GMT 19/20 lecture 24 1/6/20

Proposition 1 If T is a d-current in $\mathbb{R}^d$ and $\mathbb{M}(\partial T) < +\infty$ then

$$T = [\mathbb{R}^d, e, u]$$

standard orient. of $\mathbb{R}^d$, $e_1 \wedge \dots \wedge e_d$

where $u \in BV_{loc}(\mathbb{R}^d) \subset L^p_{loc}(\mathbb{R}^d)$ with $p = \frac{d}{d-1}$

Moreover u is locally constant in $\mathbb{R}^d \setminus \mathrm{spt}(\partial T)$

$\mathrm{spt}(U) :=$ smallest closed set C s.t.

$\mathrm{spt}(\omega) \cap C = \emptyset \Rightarrow \langle U, \omega \rangle = 0$

Sketch of proof

Define $\langle \Lambda, \phi \rangle := \langle T, \phi \, dx \rangle$.

$\phi \in C^\infty_c(\mathbb{R}^d)$, $dx = dx_1 \wedge \dots \wedge dx_d$

Then $\mathbb{M}(\partial T) < +\infty \Rightarrow D\Lambda$ is a measure.

$(\Leftarrow)$

$D\Lambda$: distributional gradient

Conclude by the following:

Lemma If Λ is a distribution on $\mathbb{R}^d$ s.t. $D\Lambda$ is a measure, then Λ is a function in $BV_{loc}(\mathbb{R}^d) \subset L^p_{loc}(\mathbb{R}^d)$

$\frac{d}{d-1}$ $\shortparallel$ p

(Λ is represented by a function...)

Proposition 2 Let Σ be a closed, oriented k-dim. surface in $\mathbb{R}^d$, and let T be a k-dim. current in $\mathbb{R}^d$ with $spt(T) \subset \Sigma$ (i.e. $\langle T, \omega \rangle = 0$ if $spt(\omega) \cap \Sigma = \emptyset$) and $\mathbb{M}(\partial T) < +\infty$.

Then $T = [\Sigma, \tau_\Sigma, m]$ with $m \in BV_{loc}(\Sigma) \subset L^p_{loc}(\Sigma)$, and m is locally constant in $\Sigma \setminus spt(\partial T)$.

In particular the support of a normal current T cannot be a negligible subset of k-surface ...

Basic operation on currents (and forms)

Product of currents

(extends the notion of Cartesian product of surfaces)

Let $T = \tau\mu$ be a k-current with finite mass in $\mathbb{R}^d$, let $\tilde{T} = \tilde{\tau}\tilde{\mu}$ be a $\tilde{k}$-current .... in $\mathbb{R}^{\tilde{d}}$.

Then $T \times \tilde{T}$ is the $(k+\tilde{k})$-current with finite mass in $\mathbb{R}^d \times \mathbb{R}^{\tilde{d}} \simeq \mathbb{R}^{d+\tilde{d}}$ defined by

$$T \times \tilde{T} := (\tau \wedge \tilde{\tau}) \cdot \underbrace{(\mu \times \tilde{\mu})}_{\text{product measure}}$$

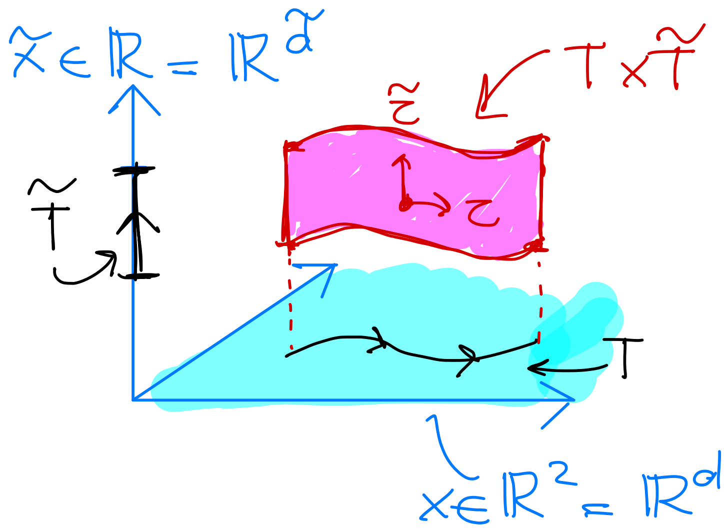
where

$$\tau \wedge \tilde{\tau}\ (\underset{\mathbb{R}^d \times \mathbb{R}^{\tilde{d}}}{\underset{\Uparrow}{x, \tilde{x}}}) := \tau(x) \wedge \tilde{\tau}(\tilde{x})$$

k-vectors in $\mathbb{R}^d$ are identified with k-vectors in $\mathbb{R}^d \times \mathbb{R}^{\tilde{d}}$ starting from the canonical identif. of $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\}$.

Remarks

1) Example



2) Note that $\langle T \times \tilde{T}, \phi \, dx_{\underline{i}} \wedge d\tilde{x}_{\underline{j}} \rangle = 0$ unless $\underline{i} \in I(d,k)$ and $\underline{j} \in I(\tilde{d}, \tilde{k})$.

3) $\mathbb{M}(T \times \tilde{T}) = \mathbb{M}(T) \cdot \mathbb{M}(\tilde{T})$

because $|\tau \wedge \tilde{\tau}| = |\tau| \cdot |\tilde{\tau}|$ if $\tau, \tilde{\tau}$ are multivectors in $\mathbb{R}^N$ supported on orthogonal supspaces (if $|\cdot|$ is either the Euclidean norm or the mass norm).

4) You can define $T \times \tilde{T}$ also if T and $\tilde{T}$ are general currents

5) If $T = [E, \tau, m]$, $\tilde{T} = [\tilde{E}, \tilde{\tau}, \tilde{m}]$ are rectifiable, then $T \times \tilde{T}$ is also rectifiable, and

$$T \times \tilde{T} := [E \times \tilde{E}, \underbrace{\tau \wedge \tilde{\tau}}_{\substack{\text{unit. simple} \\ (k+\tilde{k})\text{-vector!}}}, m \cdot \tilde{m}]$$

Key point is that

$$(\mathcal{H}^k \llcorner E) \times (\mathcal{H}^{\tilde{k}} \llcorner \tilde{E}) = \mathcal{H}^{k+\tilde{k}} \llcorner (E \times \tilde{E})$$

in particular

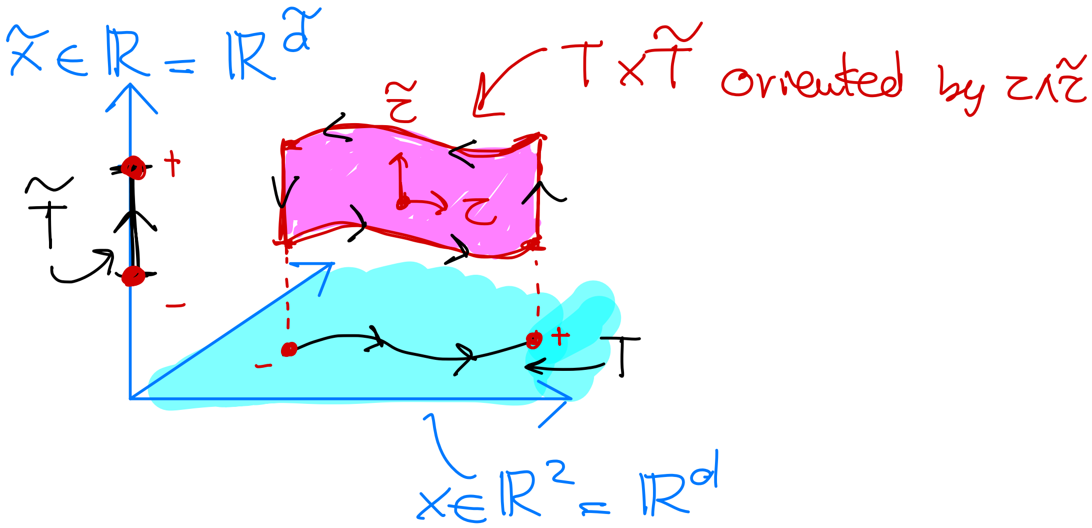
$$\mathcal{H}^k(E) \cdot \mathcal{H}^{\tilde{k}}(\tilde{E}) = \mathcal{H}^{k+\tilde{k}}(E \times \tilde{E})$$

which holds because $E, \tilde{E}$ are rectifiable (not in general)

Proposition 3 If $T, \tilde{T}$ are normal then $T \times \tilde{T}$ is normal and

$$\partial(T \times \tilde{T}) = \partial T \times \tilde{T} + (-1)^k \cdot T \times \partial \tilde{T}$$

(Leibniz rule) Not so easy to prove)



Rem If ω_1, ω_2 are h_1-, h_2-forms of class C^1 in $\mathbb{R}^N$ then

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{h_1} \omega_1 \wedge d\omega_2$$

(easy to prove)

Pull-back of forms, push-forward of currents

Purpose: define the image of a current T in $\mathbb{R}^n$ through a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Pull-back of a k-covector

Let $\alpha \in \Lambda^k(W)$, $T: V \longrightarrow W$ linear.

Then $T^\# \alpha \in \Lambda^k(V)$ is defined by:

called "pull-back of α accord. to T"

$$T^\# \alpha (v_1, \dots, v_k) := \alpha(Tv_1, \dots, Tv_k)$$

$v_i \in V$ $\qquad$ $Tv_i \in W$

If V, W are endowed with scalar products then

$$\| T^\# \alpha \| \leq \|T\|^k \|\alpha\|$$

where $\|T\|$ is the operator norm of T

(that is, the Lipschitz constant of T)

$\|\alpha\|$ is the comass norm of α

[Proof uses that

$$|Tv_1 \wedge \dots \wedge Tv_k| \leq \|T\|^k |v_1 \wedge \dots \wedge v_k|$$
]

Rem 1) If $k=1$, $T^\#$ is the adjoint T^*

2) $T^\#(\alpha \wedge \tilde{\alpha}) = (T^\# \alpha) \wedge (T^\# \tilde{\alpha})$

Push-forward of a k-vector

$T: V \to W$ linear, $v \in \Lambda_k(V)$

then $T_\# v \in \Lambda_k(W)$ defined by

called "push-forward of v acc. to T"

$$\langle T_\# v, \alpha \rangle := \langle v, T^\# \alpha \rangle$$

$\forall \alpha \in \Lambda^k(W)$

Rem 1) $T_\#(v \wedge \tilde{v}) = (T_\# v) \wedge (T_\# \tilde{v})$

in particular, if $v = v_1 \wedge \ldots \wedge v_k$ is simple

$T_\#(v_1 \wedge \ldots \wedge v_k) = Tv_1 \wedge \ldots \wedge Tv_k$

If moreover V, W are endowed with scalar products

$$\|T_\# v\| \leq \|T\|^k \|v\|$$

where $\|T\| :=$ operator norm of T

$\|v\| =$ mass norm of v.

Pull-back of forms (on $\mathbb{R}^d$)

Let $f : \mathbb{R}^d \to \mathbb{R}^m$ $\in \mathcal{C}^{h+1}$

and let ω be a k-form on $\mathbb{R}^m$ $\in \mathcal{C}^h$

Then $f^\# \omega$ is the k-form of $\mathbb{R}^d$ $\in \mathcal{C}^h$

"pull-back of ω acc. to f"

defined by

$$\left(f^\# \omega\right)(\underset{\substack{\cap \\ \mathbb{R}^d}}{x}) := (\underset{\substack{\mathcal{L}(\mathbb{R}^d, \mathbb{R}^m) \\ \Psi}}{d_x f})^\# \Big(\underset{\substack{\Lambda^k(\mathbb{R}^m) \\ \Psi}}{\omega(\underset{\substack{\cap \\ \mathbb{R}^m}}{f(x)})} \Big).$$

Then

$$d\left(f^\# \omega\right) = f^\# (d\omega).$$

if f and ω are $\mathcal{C}^1$.

Push-forward of currents

Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be $\mathcal{C}^\infty$ and proper

$f^{-1}(K)$ compact $\forall K$ compact

and T be a k-current on $\mathbb{R}^d$

Then $f_\# T$ is the k-current on $\mathbb{R}^m$

push-forw. of T acc. dif

defined by

$$\langle f_\# T, \omega \rangle := \langle T, f^\# \omega \rangle$$

$\forall\ \omega \in \mathcal{D}^k(\mathbb{R}^m)$

Then $$f_\#(\partial T) = \partial(f_\# T)$$

Rem 1) If T has finite mass then $f_\# T$ is well-defined if f is C^1 and proper

2) If T has compact support then $f_\# T$ is well-defined if f is C^∞ (but not proper)

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