GMT 19/20   last lecture!!!   5/6/20

FLAT NORM

Let T be a k-current in $\mathbb{R}^d$. We define

$k \geq 0$

flat norm

$$\mathbb{F}(T) := \inf \{ \mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S \}$$

S: $(k+1)$-current; R: k-current

Rems

- $\mathbb{F}(T) \leq \mathbb{M}(T)$ (= holds if $k=d$)
- $\mathbb{F}$ is clearly a seminorm (possibly with value $+\infty$) and also a norm:

Indeed $\forall \omega \in \mathcal{D}^k$

$$\begin{aligned} \langle T, \omega \rangle &= \langle R + \partial S, \omega \rangle \\ &= \langle R, \omega \rangle + \langle S, d\omega \rangle \\ &\leq \mathbb{M}(R) \|\omega\|_\infty + \mathbb{M}(S) \cdot \|d\omega\|_\infty \\ &\leq (\mathbb{M}(R) + \mathbb{M}(S)) \cdot (\|\omega\|_\infty + \|d\omega\|_\infty) \end{aligned}$$

$$\Rightarrow \quad |\langle T, \omega \rangle| \leq \mathbb{F}(T) (\|\omega\|_\infty + \|d\omega\|_\infty)$$

Thus $\mathbb{F}(T) = 0 \Rightarrow \langle T, \omega \rangle = 0 \ \forall \omega \in \mathcal{D}^k$.

- $\mathbb{F}(T_n - T) \to 0 \Rightarrow T_n \to T$

  ($T_n \xrightarrow[\mathbb{F}]{} T$) (i.e. $\langle T_n - T, \omega\rangle \to 0$)

  "convergence in $\mathbb{F}$ implies conv. in the sense of currents"

- $\mathbb{F}$ metrizes convergence in the sense of currents in the sense that

  Prop: Let T_n be s.t.
  - $\operatorname{supp}(T_n) \subset E$ bounded
  - $\mathbb{M}(T_n), \mathbb{M}(\partial T_n) \leq C < +\infty$

  Then $T_n \to T$ iff $\mathbb{F}(T_n - T) \to 0$

  (proof of the "only if" part follows from polyhedral def. Th.)

- $\mathbb{F}$ can be (easily) estimated from above and then is a useful tool to prove conv. (in the sense of currents)

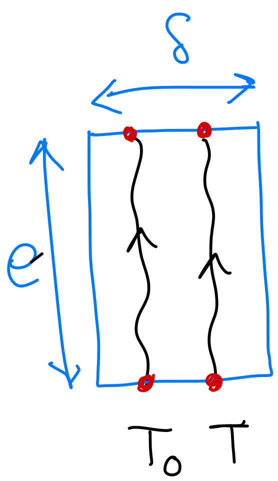
• δ_x, δ_{x_0} 0-currents (measures)

Then $$\mathbb{F}(\delta_x - \delta_{x_0}) \leq \min\{2;\ |x-x_0|\}$$

(in particular $x \to x_0 \Rightarrow \delta_x \xrightarrow[\mathbb{F}]{} \delta_{x_0}$
$\Rightarrow \delta_x \longrightarrow \delta_{x_0}$).

Proof $$\mathbb{F}(\delta_x - \delta_{x_0}) \leq \mathbb{M}(\delta_x - \delta_{x_0}) \overset{(\leq)}{=} 2.$$

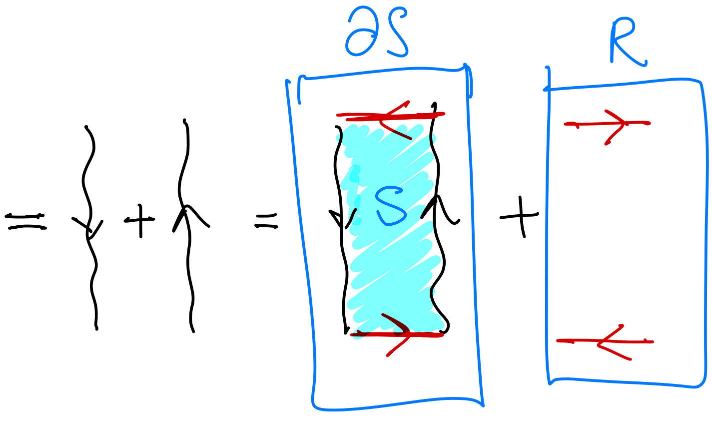
$$\delta_x - \delta_{x_0} = \partial S$$
$$\Downarrow$$
$$\mathbb{F}(\delta_x - \delta_{x_0}) \leq \mathbb{M}(S) = |x - x_0|$$

• 

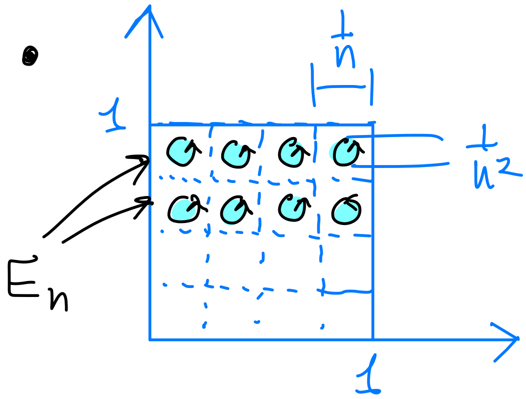
$$\mathbb{F}(T - T_0) \leq (\ell + 2)\delta$$

In particular $\delta \to 0$ implies
$T \xrightarrow{\mathbb{F}} T_0 \Rightarrow T \longrightarrow T_0$

Proof

$T - T_0 =$ 

$$\mathbb{F}(T-T_0) \leq \mathbb{M}(S)+\mathbb{M}(R) \leq C\delta + 2\delta$$

• 

$T_n := [E_n, \tau_n, 1]$

$\mathbb{M}(T_n) = \pi$

$\mathbb{F}(T_n) \leq \frac{\pi}{4n^2}$

($\Longrightarrow$ $T_n \longrightarrow 0$ as $n \to +\infty$)

Let indeed $S_n := [\,\bigcup$ blue discs, $e, 1]$

Then $T_n = \partial S_n \Rightarrow \mathbb{F}(T_n) \leq \mathbb{M}(S_n) = \frac{\pi}{4n^2}$.

Prove that same holds if

$T_n = [\bigcup_i$ closed curves $C_{n,i}$, τ_n, $1]$

and $\sum \mathcal{H}^1(C_{n,i}) = L_n \leq C < +\infty$

$\sup_i \mathcal{H}^1(C_{n,i}) \longrightarrow 0$

- Estimate for the homothopy formula

$$f_0, f_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^m$$

homotopic via $F : \underset{[a_0,a_1]}{\underset{\shortparallel}{I}} \times \mathbb{R}^d \longrightarrow \mathbb{R}^m$

Given a k-current T,

$$\underset{(f_1)_\# T}{\underset{\shortparallel}{T_1}} - \underset{(f_0)_\# T}{\underset{\shortparallel}{T_0}} := \partial \boxed{\underset{F(I\times T)}{\underset{\shortparallel}{S}}}$$

homoth. formula

Then

$$\mathbb{F}(T_1 - T_0) \leq \mathbb{M}(S) \leq (\mathrm{lip}\, F)^{k+1} (a_1 - a_0)\, \mathbb{M}(T)$$

In particular if $F|_{[0,\delta]\times \mathbb{R}^d \to \mathbb{R}^m}$ is the linear homothopy between f_0 and f_1, that is

$$F(t,x) = \frac{t}{\delta} f_1(x) + \left(1 - \frac{t}{\delta}\right) f_0(x)$$

then

$$\mathbb{F}(T_1 - T_0) \leq \|f_1 - f_0\| \, (\mathrm{lip}\, f_0 + \mathrm{lip}\, f_1)^k \, \mathbb{M}(T)$$

- variants of flat norm

(a) if I am interested in currents T which are boundaries then it is convenient to use

$$\widetilde{\mathbb{F}}(T) := \inf\{ \mathbb{M}(S) : T = \partial S \}$$

Under some ass. on the ambient space $\widetilde{\mathbb{F}}$ and $\mathbb{F}$ are equivalent norms

(b) if I am interested in integral currents T it is convenient to define $\widehat{\mathbb{F}}(T)$ as $\mathbb{F}(T)$ with the restr. that R and S are integral

$\widehat{\mathbb{F}}$ defines a distance (but is not a norm).

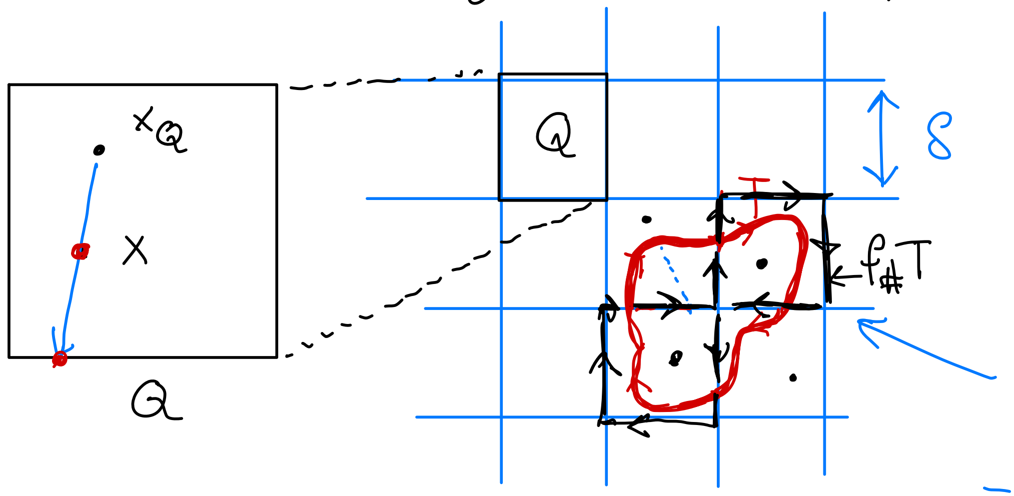
Not know if this is equivalent to the dist. induced by $\mathbb{F}$

Polyhedral deformation Theorem

Given a k-current T in $\mathbb{R}^d$ I want to find P polyhedral s.t. P is close to T (in flat norm).

Construction in the case $\partial T = 0$ (and $M(T) < +\infty$).

Fix a δ-grid in the space $\mathbb{R}^d$ (grid of cubes of side length δ)



$L_{d-1} = (d-1)$-dimensional skeleton of the grid = union of the $(d-1)$-dim. faces of the cubes

For every Q cube in the grid choose x_Q.

Let $f : \mathbb{R}^d \setminus \{x_Q : Q \in \cdots\}$ be the map that on each Q agrees with the retraction of $Q \setminus \{x_Q\}$ onto ∂Q (as in picture!)

Consider $\boxed{f_\# T}$. deformation of T on the $(d-1)$-dim. skeleton L_{d-1}

$$\partial(f_\# T) = f_\#(\partial T) = 0$$

If $k = d-1$ then $f_\# T$ is a $(d-1)$-current with bdry $= 0$ on L_{d-1}

$\triangledown$ Then the constancy lemma suggests that the "restriction" of $f_\# T$ to each face F of L_{d-1} should be

$[F, \tau_F, m_F]$

τ_F: constant orient. of F

m_F: constant!

That is

$$f_\# T = \underbrace{\sum_{F \in \dots} [F, \tau_F, m_F]}_{\text{polyhedral!}}$$

(at least if T has compact support)

"Expected estimates"

$$\mathbb{M}(f_\# T) \leq C \cdot \mathbb{M}(T)$$

and homothopy formula gives

$$f_\# T - T = \partial S$$

when $S = F_\# (I \times T)$ and in partic.

linear homothopy of f and identity

$$\boxed{\mathbb{F}(f_\# T - T) \leq \mathbb{M}(S) \leq C\delta\, \mathbb{M}(T)}$$

Difficulties

1) What if k is arbitrary?

I iterate the procedure, by taking a retraction of L_{d-1} onto L_{d-2} then a retraction of L_{d-2} onto L_{d-3} .... and so on untie I get a k-current with no bdry on L_k

2) f is not C^1!

and not even continuous

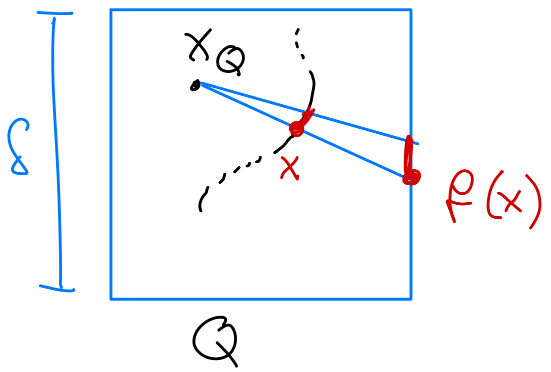
(it is "singular" at each x_Q!!)

So $f_\# T$ is not well defined

and the estimates do not work.

Let me focus on the estimate

$\mathbb{M}(f_\# T)$ ($\mathbb{F}(f_\# T - T)$ is similar)



$$\|df(x)\| \lesssim \frac{\delta}{|x - x_Q|}$$

ineq. holds up to const.

$T = \tau\mu$

$$\mathbb{M}(f_\# T) \le \int_{\mathbb{R}^d} \|d_x f\|^k \, d\mu(x)$$

$$\lesssim \int_{\mathbb{R}^d} |g(x)|^k \, d\mu(x)$$

where $g(x) = \dfrac{\delta}{|x - x_Q|}$ in each Q

Key lemma

Given μ (positive finite measure on $\mathbb{R}^d$), I can choose the points x_Q so that

$$(*) \qquad \int_{\mathbb{R}^d} |g(x)|^k \, d\mu(x) \lesssim M(\mu)$$

Proof $\forall\, Q$ cube of sidelength δ

Claim

$$\fint_{\bar{x}\in Q} \left(\int_{x\in Q} \left(\frac{\delta}{|x-\bar{x}|} \right)^k d\mu(x) \right) \underbrace{d\bar{x}}_{= d\mathcal{L}^d(\bar{x})} \lesssim \mu(Q)$$

then I can choose $x_Q = \bar{x}$ s.t.

$$\int_{x\in Q} \left(\frac{\delta}{|x-\bar{x}|} \right)^k d\mu(x) \lesssim \mu(Q)$$

and this proves $(*)$.

Proof of claim

$$\int_{\bar{x}\in Q}\left(\int_{x\in Q}\left(\frac{\delta}{|x-\bar{x}|}\right)^{k} d\mu(x)\right) d\bar{x}$$

$$= \delta^{k-n}\int_{x\in Q}\left(\int_{\bar{x}\in Q}\frac{1}{|x-\bar{x}|^{k}}\, d\bar{x}\right) d\mu(x)$$

$$\leq \delta^{k-n}\int_{x\in Q}\left(\int_{\bar{x}\in B(x,\sqrt{d}\cdot\delta)}\frac{1}{|x-\bar{x}|^{k}}\, d\bar{x}\right) d\mu(x)$$

$$\lessapprox \delta^{k-n}\int_{x\in Q}\underbrace{\left(\int_{0}^{\sqrt{d}\,\delta}\frac{1}{\rho^{k}}\,\rho^{d-1}\, d\rho\right)}_{\cong\ \delta^{d-k}} d\mu(x)$$

$$\lessapprox \int_{x\in Q} d\mu(x) = \mu(Q)$$

$\square$

Using this lemma you prove that $f_{\#}T$ is well-defined and

$$\mathbb{M}(f_{\#}T) \leq C\,\mathbb{M}(T)$$

then in a similar way you also prove that

$$S := f_\# (I \times T)$$

linear homot. between id. and f

is well-defined and

$$\mathbb{F}(f_\# T - T) \leq C \delta \, \mathbb{M}(T)$$

Polyhedral def. Th. (case $\partial T \neq 0$)

Let T be a k-current in $\mathbb{R}^d$ (with compact support) $\partial T = 0$, $\mathbb{M}(T) < +\infty$.

Choose a δ-grid.

Then there exists

const. orient. constant mult.

$$P = \sum_{F} [F, \tau_F, m_F]$$

F k-dim. face of the grid

s.t.

- $\partial P = 0$
- $\mathbb{M}(P) \leq C \, \mathbb{M}(T)$
- $P - T = \partial S$, $\mathbb{M}(S) \leq C \delta \, \mathbb{M}(T)$

in particular $\mathbb{F}(P-T) \le C\,\delta\,\mathbb{M}(T)$.

- If T is integral, so is P.