Asymptotic Analysis 8 (1994), 247-258
North-Holland

Singular perturbation problems with a compact support semilinear term

Giovanni Alberti and Luigi Ambrosio
Istituto di Matematiche Applicate, v. Bonanno 25/B, 56126 Pisa, Italy

Giuseppe Buttazzo
Dipartimento di Matematica, v. Buonarroti 2, 56127 Pisa, Italy

Received 16 September 1991

Abstract. In this paper we study the asymptotic behavior of the functional

$$F_\varepsilon(u) := \int_\Omega \Big[\varepsilon|\nabla u|^2 + \varepsilon^{-3}\beta\Big(\frac{u}{\varepsilon}\Big)\Big]\,dx$$

where β is a non-negative lower semicontinuous function with compact support. When ε tends to 0, the limit functional corresponds to a least area problem with an obstacle.

Introduction

In the last years, many papers have been devoted to the asymptotic behavior of the functionals

$$\int_\Omega \Big[\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\Big]\,dx$$

as $\varepsilon \downarrow 0$. The first result, proved by Modica & Mortola in [16] (see also Modica [14]), deals with real valued functions u. If the function W is nonnegative and vanishes only at $a, b \in \mathbb{R}$ with $a < b$, they proved that the limit functional, in a suitable variational sense, is

$$2\left(\int_a^b W^{1/2}(s)\,ds\right) P\big(\{u = a\}, \Omega\big) \tag{1}$$

if $u(x) \in \{a, b\}$ almost everywhere, and $+\infty$ otherwise. In (1), $P\big(\{u = a\}, \Omega\big)$ denotes the perimeter in Ω of the interface between the regions $\{u = a\}$ and $\{u = b\}$. The result has been later extended in many directions:

(a) by introducing additional constraints [13, 15];
(b) by considering vector valued functions and different kinds of zero sets of W [1, 3, 9, 17, 20];
(c) by considering fully non-linear integrands of the form $\varepsilon^{-1}f(x, u, \varepsilon\nabla u)$ [4, 18, 19].

In this paper we deal with functionals of the form

$$F_\varepsilon(u) = \int_\Omega \left[\varepsilon|\nabla u|^2 + \varepsilon^{-3}\beta\left(\frac{u}{\varepsilon}\right)\right] dx \ ,$$

where β is a nonnegative lower semicontinuous function with compact support. We show in Theorem 1.3 that the limit functional is

$$F(u) = 2\left(\int_{-\infty}^{+\infty} \beta^{1/2}(s)\, ds\right) \min\big\{P(B,\Omega) \ : \ \{u>0\} \subset B \subset \{u \geq 0\}\big\} \ . \qquad (2)$$

The meaning of the constraint condition on B in (2) is the following: if the set $\{u=0\}$ is negligible, then the only admissible choice for B is $B = \{u > 0\}$, which gives

$$F(u) = 2\left(\int_{-\infty}^{+\infty} \beta^{1/2}(s)\, ds\right) P\big(\{u>0\},\Omega\big) \ .$$

On the contrary, if the set $\{u = 0\}$ is not negligible, then we have to choose a subset H of its in order to get, for $B = \{u > 0\} \cup H$, the least perimeter $P(B,\Omega)$ (a comprehensive treatment of the theory of functions of bounded variation and sets of finite perimeter can be found in [8] and [10]).

We also study (see Theorem 1.1 and Proposition 1.6) the asymptotic behavior of the minimizers of the problems

$$\int_\Omega \left[\varepsilon|\nabla u|^2 + \varepsilon^{-3}\beta\left(\frac{u}{\varepsilon}\right)\right] dx + \int_\Omega g(x,u)\, dx$$

for suitable functions $g : \Omega \times \mathbb{R} \to \mathbb{R}$.

1. – Statement of the Results

Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. As usual we denote by $H^1(\Omega)$ the Sobolev space of all functions $u \in L^2(\Omega)$ with distributional derivatives in $L^2(\Omega)$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a nonnegative lower semicontinuous function with compact support and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Borel function such that

(i) for a.e. $x \in \Omega$ the function $g(x,\cdot)$ is continuous on $\mathbb{R}$;

(ii) there exist $p_1, p_2 \in L^1(\Omega)$ and $C_1, C_2 > 0$ such that for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$

$$p_1(x) + C_1|s|^2 \leq g(x,s) \leq p_2(x) + C_2|s|^2.$$

Then, for every $\varepsilon > 0$, the well-known direct method of the Calculus of Variations (see for instance [6]) ensures the existence of a solution u_ε of the problem

$$\min\left\{\int_\Omega \left[\varepsilon|\nabla u|^2 + \varepsilon^{-3}\beta\left(\frac{u}{\varepsilon}\right)\right] dx + \int_\Omega g(x,u)\, dx \ : \ u \in H^1(\Omega)\right\} \ . \qquad (\mathscr{P}_\varepsilon)$$

The present paper is devoted to the characterization of the asymptotic behavior of the solutions u_ε and of the minimum values $\min(\mathscr{P}_\varepsilon)$ as $\varepsilon \to 0$. In order to describe the limit problem, we introduce the functions

$$g_-(x) = \min\big\{g(x,t) \ : \ t \leq 0\big\}, \qquad g_+(x) = \min\big\{g(x,t) \ : \ t \geq 0\big\} \ ,$$

we set

$$c = 2\int_{-\infty}^{+\infty} \beta^{1/2}(s)\, ds \ .$$

Let $(\mathscr{P})$ be the following problem

$$\min\left\{cP(B,\Omega) + \int_B g_+\, dx + \int_{\Omega\setminus B} g_-\, dx \ : \ B \text{ Borel set}\right\} \ . \qquad (\mathscr{P})$$

Well known compactness and lower semicontinuity properties of sets of finite perimeter (see for instance Giusti [10]) ensure that the infimum of $(\mathscr{P})$ is attained. The following theorem is the main result of this paper.

Theorem 1.1. *We have*

$$\lim_{\varepsilon\to 0^+} \big[\min(\mathscr{P}_\varepsilon)\big] = \min(\mathscr{P}). \qquad (1.2)$$

Moreover, given any sequence (ε_h) converging to 0 and any sequence (u_h) of solutions of $(\mathscr{P}_{\varepsilon_h})$, there exist a subsequence (u_{h_k}) and a minimizer B of $(\mathscr{P})$ such that, for almost every $x \in \Omega$, either $u_{h_k}(x)$ converges to 0, or $\operatorname{sign} u_{h_k}(x)$ converges to $\chi_B(x) - \chi_{\Omega\setminus B}(x)$. Finally, if

$$g_-(x) < g(x,0) \quad \text{and} \quad g_+(x) < g(x,0) \qquad \text{a.e. in } \Omega, \qquad (1.3)$$

then

$$\operatorname{sign}(u_{h_k}) \to \chi_B - \chi_{\Omega\setminus B} \qquad \text{a.e. in } \Omega.$$

Remark 1.2. By using (ii) and the projection theorem (see for instance Castaing & Valadier [7], Theorem III.22), it can be shown that g_- and g_+ belong to $L^1(\Omega)$. Moreover, Aumann's measurable selection theorem (see [7], Theorem III.23) allows us to find two functions $a, b \in L^2(\Omega)$ such that $a \leq 0$, $b \geq 0$ and

$$g_-(x) = g(x,a(x)) \quad \text{and} \quad g_+(x) = g(x,b(x)) \qquad \text{for a.e. } x \in \Omega.$$

Notice that, if for every $h \in \mathbb{N}$ we set $a_h = (a \wedge h) \vee -h$ and $(b_h \wedge h) \vee -h$, we have

$$g_-(x) = \lim_{h\to+\infty} g(x,a_h(x)) \qquad \text{and} \qquad g_+(x) = \lim_{h\to+\infty} g(x,b_h(x))$$

in the strong convergence of $L^1(\Omega)$.

If we denote by $\mathscr{M}(\Omega)$ the space of all real valued Borel functions defined in Ω and endow $\mathscr{M}(\Omega)$ with a distance inducing convergence in measure, then Theorem 1.1

can be partially rephrased in terms of Γ-convergence (see for instance [2] for the basic definitions) as follows.

Theorem 1.3. *The functionals*

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big[\varepsilon|\nabla u|^2 + \varepsilon^{-3}\beta\Big(\frac{u_\varepsilon}{\varepsilon}\Big)\Big]\,dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{if } u \in \mathscr{M}(\Omega)\setminus H^1(\Omega) \end{cases}$$

Γ-converge in $\mathscr{M}(\Omega)$, as $\varepsilon \to 0$, to the functional

$$F(u) = \min\big\{cP(B,\Omega) \ :\ \{u>0\} \subset B \subset \{u \geq 0\}\big\}. \tag{1.4}$$

It is well-known that Γ-convergence and equicoercivity ensure convergence of minimizers to minimizers and of minimum values to minimum values. However, it can be easily seen that the functionals F_ε are not equicoercive in $\mathscr{M}(\Omega)$. For instance, if Ω is the interval $]0,1[$ and $u_\varepsilon(x) = 2 + \sin(x/\sqrt{\varepsilon})$, for every $\varepsilon > 0$ we have $F_\varepsilon(u) \leq C$ for a suitable constant C, but it is impossible to extract subsequences (u_{ε_h}) which converge almost everywhere. In particular, Theorem 1.1 cannot be deduced from Theorem 1.3. In order to prove (1.2) we need to consider a weaker form of convergence (see Proposition 2.1).

Remark 1.4. Since the perimeter $P(B,\Omega)$ is not affected by modifications of B in negligible sets, an equivalent definition of F can be given by requiring the inclusions to hold only almost everywhere. We also remark that $F(u) = cP\big(\{u>0\},\Omega\big)$ if $u \neq 0$ almost everywhere, and $F(u) = 0$ if either $u \geq 0$ or $u \leq 0$ in Ω.

Remark 1.5. When we consider F_ε as functions also of the the domain of integration Ω (and then we write $F_\varepsilon(u,\Omega)$ instead of $F_\varepsilon(u)$), it is interesting to notice that even if all the functionals F_ε are σ-additive measures as functions of Ω, the Γ-limit F does not share this property. In the one dimensional case, it suffices to take a function u equal to 1 in $]0,1/3[$, equal to 0 in $[1/3,2/3]$, and equal to -1 in $]2/3,1[$. Then, $F(u,]0,2/3[) = 0$, $F(u,]1/3,1[) = 0$ but $F(u,]0,1[) = 1$, so that $A \mapsto F(u,A)$ is not an additive set function.

Remark 1.6. In general, there is no hope for uniqueness of the minimizing set B of the problem $(\mathscr{P})$. Indeed take $\Omega =]0,1[$, the same function u of Remark 1.5, and any function g such that $g_+ = -u$ and $g_- = +u$. Then, $\min(\mathscr{P}) = 1/3$ and any set $B =]0,t[$ with $1/3 \leq t \leq 2/3$ is a minimizer. This phenomenon forced us to consider only the convergence of the signs of suitable subsequences of minimizers. The pointwise convergence of the minimizers is ensured under stronger assumptions on g, as the following proposition shows.

Proposition 1.7. *Let us assume that for almost every $x \in \Omega$ the function $g(x,\cdot)$ has a unique minimizer $u_+(x) > 0$ when restricted to the half line $[0,+\infty[$ and a unique minimizer $u_-(x) < 0$ when restricted to the half line $]-\infty,0]$. Then, given any sequence (ε_h) converging to 0 and any sequence (u_h) of minimizers of $(\mathscr{P}_{\varepsilon_h})$, there is a subsequence (u_{h_k}) converging in measure to a function $u \in \mathscr{M}(\Omega)$ such that $u(x) \in \big\{u_-(x), u_+(x)\big\}$ for any x and the set $\{u>0\}$ minimizes $(\mathscr{P})$.*

2. – Proof of the Results

The proof of Theorems 1.1 and 1.3 essentially relies on the following two propositions. We define

$$m = \inf\big\{s \ :\ \beta(s) \neq 0\big\}, \qquad M = \sup\big\{s \ :\ \beta(s) \neq 0\big\}.$$

Proposition 2.1. *Let $(\varepsilon_h) \subset]0,+\infty[$ be converging to 0, and let $(u_h) \subset H^1(\Omega)$, $u \in \mathscr{M}(\Omega)$ such that*

$$\limsup_{h\to+\infty} \frac{u_h}{\varepsilon_h} \geq M$$

almost everywhere in the set $\big\{x \in \Omega \ :\ u(x) > 0\big\}$ and

$$\liminf_{h\to+\infty} \frac{u_h}{\varepsilon_h} \leq m$$

almost everywhere in the set $\big\{x \in \Omega \ :\ u(x) < 0\big\}$. Then,

$$\liminf_{h\to+\infty} \int_\Omega \Big[\varepsilon_h|\nabla u_h|^2 + \varepsilon_h^{-3}\beta\Big(\frac{u_h}{\varepsilon_h}\Big)\Big]\,dx \geq F(u),$$

with $F(u)$ defined in (1.4).

Proof. We may assume, possibly passing to subsequences, that the liminf in the statement is a finite limit, say L. Let I be the primitive of $\beta^{1/2}$ which vanishes for $t \leq m$, and let $v_h = 2I(u_h/\varepsilon_h)$. By the chain rule and the inequality $2ab \leq a^2 + b^2$ we infer

$$\int_\Omega |\nabla v_h|\,dx = \frac{2}{\varepsilon_h}\int_\Omega \beta^{1/2}\Big(\frac{u_h}{\varepsilon_h}\Big)|\nabla u_h|\,dx \leq \int_\Omega \Big[\varepsilon_h|\nabla u_h|^2 + \varepsilon_h^{-3}\beta\Big(\frac{u_h}{\varepsilon_h}\Big)\Big]\,dx.$$

Since $0 \leq v_h \leq c$, it follows that the sequence v_h is bounded in $BV(\Omega)$, and we can assume, by Rellich's theorem, that it converges almost everywhere to a function $v \in BV(\Omega)$. Moreover passing to the limit as $h \to +\infty$ in the foregoing inequality, and recalling the lower semicontinuity of the total variation (see for instance [10], Theorem 1.9), we get $|Dv|(\Omega) \leq L$. We need only to show that $F(u) \leq |Dv|(\Omega)$. Since $0 \leq v \leq c$, by using the Fleming-Rishel formula (see [10], Theorem 1.23)

$$|Dv|(\Omega) = \int_0^c P\big(\{x \in \Omega \ :\ v(x) > t\},\Omega\big)\,dt$$

we can find $t \in]0,c[$ such that $B = \{x \in \Omega \ :\ v > t\}$ is a set of finite perimeter in Ω and $cP(B,\Omega) \leq |Dv|(\Omega)$. By our assumption on (u_h), for almost every x in the set $\big\{x \in \Omega \ :\ u(x) > 0\big\}$ the sequence $v_h(x)$ converges to c. In particular, $\big\{x \in \Omega \ :\ u(x) > 0\big\} \subset B$ up to a negligible set. A similar argument shows that $B \subset \big\{x \in \Omega \ :\ u(x) \geq 0\big\}$ up to a negligible set. Hence,

$$F(u,\Omega) \leq cP(B,\Omega) \leq |Dv|(\Omega) \leq L,$$

and the proposition is proved. □

In the proposition below it will be very useful the so-called coarea formula (see for instance [8], 3.2.12)

$$\int_B |\nabla\varphi|\,dx = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}\big(\{x\in B\ :\ \varphi(x)=t\}\big)\,dt \tag{2.1}$$

which holds for every Borel set $B\subset\mathbb{R}^n$ and every Lipschitz function φ.

Proposition 2.2. *For any function $u\in\mathscr{M}(\Omega)$ it is possible to find functions $(u_\varepsilon)\subset H^1(\Omega)$ converging to u almost everywhere, such that*

$$\lim_{\varepsilon\to0^+}\int_\Omega \left[\varepsilon|\nabla u_\varepsilon|^2+\varepsilon^{-3}\beta\left(\frac{u_\varepsilon}{\varepsilon}\right)\right]dx = F(u). \tag{2.2}$$

Moreover, the sequence u_ε is bounded in $L^\infty(\Omega)$ if $u\in L^\infty(\Omega)$.

Proof. The proof is achieved in three steps. We define the functional

$$F^+(u)=\inf\left\{\limsup_{\varepsilon\to0^+}\int_\Omega \left[\varepsilon|\nabla u_\varepsilon|^2+\varepsilon^{-3}\beta\left(\frac{u_\varepsilon}{\varepsilon}\right)\right]dx\ :\ u_\varepsilon\to u \text{ a.e. in } \Omega\right\}.$$

A diagonal argument shows that the infimum in the definition of F^+ is achieved, and F^+ is lower semicontinuous with respect to the almost everywhere convergence. Then, the statement of the proposition is equivalent to the inequality $F^+(u)\le F(u)$ (the "liminf" inequality in (2.2) follows by Proposition 2.1).

Step 1. We assume that u is bounded, $u\neq0$ almost everywhere, and there is a bounded open set $C\subset\mathbb{R}^n$ with a smooth boundary such that

$$C\cap\Omega=\big\{x\in\Omega\ :\ u(x)>0\big\}. \tag{2.3}$$

We denote by τ the distance function from ∂C, by C_η the set $\big\{x\in C\ :\ \tau(x)<\eta\big\}$, and by K a positive number such that $[m,M]\subset[-K,K]$. We want to define $u_\varepsilon=a_\varepsilon\sigma_\varepsilon$, where a_ε and σ_ε are functions which fulfil suitable conditions. Let $\psi_\varepsilon\in C^\infty(\mathbb{R}^n)$ such that $0\le\psi_\varepsilon\le1$ and

$$\psi_\varepsilon\equiv0 \ \text{ in } C_{\sqrt{\varepsilon}} \qquad \psi_\varepsilon\equiv1 \ \text{ in } \mathbb{R}^n\setminus C_{2\sqrt{\varepsilon}} \qquad |\nabla\psi_\varepsilon|\le\frac{2}{\sqrt{\varepsilon}}\,.$$

We define

$$a_\varepsilon(x)=\psi_\varepsilon(x)\int_\Omega \frac{|u|(y)}{K\rho_\varepsilon^n}h\left(\frac{y-x}{\rho_\varepsilon}\right)dy+\varepsilon,$$

where h is any fixed convolution kernel. The functions a_ε are in $C^\infty\big(\mathbb{R}^n\big)$, are greater than ε, converge to $|u|/K$ almost everywhere, and are equal to ε on $C_{\sqrt{\varepsilon}}$. In addition, the upper bound on $|\nabla\psi_\varepsilon|$ yields

$$\lim_{\varepsilon\to0^+}\varepsilon\int_\Omega|\nabla a_\varepsilon|^2\,dx=0 \tag{2.4}$$

provided ρ_ε converges to 0 slowly enough. Now we turn to the construction of σ_ε. Let $\eta_\varepsilon>0$, γ_ε be solutions of the problem

$$\gamma'=\frac{\beta^{1/2}\big(\gamma(t)\big)+\varepsilon^{3/2}}{\varepsilon^3},\qquad \gamma(0)=-K,\qquad \gamma(\eta_\varepsilon)=K.$$

By a change of variables we infer

$$\eta_\varepsilon=\int_{-K}^{K}\frac{\varepsilon^3}{\beta^{1/2}(t)+\varepsilon^{3/2}}\,dt\le 2K\varepsilon^{3/2}. \tag{2.5}$$

The functions σ_ε are defined as follows:

$$\sigma_\varepsilon(x)=\begin{cases}K & \text{if } x\in C \text{ and } \tau(x)\ge\eta_\varepsilon\\ \gamma_\varepsilon\big(\tau(x)\big) & \text{if } x\in C \text{ and } \tau(x)<\eta_\varepsilon\\ -K & \text{if } x\notin C.\end{cases}$$

By (2.5), for ε small enough the functions a_ε are equal to ε in C_{η_ε} and, setting $u_\varepsilon=a_\varepsilon\sigma_\varepsilon$, we have $\beta(u_\varepsilon/\varepsilon)=0$ outside C_{η_ε}. By (2.4) we infer

$$\limsup_{\varepsilon\to0^+}\int_\Omega\left[\varepsilon|\nabla u_\varepsilon|^2+\varepsilon^{-3}\beta\left(\frac{u_\varepsilon}{\varepsilon}\right)\right]dx=\limsup_{\varepsilon\to0^+}\int_{B_\varepsilon}\left[\varepsilon^3|\nabla\sigma_\varepsilon|^2+\varepsilon^{-3}\beta(\sigma_\varepsilon)\right]dx,$$

where we have set for simplicity $B_\varepsilon=C_{\eta_\varepsilon}$. Since $|\nabla\tau|=1$ almost everywhere (see for instance [8], 3.2.34), by using the coarea formula (2.1) and our special choice of γ_ε we get

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\int_{B_\varepsilon}\left[\varepsilon^3|\nabla\sigma_\varepsilon|^2+\varepsilon^{-3}\beta(\sigma_\varepsilon))\right]dx=\\ &=\limsup_{\varepsilon\to0^+}\int_0^{\eta_\varepsilon}\left[\varepsilon^3|\gamma_\varepsilon'|^2+\varepsilon^{-3}\beta(\gamma_\varepsilon)\right]\mathscr{H}^{n-1}\big(\{x\in C\cap\Omega\ :\ \tau(x)=t\}\big)\,dt=\\ &=\limsup_{\varepsilon\to0^+}\int_0^{\eta_\varepsilon}\left[2\gamma_\varepsilon'\beta^{1/2}(\gamma_\varepsilon)+1\right]\mathscr{H}^{n-1}\big(\{x\in C\cap\Omega\ :\ \tau(x)=t\}\big)\,dt.\end{aligned}$$

Since ∂C is smooth, for t small enough we have $\{x\in C\ :\ \tau(x)=t\}=\{y+t\nu(y)\ :\ y\in\partial C\}$, where ν is the inner normal to ∂C. In particular,

$$\begin{aligned}\mathscr{H}^{n-1}\big(\{x\in C\cap\Omega\ :\ \tau(x)=t\}\big)&\le\mathscr{H}^{n-1}\big(\{y+t\nu(y)\ :\ y\in\partial C,\ \operatorname{dist}(y,\Omega)<t\}\big)\\ &\le(1+Lt)^{n-1}\mathscr{H}^{n-1}\big(\{x\in\partial C\ :\ \operatorname{dist}(x,\Omega)<t\}\big).\end{aligned}$$

In the above formula, $1+Lt$ is greater than the Lipschitz constant of the map $y\mapsto y+t\nu(y)$ defined in ∂C, and L is a suitable constant depending on the curvatures of C. Finally, assembling the previous inequalities we get

$$\limsup_{\varepsilon\to0^+}\int_\Omega\left[\varepsilon|\nabla u_\varepsilon|^2+\varepsilon^{-3}\beta\left(\frac{u_\varepsilon}{\varepsilon}\right)\right]dx\le2\left(\int_{-\infty}^{+\infty}\beta^{1/2}(s)\,ds\right)\mathscr{H}^{n-1}(C\cap\overline{\Omega}). \tag{2.6}$$

Step 2. We now remove the regularity assumption on the set $C = \{x \in \Omega \ : \ u(x) > 0\}$, assuming only that $P(C,\Omega) < +\infty$. By using Proposition 2.16 and Remark 2.13 of [10], it is possible to extend the characteristic function of C to a function $v \in BV(\mathbb{R}^n)$ with compact support such that $0 \le v \le 1$ and $|Dv|(\partial\Omega) = 0$. Let (ρ_h) be a sequence of mollifiers, and let $v_h = v * \rho_h$; by Sard's theorem, almost every level set of v_h is smooth. Moreover, given any $\eta \in]0, 1/2[$, by the coarea formula (2.1) for every $h \in \mathbb{N}$ we can find $t_h \in]\eta, 1-\eta[$ such that the set

$$C_h = \{x \in \mathbb{R}^n \ : \ v_h(x) > t_h\}$$

is smooth and

$$\mathscr{H}^{n-1}\big(C_h \cap \overline{\Omega}\big) \le \frac{1}{1-2\eta}\int_\Omega |\nabla v_h|\, dx = \frac{1}{1-2\eta}|Dv_h|(\Omega). \tag{2.7}$$

Now we set

$$u_h(x) = \begin{cases} |u|(x) & \text{if } x \in C_h \\ -|u|(x) & \text{if } x \in \Omega \setminus C_h. \end{cases}$$

Since v_h converges to χ_C almost everywhere in Ω, it can be easily seen that u_h converges to u almost everywhere. Since u_h fulfil condition (2.3), by (2.6) and (2.7) we infer

$$F^+(u_h) \le \frac{c}{1-2\eta}|Dv_h|(\Omega).$$

Since $|Dv|(\partial\Omega) = 0$, the sequence $|Dv_h|(\Omega)$ converges to $|Dv|(\Omega) = P(C,\Omega)$ (see for instance [10], Proposition 1.15). By letting $h \to +\infty$ we get

$$F^+(u) \le \frac{1}{1-2\eta}F(u),$$

and the inequality follows by letting $\eta \to 0$.

Step 3. Let B be a minimizing set in the definition of $F(u)$. Let u_h be the functions defined by

$$u_h(x) = \begin{cases} \big(h \wedge u(x)\big) \vee -h & \text{if } u(x) \ne 0 \\ 1/h & \text{if } u(x) = 0 \text{ and } x \in B \\ -1/h & \text{if } u(x) = 0 \text{ and } x \notin B. \end{cases}$$

Since the functions u_h are bounded and are nowhere equal to zero, the first two steps yield

$$F^+(u_h) \le F(u_h) = F(u).$$

By letting $h \to +\infty$ and using the lower semicontinuity of F^+ we obtain the desired inequality. □

Proof of Theorem 1.1

Let B be a minimizer of $(\mathscr{P})$, and let a_h, b_h be the functions in Remark 1.2. We define

$$u_h(x) = \begin{cases} b_h(x) & \text{if } x \in B \\ a_h(x) & \text{if } x \notin B. \end{cases}$$

By Proposition 2.2, for every integer h we can find a sequence (u_ε) converging to u_h almost everywhere, bounded in $L^\infty(\Omega)$, and such that

$$\lim_{\varepsilon\to0^+}\left[\int_\Omega \left[\varepsilon|\nabla u_\varepsilon|^2 + \varepsilon^{-3}\beta\left(\frac{u_\varepsilon}{\varepsilon}\right)\right] dx + \int_\Omega g(x,u_\varepsilon)\, dx\right] =$$

$$= F(u_h) + \int_\Omega g(x,u_h) \le cP(B,\Omega) + \int_\Omega g(x,u_h)\, dx.$$

In particular,

$$\limsup_{\varepsilon\to0^+}\big[\min(\mathscr{P}_\varepsilon)\big] \le \lim_{h\to+\infty}\left[cP(B,\Omega) + \int_\Omega g(x,u_h)\, dx\right] =$$

$$= cP(B,\Omega) + \int_B g_+\, dx + \int_{\Omega\setminus B} g_-\, dx = \min(\mathscr{P}).$$

In order to show the inequality

$$\liminf_{\varepsilon\to0^+}\big[\min(\mathscr{P}_\varepsilon)\big] \ge \min(\mathscr{P}), \tag{2.8}$$

we choose a sequence (ε_h) converging to 0 such that

$$\liminf_{\varepsilon\to0^+}\big[\min(\mathscr{P}_\varepsilon)\big] = \lim_{h\to+\infty}\big[\min(\mathscr{P}_{\varepsilon_h})\big] = L$$

and we assume $L < +\infty$ (the inequality being trivial if $L = +\infty$). Let $u_h \in H^1(\Omega)$ be a minimizer of $(\mathscr{P}_{\varepsilon_h})$, and let $v_h = 2I(u_h/\varepsilon_h)$ be as in the proof of Proposition 2.1. The same argument of Proposition 2.1 shows that the sequence (v_h) is bounded in $BV(\Omega)$. Hence, it it not restrictive to assume that v_h converges almost everywhere to a function $v \in BV(\Omega)$. We define

$$B_1 = \left\{x \in \Omega \ : \ \limsup_{h\to+\infty}\frac{u_h(x)}{\varepsilon_h} \ge M\right\}$$

$$B_2 = \left\{x \in \Omega \ : \ \liminf_{h\to+\infty}\frac{u_h(x)}{\varepsilon_h} \le m\right\}$$

$$u(x) = \begin{cases} 1 & \text{if } x \in B_1 \\ -1 & \text{if } x \in B_2 \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.1 we get

$$\liminf_{h\to+\infty}\int_\Omega \left[\varepsilon_h|\nabla u_h|^2 + \varepsilon_h^{-3}\beta\left(\frac{u_h}{\varepsilon_h}\right)\right] dx \ge F(u). \tag{2.9}$$

If $x \in B_1$ and $v_h(x)$ converges, then necessarily the limit of v_h is equal to c, and

$$\liminf_{h\to+\infty}\frac{u_h(x)}{\varepsilon_h} \ge M,$$

because $2I(t) < c$ for all $t < M$. In particular,

$$\liminf_{h\to+\infty} u_h(x) \geq 0$$

for almost every $x \in B_1$, hence

$$\liminf_{h\to+\infty} \int_{B_1} g(x, u_h)\, dx \geq \int_{B_1} \liminf_{h\to+\infty} g(x, u_h)\, dx \geq \int_{B_1} g_+\, dx.$$

A similar argument gives

$$\liminf_{h\to+\infty} \int_{B_2} g(x, u_h)\, dx \geq \int_{B_2} g_-\, dx.$$

Moreover, for every $x \in \Omega \setminus \big(B_1 \cup B_2\big)$ the sequence (u_h) converges to 0, so that

$$\liminf_{h\to+\infty} \int_{\Omega\setminus(B_1\cup B_2)} g(x, u_h)\, dx \geq \int_{\Omega\setminus(B_1\cup B_2)} g(x, 0)\, dx.$$

The last three inequalities yield

$$\liminf_{h\to+\infty} \int_{\Omega} g(x, u_h)\, dx \geq \int_{B} g_+\, dx + \int_{\Omega\setminus B} g_-\, dx \tag{2.10}$$

for any Borel set B containing B_1 and contained in $\Omega\setminus B_2$. By taking as B the minimizing set in the definition of $F(u)$, the inequality (2.8) follows by (2.9) and (2.10).

This proves (1.2). The last statements of the theorem can be shown by repeating the same argument leading to (2.9) with an arbitrary sequence (ε_h). Finally, if (1.3) holds and if the sequence (u_{h_k}) were converging to 0 in a set of positive measure, then the strict inequality in (2.10) would imply a strict inequality in (2.8), that is a contradiction. □

Proof of Theorem 1.3

By Proposition 2.1 we infer

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq F(u)$$

whenever $u_\varepsilon \to u$ in measure. In fact, sequences converging in measure admit subsequences converging almost everywhere. By Proposition 2.2 we get a sequence (u_ε) converging to u in measure such that

$$\lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = F(u).$$

This proves (see for instance [2]) the Γ-convergence of F_ε to F. □

Proof of Proposition 1.7

By Theorem 1.1, we may assume, with no loss of generality, that $\operatorname{sign}(u_h)$ converges almost everywhere to $\chi_B - \chi_{\Omega\setminus B}$ for a suitable minimizer B of $\mathscr{P}$. We shall prove that u_h converges in measure to the function defined by

$$u(x) = \begin{cases} u_+(x) & \text{if } x \in B \\ u_-(x) & \text{if } x \in \Omega \setminus B. \end{cases}$$

Let $\delta > 0$ be given; by the assumptions on g there exists $\gamma(x) > 0$ such that

$$\begin{aligned} g(x,t) &\geq \gamma(x) + g_+(x) && \text{if } t \geq 0 \text{ and } |t - u_+(x)| \geq \delta \\ g(x,t) &\geq \gamma(x) + g_-(x) && \text{if } t \leq 0 \text{ and } |t - u_-(x)| \geq \delta. \end{aligned}$$

Therefore, setting

$$\Omega_h = \big\{x \in \Omega \;:\; \operatorname{sign} u_h(x) = \operatorname{sign} u(x),\ |u_h(x) - u(x)| \geq \delta\big\}$$

it is easy to obtain

$$\int_\Omega g(x, u_h)\, dx \geq \int_{\Omega_h} \gamma\, dx + \int_{\{u_h\geq 0\}\cap B} g_+\, dx + \int_{\{u_h<0\}\setminus B} g_-\, dx. \tag{2.11}$$

By Theorem 1.1 and Proposition 2.1 we get

$$\limsup_{h\to+\infty} \int_\Omega g(x, u_h)\, dx \leq \int_B g_+\, dx + \int_{\Omega\setminus B} g_-\, dx \tag{2.12}$$

so that, by (2.11) and (2.12),

$$\lim_{h\to+\infty} \int_{\Omega_h} \gamma\, dx = 0.$$

Since $\gamma > 0$ and $\operatorname{sign}(u_h) \to \operatorname{sign}(u)$ a.e. in Ω, this implies

$$\lim_{h\to+\infty} \operatorname{meas}\big(\{x \in \Omega \;:\; |u_h(x) - u(x)| \geq \delta\}\big) = 0$$

and, since $\delta > 0$ is arbitrary, we obtain the convergence in measure of u_h to u. □

Acknowledgements: The third author gratefully acknowledges the Institute for Mathematics and its Applications of the University of Minnesota for its support during the completion of this paper.

References

[1] L. AMBROSIO: *Metric space valued functions of bounded variation.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., (to appear).

[2] H. ATTOUCH: *Variational Convergence for Functions and Operators.* Pitman, Boston, 1984.

[3] S. BALDO: *Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids.* Ann. Inst. H. Poincare Anal. Non Lineaire, **7** (2) (1990), 67–90.

[4] G. BOUCHITTE: *Singular perturbation of variational problems arising from a two-phase transition model.* Appl. Math. Optim., **21** (1990), 289–314.

[5] H. BREZIS: *Analyse Fonctionelle.* Masson, Paris, 1983.

[6] G. BUTTAZZO: *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Res. Notes Math. Ser., **207**, Longman, Harlow, 1989.

[7] C. CASTAING & M. VALADIER: *Convex Analysis and measurable Multifunctions.* Lecture Notes in Math., **590**, Springer-Verlag, Berlin, 1977.

[8] H. FEDERER: *Geometric Measure Theory.* Springer-Verlag, Berlin, 1969.

[9] I. FONSECA & L. TARTAR: *The gradient theory of phase transitions for systems with two potential wells.* Proc. Roy. Soc. Edinburgh, **A-111** (1989), 89–102.

[10] E. GIUSTI: *Minimal Surfaces and Functions of Bounded Variation.* Birkäuser, Boston, 1984.

[11] M. E. GURTIN: *On a theory of phase transitions with interfacial energy.* Arch. Rational Mech. Anal., **87** (1984), 187–212.

[12] R. V. KOHN & P. STERNBERG: *Local minimizers and singular perturbations.* Proc. Roy. Soc. Edinburgh, **A-111** (1989), 69–84.

[13] S. LUCKHAUS & L. MODICA: *The Gibbs-Thompson relation within the gradient theory of phase transitions.* Arch. Rational Mech. Anal., **107** (1989), 71–83.

[14] L. MODICA: *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rational Mech. Anal., **98** (1987), 123–142.

[15] L. MODICA: *Gradient theory of phase transitions with boundary contact energy.* Ann. Inst. H. Poincare Anal. Non Lineaire, **5** (1987), 453–486.

[16] L. MODICA & S. MORTOLA: *Un esempio di Γ-convergenza.* Boll. Un. Mat. Ital., (5) **14-B** (1977), 285–299.

[17] L. MODICA & S. MORTOLA: *Il limite nella Γ-convergenza di una famiglia di funzionali ellittici.* Boll. Un. Mat. Ital., (3) **14-A** (1977), 526–529.

[18] N. C. OWEN: *Nonconvex variational problems with general singular perturbations.* Trans. Amer. Math. Soc., **310** (1988), 393–404.

[19] N. C. OWEN & P. STERNBERG: *Nonconvex problems with anisotropic perturbations.* Nonlinear Anal., **16** (1991), 705–718.

[20] P. STERNBERG: *The effect of a singular perturbation on nonconvex variational problems.* Arch. Rational Mech. Anal., **101** (1988), 209–260.