GMT 19/20 Lecture 16 14/5/20

Setting: E is d-rectif. set in $\mathbb{R}^n$
$T^w_x E$ is the weak tangent bundle to E

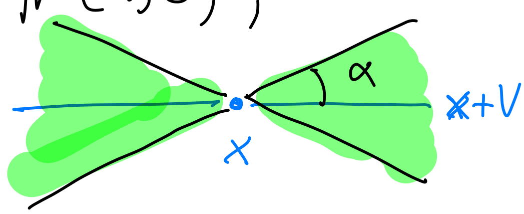
Prop. 1 Assume that E is $\mathcal{H}^d$-loc. finite.
Then $T^w_x E$ is the approximate tangent plane to E at x, that is,

$$\mathcal{H}^d \llcorner \underbrace{E_{x,r}}_{= \frac{1}{r}(E-x)} \longrightarrow \mathcal{H}^d \llcorner T^w_x E$$

"flat d-dim. measure"

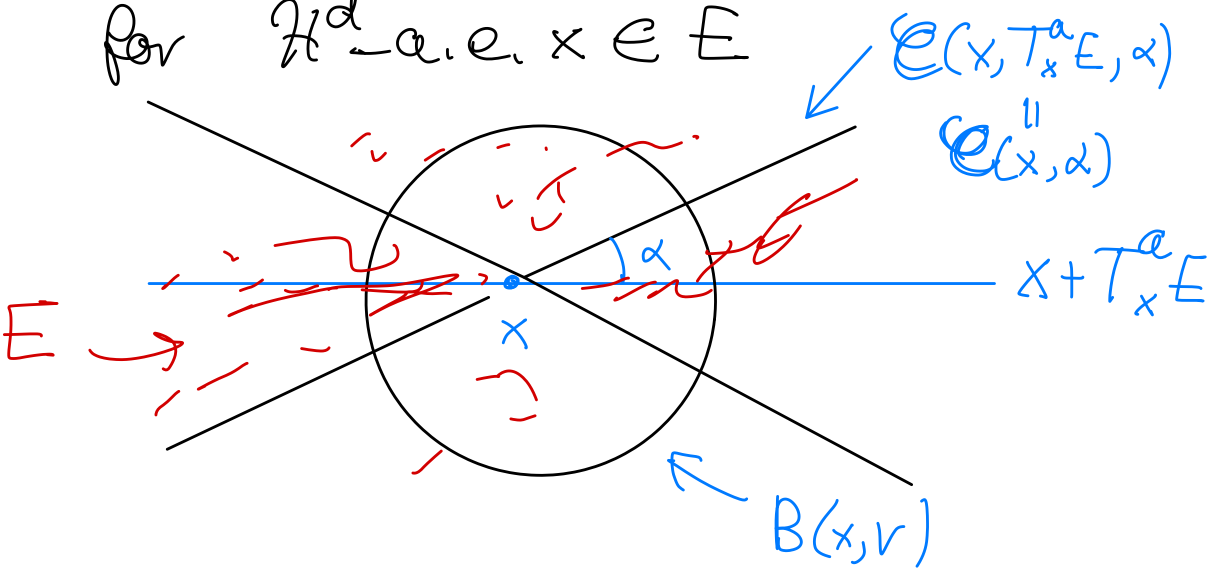
locally in the sense of measure (i.e. test functions in $\mathcal{C}_c(\mathbb{R}^n)$)

Def Given $x \in \mathbb{R}^n$, $V \in Gr(n,d)$, $\alpha \in (0, \frac{\pi}{2})$ then



$$\mathcal{C}(x,V,\alpha) := x + \{ y \mid \operatorname{dist}(y,V) \le |y| \sin\alpha \}$$

Corollary 2 Take E as in Prop. 1 then for $\mathcal{H}^d$-a.e. $x \in E$



(i) $\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}(x,\alpha)) \sim \alpha_d r^d$

(as $r \to 0$)

(ii) $\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}^c(x,\alpha)) = o(r^d)$

Proof (i) & (ii) holds for every x s.t. $T_x^w E$ is the approximate tangent space to x at E.

Proof of (i) (only, (ii) I don't prove)

$$\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}(x,\alpha)) =$$
$$= r^d \cdot \boxed{\mathcal{H}^d(E_{x,r} \cap B(0,1) \cap \mathcal{C}(0, T_x E, \alpha))}$$

It is enough to show that

$$\mathcal{H}^d(E_{x,r} \cap B(0,1) \cap \mathcal{C}(\underbrace{0, T_xE, \alpha}_{\mathcal{C}}))$$

$$\| $$

$$\int_{E_{x,r}} g \, d\mathcal{H}^d \longrightarrow \int_{T_xE} g \, d\mathcal{H}^d$$

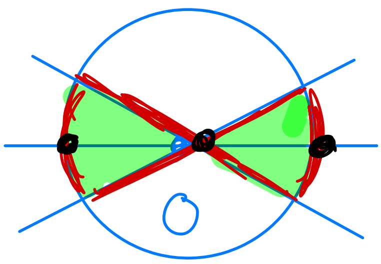
with $g = \mathbb{1}_{B(0,1) \cap \mathcal{C}}$

$$\int_{T_xE} g\, d\mathcal{H}^d = \mathcal{H}^d(T_xE \cap \mathcal{C} \cap B(0,1)) = \mathcal{H}^d(T_xE \cap B(0,1)) = \alpha_d$$

use the def. of approx. tangent space

+ g is bounded, compact support,

& $\mathcal{H}^d(\underbrace{\mathrm{Sing}(g) \cap T_xE}) = 0$

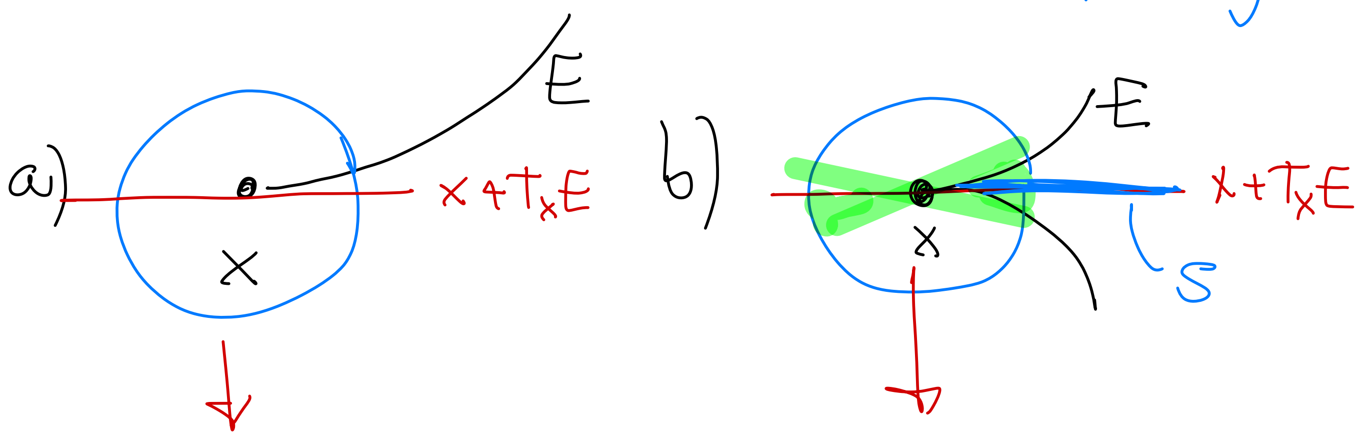


T_xE

$$\partial(B(0,1) \cap \mathcal{C}) \cap T_xE$$
$$\cap$$
$$(\partial B(0,1) \cup \partial \mathcal{C}) \cap T_xE$$
$$\|$$
$\{0\} \cup$ sphere of dim. $d-1$ $\square$

Remarks (for $d=1$, $n=2$)

(1) The following behaviours is "not generic" (it only happens for an $\mathcal{H}^d$-null set of points)



Contradicts (i) in Cor. 2

Contradicts Prop. 1

$$\mathcal{H}^1 \llcorner E_{x,r} \longrightarrow 2\cdot\mathcal{H}^1 \llcorner S$$

(2) It can happen that for $\mathcal{H}^d$ a.e. $x \in E$

$$\mathcal{H}^d\big(E \cap B(x,r)) \cap \mathcal{C}^c(x,\alpha)\big) > 0 \qquad \forall \alpha > 0 \quad \forall r > 0$$

Example $d=1$, $n=2$

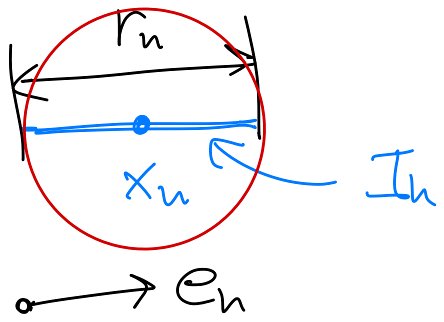
Let x_n dense seq. in $\mathbb{R}^2$

Let $r_n > 0$ s.t. $\sum r_n < +\infty$

Let e_n be a seq. of unit vectors

Let

$$E := \bigcup_n \underbrace{\left[x_n - \frac{r_n}{2} e_n,\ x_n + \frac{r_n}{2} e_n\right]}_{I_n}$$

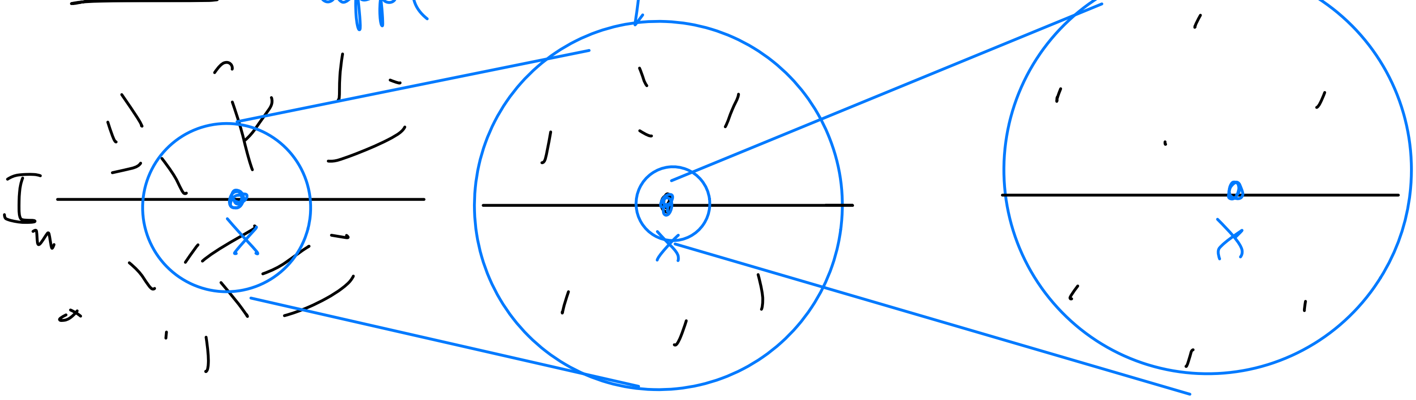


Then E is 1-rectif.

E is $\mathcal{H}^1$-finite

$T^W_x E = \mathrm{span}(e_n)$ for $\mathcal{H}^1$-a.e. $x \in I_n$ $\forall n$

AND $\mathrm{supp}(\mathcal{H}^1 \llcorner E) = \mathbb{R}^2$



$$\boxed{\text{for } \mathcal{H}^1\text{-a.e. } x \in I_n \qquad \mathcal{H}^1(B(x,r) \setminus I_n) = o(r)}$$

try a direct proof!

Rectifiability Criteria:

(When is E rectifiable?)

2 easy results

Prop 3

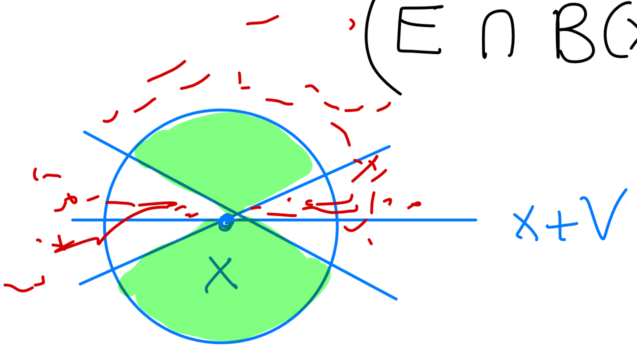
Let E (Borel) $\subset \mathbb{R}^n$ and assume that

$\boxed{\forall x}\ \exists\ \mathcal{C}(x, V, \alpha)\ ,\quad r = r(x) > 0$

s.t.

$V = V(x)$ is d-plane
$\alpha = \alpha(x) \in (0, \frac{\pi}{2})$

$$\left(E \cap B(x,r)\right) \subset \mathcal{C}(x, V, \alpha)$$



$\mathcal{C}(x, V, \alpha)$ is "tangent" to E at x.

Then E is contained in a countable union of d-dim. Lipschitz graph (graphs of Lip. maps from some d-dim. V to $V^\perp$)

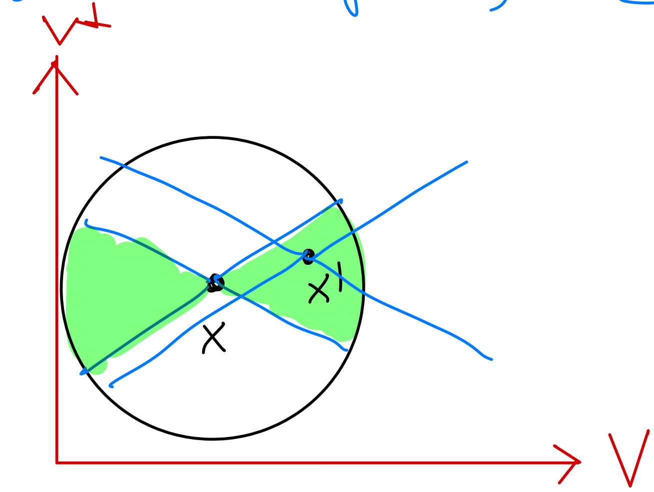
<u>Rem</u> if ass. holds for $\mathcal{H}^d$-a.e. x, you still get E rectifiable

Lemma Let F be a set in $\mathbb{R}^n$ s.t. $\exists$ $V \in Gr(n,d)$, $\alpha \in (0,\frac{\pi}{2})$, $r > 0$ s.t. $\boxed{F \cap \cancel{B(x,r)} \subset C(x,V,\alpha)}$ $\forall x \in F$ & $\operatorname{diam} F < r$

Then $F \subset$ graph of $g: V \to V^\perp$ with $\operatorname{lip}(g) \le \tan\alpha$

Proof Indeed F is the graph of a Lipschitz map $g: \pi_V(F) \to V^\perp$

Indeed if $x, x' \in F$



use McShane or Kirszbraun to extend g to V

$\square$

Proof of Prop 3

Idea: write E as countable union of subsets that satisfy the ass. in the Lemma:

$\forall \alpha, V, r$ define

$$E_{\alpha,V,r} := \{x \in E \text{ s.t. } E \cap B(x,r) \subset C(x,V,\alpha)\}$$

Then $E_{\alpha,V,r} \cap Q$ satisf. the ass. of the Lemma if Q has $\operatorname{diam}(Q) \leq r$.

Then

$$E = \bigcup_{\substack{\alpha = \frac{\pi}{2} - \frac{1}{n} \\ n=1,2,\dots}} \bigcup_{V \in \mathcal{G}_m} \bigcup_{\substack{r = \frac{1}{m} \\ m=1,2,\dots}} \bigcup_{Q \in \mathcal{F}_r} \overbrace{E_{\alpha,V,r}}^{E_{\alpha,V,r}} \cap Q$$

$\mathcal{G}_m$: finite family of d-planes s.t. every d-plane $\tilde{V}$ is contained in $C(V, \frac{1}{2n})$ same V in $\mathcal{G}_m$

$\mathcal{F}_r$: family of sets Q with $\operatorname{diam}(Q) \leq r$ that covers $\mathbb{R}^n$

$\square$

Def. Let E be a Borel set in $\mathbb{R}^n$,
let $\mathcal{C} = \mathcal{C}(x, V, \alpha)$ be a cone.
$V \in Gr(n,d)$, $\alpha \in (0, \frac{\pi}{2})$

We say that $\mathcal{C}$ is approximately tangent to E at x if

$$\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}^c) = o(r^d) \quad \text{as } r \to 0$$

Theorem 4
Let $E \subset \mathbb{R}^n$ s.t. at every $x \in E$
(i) E admits a tangent cone $\mathcal{C}(x, V, \alpha)$
(α, V depends on x)
(ii) $\Theta^d_*(E, x) > 0$ lower d-dim. density of E at x

Then E is contained in a countable union of d-dim. Lipschitz graphs
($\Rightarrow$ E is d-rectif.)

Rem If ass. holds only for $\mathcal{H}^d$-a.e. x
E is still d-rectif.