*(y)=x$ for all $y\in A$.*

Proof. By Proposition 1 we have that $\partial f^*(y)=\{x':y\in\partial f(x')\}$ for all $y\in\mathbb{R}^k$. Hence $x\in\partial f^*(y)$ for all $y\in A$. Let $y\in A$ and $x'\in\partial f^*(y)$ be fixed. I want to show that $x'=x$. Since A is open, there exist $r>0$ such that $y+r(x'-x)$ belongs to A. Then x belongs to $\partial f^*(y+r(x'-x))$ and, taking into account that ∂f^* is a monotone function,

$$0\leq\big\langle x-x';(y+r(x'-x))-y\big\rangle=-r|x-x'|^2.$$

Hence $x=x'$. $\square$

Proposition 2.3. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function such that $f(x') \geq |x'|^2$ for all x' and let $x, y \in \mathbb{R}^k$ and $\varepsilon > 0$ be given so that $\partial f(x) \supset B(y, 2\varepsilon)$. Then*

$$x_i = \frac{1}{\varepsilon}\big[f^*(y + \varepsilon e_i) - f^*(y)\big] \quad \textit{for } i = 1, \ldots, k \tag{2.2}$$

where $x = (x_1, \ldots, x_k)$ and $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{R}^k$.

Proof. By Lemma 2.2, $\partial f^*(y') = x$ for all $y' \in B(y, 2\varepsilon)$ and then f^* is differentiable with gradient x in every point of $B(y, 2\varepsilon)$ (cf. statement (i) of Proposition 1.6) and (2.2) immediately follows. □

In the following we shall consider convex functions defined on the product space $\mathbb{R}^n \times \mathbb{R}^k$ where n and k are positive integers. We denote by π the projection of $\mathbb{R}^n \times \mathbb{R}^k$ on $\mathbb{R}^k$, and when f is a function of $\mathbb{R}^n \times \mathbb{R}^k$, for every $t \in \mathbb{R}^n$ we denote by f_t the function of $\mathbb{R}^k$ given by $f_t : x \mapsto f(t, x)$.

Our strategy is the following: when f is a convex function of $\mathbb{R}^n \times \mathbb{R}^k$, we consider the function $\hat{f}$ given by

$$\hat{f}(t, y) = [f_t]^*(y) \qquad \text{for all } (t, y) \in \mathbb{R}^n \times \mathbb{R}^k. \tag{2.3}$$

If we denote by S the set of all points (t, x) such that $\partial f_t(x)$ includes a fixed open ball $B(y, 2\varepsilon)$, by Proposition 2.3 we obtain that S is included in the graph of the function $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ which is given by

$$g_i(t) = \frac{1}{\varepsilon}\big[\hat{f}(t, y + \varepsilon e_i) - \hat{f}(t, y)\big] \quad \text{for all } t \in \mathbb{R}^n \text{ and } i = 1, \ldots, k \tag{2.4}$$

where $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{R}^k$ (cf. formula (2.2)). Then we prove that g is a Lipschitz function which belongs to BV^2_{loc}, we show that its graph is a C^2 rectifiable set and then we obtain that S is a C^2 rectifiable set. Eventually we prove that the set $\Sigma^k(f)$ may be covered by countably many sets of the kind of S.

We begin with a description of the relation between the subdifferential of a convex function f of $\mathbb{R}^n \times \mathbb{R}^k$ and the subdifferential of f_t.

Proposition 2.4. *Let f be a convex function of $\mathbb{R}^n \times \mathbb{R}^k$ and let (t, x) be a point of $\mathbb{R}^n \times \mathbb{R}^k$. Then $\partial f_t(x) = \pi[\partial f(t, x)]$.*

Proof. It is obvious that $\partial f_t(x) \supset \pi[\partial f(t, x)]$; let's prove the opposite inclusion.

Let $v \in \partial f_t(x)$ be fixed and set $\lambda(x') := f(t, x) + \langle v; x' - x\rangle$ for all $x' \in \mathbb{R}^k$. By definition of subdifferential, $f_t \geq \lambda$ and then the open epigraph of f (i.e. the set of all points $(t', x', s) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ such that $s > f(t', x')$) and the graph of λ (i.e. the set of all points (t', x', s) such that $t' = t$ and $s = \lambda(x')$) are disjoint convex sets. Hence we may apply Hahn-Banach theorem to find an affine function ϕ such that $f \geq \phi$ and $\phi_t \geq \lambda$ everywhere.

Then $f(t, x) \geq \phi(t, x) \geq \lambda(x)$ and since $\lambda(x) = f(t, x)$ by construction, $\phi(t, x) = f(t, x)$ and then the gradient of ϕ, w, belongs to $\partial f(t, x)$ because $f \geq \phi$. Furthermore, the inequality $\phi_t \geq \lambda$ can hold if and only if the gradients of ϕ_t and λ are the same, i.e. $\pi(w) = v$. Hence the thesis. □

Proposition 2.5. *Let $f : \mathbb{R}^n \times \mathbb{R}^k$ be a convex function such that $f(t, x) \geq |x|^2$ for all t, x and let $\hat{f}$ be the function given by formula* (2.3)*, i.e.*

$$\hat{f}(t, y) = [f_t]^*(y) \quad \textit{for all } t \in \mathbb{R}^n,\ y \in \mathbb{R}^k.$$

Then $\hat{f}$ is a real function which is concave with respect to the first variable and is convex with respect to to the second one.

Proof. By Proposition 2.1 we have that for all $t \in \mathbb{R}^n$, $\hat{f}_t = [f_t]^*$ is a real convex function of $\mathbb{R}^k$. It remains to prove that $t \mapsto \hat{f}(t, y)$ is concave for every $y \in \mathbb{R}^k$.

Let $y \in \mathbb{R}^k$ be fixed; then

$$\hat{f}(t, y) = -\inf_{x \in \mathbb{R}^k} g(t, x) \quad \text{for all } t \in \mathbb{R}^n$$

where g is the function given by $g(t, x) = f(t, x) - \langle y; x \rangle$ for all t, x. Since g is convex and satisfies $\lim\limits_{|x| \to \infty} g(t, x) = +\infty$ for every t, it is enough to apply the following lemma. □

Lemma 2.6. *Let $g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a convex function such that*

$$\lim_{|x| \to \infty} g(t, x) = +\infty \quad \textit{for every } t \in \mathbb{R}^n. \tag{2.5}$$

Then the function h given by $t \mapsto \inf\{g(t, x) : x \in \mathbb{R}^k\}$ is a convex real function of $\mathbb{R}^n$.

Proof. Let $t \in \mathbb{R}^n$ be fixed. By (2.5) we have that g_t is a coercive convex function and then it admits at least one minimum point x. Then $h(t) = g(t, x)$ and $0 \in \partial g_t(x)$ (statement (ii) of Proposition 1.6). By Proposition 2.4 there exists $v \in \mathbb{R}^n$ such that $(v, 0) \in \partial g(t, x)$ and this means that

$$g(t', x') \geq g(t, x) + \langle v, t' - t \rangle = h(t) + \langle v, t' - t \rangle \quad \text{for all } (t', x') \in \mathbb{R}^n \times \mathbb{R}^k$$

and then

$$h(t') \geq h(t) + \langle v, t' - t \rangle \quad \text{for all } t' \in \mathbb{R}^n.$$

Hence h is a convex function (cf. Remark 1.3). □

Let Ω be an open subset of $\mathbb{R}^n$, we recall that function g in the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^k)$ belongs to $BV^2(\Omega, \mathbb{R}^k)$ when its distributional second order derivative is (represented by) a finite Radon measure on Ω (see for instance [8] and [9]), and this means that for all integers i, j, h with $1 \leq i, j \leq n$ and $1 \leq h \leq k$, there exists a finite Radon measure on Ω, which we denote by $D_{ij} g_h$, such that

$$\int_\Omega \phi \, d(D_{ij} g_h) = \int_\Omega (D_{ij} \phi) g_h \, d\mathscr{L}_n \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

A function g belongs to $BV^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^k)$ when it belongs to $BV^2(\Omega, \mathbb{R}^k)$ for every bounded open set $\Omega \subset \mathbb{R}^n$.

Lemma 2.7. *Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a convex function such that $f(t, x) \geq |x|^2$ for all t, x. Let $y \in \mathbb{R}^k$ and $\varepsilon > 0$ be given and let S be the set of all points $(t, x) \in \mathbb{R}^n \times \mathbb{R}^k$ such that $\pi[\partial f(t, x)] \supset B(y, 2\varepsilon)$. Then S is included in the graph of a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^k$ which belongs to BV^2_{loc}.*

Proof. Let $g = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ be the function given by (cf. formula (2.4))

$$g_i(t) = \frac{1}{\varepsilon} [\hat{f}(t, y + \varepsilon e_i) - \hat{f}(t, y)] \quad \text{for all } t \in \mathbb{R}^n \text{ and } i = 1, \ldots, k.$$

where $\hat{f}$ is given as in formula (2.3) and Proposition 2.5, and $\{e_1, \dots, e_k\}$ is the standard basis of $\mathbb{R}^k$. By proposition 2.4, $\pi[\partial f(t,x)] = \partial f_t(x)$ for every t, x and then $(t,x) \in S$ yields $\partial f_t(x) \supset B(y, 2\varepsilon)$ and Proposition 2.3 yields $x = g(t)$ because $\hat{f}_t = [f_t]^*$ for all t.

By Proposition 2.5, for every $y \in \mathbb{R}^k$, $t \mapsto \hat{f}(t,y)$ is a concave real function and then it is locally Lipschitz (see [10], Theorem 10.4) and belongs to $BV^2_{\text{loc}}(\mathbb{R}^n)$ (see [9]). Hence g too is locally Lipschitz and belongs to $BV^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^k)$. $\square$

Now we want to use Lemma 2.7 to prove that given a convex function f of $\mathbb{R}^n \times \mathbb{R}^k$, the set of all points (t,x) such that $\pi[\partial f(t,x)]$ has dimension k is a C^2 rectifiable set. In order to do this we need two preliminary lemmas.

Lemma 2.8. *Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function in BV^2_{loc}. Then for every $R > 0$ and every $\varepsilon > 0$ there exists a function $u : \mathbb{R}^n \to \mathbb{R}^k$ of class C^2 and an open set A such that $\mathscr{L}_n(A) < \varepsilon$ and $u(t) = f(t)$ for all $t \in B_R \setminus A$, where B_R is the closed ball with center 0 and radius R (cf.* [5]*, section 3).*

Proof. It is enough to consider the case $k = 1$ only.

Since g is a (continuous) function in BV^2_{loc}, by statement (i) of Theorem 1 of [8], for $\mathscr{L}_n$ almost every point of $\mathbb{R}^n$ g has a second order $L^{n/n-2}$ differential (an ordinary second order differential if $n \le 2$) and this means that for $\mathscr{L}_n$ almost every $x \in \mathbb{R}^n$ there exists a polynomial P_x with degree less than or equal to 2 such that

$$\Big(\fint_{B(x,r)} |g(t) - P_x(t)|^{n/n-2} dt \Big)^{n-2/n} = o(r^2) \tag{2.6}$$

where the barred integral stands for the mean on the set $B(x,r)$ (with respect to Lebesgue measure) and the left term of (2.6) must be replaced by $\sup \big\{ |g(t) - P_x(t)| : t \in B(x,r) \big\}$ if $n \le 2$.

Then $P_x(x) = g(x)$ for all x such that (2.6) holds because g is continuous and by Egoroff's theorem we may find an open set A such that $\mathscr{L}_n(A) \le \varepsilon$ and (2.6) holds uniformly in x for all $x \in B_R \setminus A$, where B_R is the closed ball with center 0 and radius R. Hence Theorem 3.6.3 (and definition 3.5.3) of [12] yields a function u of class C^2 such that $u(x) = g(x)$ for all $x \in B_R \setminus A$ (see also [7]). $\square$

Lemma 2.9. *Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a locally Lipschitz function which belongs to BV^2_{loc}. Then the graph of g is a $(\mathscr{H}^n, n)$ rectifiable subset of $\mathbb{R}^n \times \mathbb{R}^k$ of class C^2.*

Proof. It is enough to show that for every $R > 0$ the set $\big\{ (t, g(t)) : t \in B(0,R) \big\}$ is a C^2 rectifiable set.

Let $R > 0$ be fixed. By Lemma 2.8 we may find functions $u_m : \mathbb{R}^n \to \mathbb{R}^k$ of class C^2 and open sets $A_m \subset B(0,R)$ for $m = 1, 2, \dots$ such that $g(x) = u_m(x)$ for all $x \in B(0,R) \setminus A_m$ and $\mathscr{L}_n(\cap A_m) = 0$.

Set $B = \cap A_m$. Then the graph of the restriction of g on $B(0,R)$ may be written as $D_1 \cup D_2$ where D_1 and D_2 are the sets of all points $(t, g(t))$ with $t \in B(0,R) \setminus B$ and $t \in B$ respectively. Since D_1 is included in the union of the graphs of the functions u_m (which are n-dimensional submanifolds of class C^2 of $\mathbb{R}^n \times \mathbb{R}^k$), it is enough to prove that $\mathscr{H}^n(D_2) = 0$.

But $D_2 = g'(B)$ where $g' : B(0,R) \to \mathbb{R}^n \times \mathbb{R}^k$ is the Lipschitz function given by $g'(t) = (t, g(t))$ for all $t \in B(0,R)$. Hence a well-known formula gives

$$\mathscr{H}^n(D_2) = \mathscr{H}^n(g'(B)) \le (\text{Lip}(g'))^n \mathscr{L}_n(B) = 0. \qquad \square$$

Theorem 2.10. *Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a convex function and let S be the set of all points (t,x) such that $\pi[\partial f(t,x)]$ has dimension k. Then S is a $(\mathscr{H}^n, n)$ rectifiable set of class C^2.*

Proof. We begin with proving that this statement holds when f satisfies $f(t,x) \geq |x|^2$ for all t, x.

For every $y \in \mathbb{R}^k$ and every $\varepsilon > 0$ let $S(y,\varepsilon)$ be the set of all points (t,x) such that $\pi[\partial f(t,x)] \supset B(y, 2\varepsilon)$ and let $\mathscr{F}$ be a countable dense subset of $\mathbb{R}^k$. Since a convex set in $\mathbb{R}^k$ has dimension k if and only if its interior is not empty, S is included in the (countable) union of all $S(y, 1/m)$ with $y \in \mathscr{F}$ and m positive integer. Hence it is enough to prove that $S(y,\varepsilon)$ is a C^2 rectifiable set for every $y \in \mathbb{R}^k$ and $\varepsilon > 0$.

Let y and ε be fixed. By Lemma 2.7 there exists a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^k$ which belongs to BV^2_{loc} such that S is included in the graph of g and then $S(y,\varepsilon)$ is a C^2 rectifiable set by Lemma 2.9.

Suppose now that f does not satisfy $f(t,x) \geq |x|^2$ for all t, x. Since f is convex, there exists an affine function λ with gradient w such that $f \geq \lambda$ everywhere and then we may consider the function $f' : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ given by

$$f'(t,x) = f(t,x) - \lambda(t,x) + |x|^2 \quad \text{for all } t, x;$$

f' is a convex function such that, for all t, x, $f'(t,x) \geq |x|^2$, $\partial f'(t,x) \supset \partial f(t,x) - w + (0, 2x)$ (statement (iii) of Proposition 1.6) and $\pi[\partial f'(t,x)] \supset \pi[\partial f(t,x)] - \pi(w) + 2x$. Hence $\pi[\partial f(t,x)]$ has dimension k only if $\pi[\partial f'(t,x)]$ has dimension k and then it is enough to apply Theorem 2.10 to the function f'. □

Proof of Theorem 1. Let k be a fixed integer $0 < k < h$.

Let E be a k-dimensional linear subspace of $\mathbb{R}^h$, let π_E be the projection of $\mathbb{R}^h$ on E and set

$$S(E) := \big\{ x : \dim\big[\pi_E(\partial f(x))\big] = k \big\}.$$

Then we may find a linear isometry $\Phi : \mathbb{R}^{h-k} \times \mathbb{R}^k \to \mathbb{R}^h$ which takes $\{0\} \times \mathbb{R}^k$ into E and $\mathbb{R}^{h-k} \times \{0\}$ into $E^\perp$. Hence we may apply Theorem 2.10 to the function $f \circ \Phi$ to obtain that $S(E)$ is an $(\mathscr{H}^{h-k}, h-k)$ rectifiable set of class C^2 (we must use the obvious identity $\partial f(\Phi(x)) = \Phi^*(\partial(f \circ \Phi)(x))$, see [10], Theorem 23.9).

Furthermore, let $\mathscr{F}$ be a countable dense subset of the Grassmann manifold of all k-planes of $\mathbb{R}^h$. Then a convex set C has dimension greater than or equal to k if and only if there exists $E \in \mathscr{F}$ such that $\pi_E(C)$ has dimension k. Hence

$$\Sigma^k(f) := \big\{ x : \dim(\partial f(x)) \geq k \big\} = \bigcup_{E \in \mathscr{F}} S(E)$$

and then $\Sigma^k(f)$ is a $(\mathscr{H}^n, n)$ rectifiable set of class C^2. □

Lemma 2.11. *Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a function of class C^2 with bounded second order derivative. Then there exists a convex function $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ such that $\Sigma^k(f)$ includes the graph of g.*

Proof. The proof is divided in two steps.

First we suppose that $k = 1$. Let c be a constant such that $2c \geq |D^2 g(t)|$ for all $t \in \mathbb{R}^n$ and consider the function $t \mapsto c|t|^2 + g(t)$. This is a function of class C^2, its Hessian matrix in every t is $2cI + D^2 g(t)$ and for every $v \in \mathbb{R}^n$,

$$\big\langle (2cI + D^2 g(t))v; v \big\rangle \geq 2c|v|^2 - |D^2 g(t)|\,|v|^2 \geq 0.$$

This means that $2cI + D^2g(t)$ is a positive semi-definite quadratic form for every t and then $t \mapsto c|t|^2 + g(t)$ is a convex function (cf. [10], Theorem 4.5). Now take

$$f(t,x) := (x + c|t|^2) \vee (g(t) + c|t|^2) \quad \text{for all } t, x \in \mathbb{R}^n \times \mathbb{R}: \tag{2.7}$$

f is convex because it is the supremum of two convex functions and for every t we have that $f_t(x)$ is equal to $x + c|t|^2$ when $t \geq g(t)$ and to $g(t) + c|t|^2$ when $t \leq g(t)$. Hence

$$\partial f_t(x) = \begin{cases} \{1\} & \text{if } x > g(t), \\ [0,1] & \text{if } x = g(t), \\ \{0\} & \text{if } x < g(t), \end{cases}$$

$\pi[\partial f(t, g(t))] = [0,1]$ for all t by Proposition 2.4 and then $\Sigma^1(f)$ includes the graph of g. Let now k be taken without restriction and write $g = (g_1, \dots, g_k)$.

Take c such that $2c \geq |D^2 g_i(t)|$ for all $t \in \mathbb{R}^n$, $i = 1, \dots, k$, and then set (cf. formula (2.7))

$$f_i(t,s) := (s + c|t|^2) \vee (g_i(t) + c|t|^2) \quad \text{for all } (t,s) \in \mathbb{R}^n \times \mathbb{R}:$$

we have just proved that every f_i is a convex function of $\mathbb{R}^n \times \mathbb{R}$ which satisfies $\pi[\partial f_i(t, (g_i(t))] = [0,1]$ for all $t \in \mathbb{R}^n$. Eventually we take

$$f(t,x) := \sum_{i=1}^k f_i(t, x_i) \quad \text{for all } (t,x) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Then f is a convex function of $\mathbb{R}^n \times \mathbb{R}^k$ such that $\pi[\partial f(t, (g(t))] = [0,1]^k$ for all $t \in \mathbb{R}^n$ and then $\Sigma^k(f)$ includes the graph of g. $\square$

Proof of Theorem 2. We say that a set $G \subset \mathbb{R}^h$ is a graph if there exists a function $g : \mathbb{R}^{h-k} \to \mathbb{R}^k$ of class C^2 with bounded second order derivative, and a linear isometry Φ on $\mathbb{R}^h$ such that $\Phi(G) = \big\{(t,x) \in \mathbb{R}^h : \ t \in \mathbb{R}^{h-k}, x = g(t)\big\}$. If D is a subset of $\mathbb{R}^h$ which may be covered by countably many $(h-k)$-dimensional submanifolds of $\mathbb{R}^h$ of class C^2, we may find graphs G_m for $m = 1, 2, \dots$ such that $\big[\cup G_m\big] \supset D$.

By Lemma 2.11, for every m there exists a convex function f_m such that $\Sigma^k(f_m) \supset G_m$. Then we may find positive real numbers ε_m so that

$$\varepsilon_m \sup_{x \in B(0,m)} |f_m(x)| \leq 2^m \qquad \text{for all } m.$$

Hence the series $\sum \varepsilon_m f_m$ converges uniformly on every compact set to a convex function f and since $\big[\cup \Sigma^k(f_m)\big] \supset \big[\cup G_m\big] \supset D$, the proof is complete if we show that

$$\Sigma^k(f) \supset \Big[\bigcup_m \Sigma^k(f_m)\Big].$$

Let m and $x \in \Sigma^k(f_m)$ be fixed; we may write $f = \varepsilon_m f_m + f'_m$ where f'_m is the convex function given by the sum of all $\varepsilon_n f_n$ with $n \neq m$. Then the dimension of $\partial f(x)$ is not less than the dimension of $\partial f_m(x)$ (statement (iii) of Proposition 1.6) and so $x \in \Sigma^k(f)$. $\square$

Proof of Theorem 3. Let C be a closed convex subset of $\mathbb{R}^h$ of dimension h and take $f(x) := \mathrm{dist}(,Cx)$ for all $x \in \mathbb{R}^h$. Then f is a convex function (cf. [10], sections 4 and 5) and we claim that

$$\partial f(x) \supset \{ty : t \in [0,1], y \in \mathscr{N}(C,x)\} \quad \text{for all } x \in \partial C. \tag{2.8}$$

Hence $\Sigma^{k+1}(f) \supset \Sigma^k(C)$ and the thesis follows from Theorem 1.

Let $x \in \partial C$, $t \in [0,1]$ and $y \in \mathscr{N}(C,x)$ be fixed.

We want to show that $f(x') \geq \langle ty; x'-x\rangle$ for all $x' \in \mathbb{R}^h$. By definition of $\mathscr{N}(C,x)$, $\langle y; x-x''\rangle \geq 0$ for all $x'' \in C$; then the half-space $M := \{x'' : \langle y; x-x''\rangle \geq 0\}$ includes C and this yields, for every $x' \in \mathbb{R}^h$,

$$f(x') := \mathrm{dist}(x',C) \geq \mathrm{dist}(x',M) = 0 \vee \langle y, x'-x\rangle \geq \langle ty, x'-x\rangle. \qquad \square$$

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