GMT 19/20 lecture 14 7/5/20

Th. 1 Let Σ be a d-dim. surface of class C^1 in $\mathbb{R}^n$ parametrised by $\Phi : \Omega \to \Sigma$ of class C^1, $\Omega \subset \mathbb{R}^d$. Then $\forall F \subset \Sigma$

(AF1) $$\mathcal{H}^d(F) = \int_{\Phi^{-1}(F)} J\phi(x)\, dx$$

where $J\phi(x) := |\det d_x\phi| := |\det M|$, $M \in \mathbb{R}^{d\times d}$ associated to $d_x\phi : \mathbb{R}^d \to T_{\phi(x)}\Sigma$ by a choice of an orthon. basis $R = (v_1, \dots, v_d)$ of $T_{\phi(x)}\Sigma$

$$= \sqrt{\det(\nabla^t\phi(x)\, \nabla\phi(x))}$$

later

$$= \sqrt{\sum (\det N)^2}$$

N is a $d\times d$ minor of $\nabla\phi(x)$

since $\nabla\phi(x) = R\,M$ then

$$\underset{n\times d}{\nabla\phi(x)} = \underset{n\times d}{R}\;\underset{d\times d}{M}$$

$$\nabla^t\phi\, \nabla\phi = (RM)^t RM = M^t \underbrace{R^t R}_{= I} M = M^t M$$

$$\det(\nabla^t\phi\nabla\phi) = \det(M^t M) = (\det M)^2$$

Th 2 Given $f : \Omega \xrightarrow{} \mathbb{R}^n$ Lipschitz (Locally Lipschitz)

open in $\mathbb{R}^d$

then $\forall F \subset \cancel{f(\Omega)}\ \mathbb{R}^n$ Borel

$$\text{(AF2)} \qquad \underbrace{\int_{y \in F} \# f^{-1}(y)\, d\mathcal{H}^d(y)}_{\mathcal{H}^d\text{-measure of } F \text{ "with multiplicity"}} = \int_{f^{-1}(F)} Jf(x)\,dx$$

$\boxed{y \mapsto \# f^{-1}(y) \text{ is Borel} - \text{no measurability issue}}$

where $Jf(x)$ is defined at every $x \in \Omega$ where f is differentiable by

$$Jf(x) := \begin{cases} 0 & \text{if } \operatorname{rank}(d_x f) < d \\ |\det(d_x f)| := |\det M| & \text{with } M \in \mathbb{R}^{d\times d} \end{cases}$$

$$= \sqrt{\det(\nabla^t f(x)\, \nabla f(x))} = \ldots$$

More general variant

$\forall E \subset \Omega$ Borel

$$\text{(AF3)} \qquad \underbrace{\int_{y \in f(E)\ \cancel{y\in\mathbb{R}^n}} \#(f^{-1}(y) \cap E)\, d\mathcal{H}^d(y)}_{\mathcal{H}^d\text{-measure of } f(E) \text{ counting multiplicity}} = \int_E Jf(x)\,dx$$

$\boxed{y \mapsto \#(f^{-1}(y) \cap E) \text{ is not necc. Borel but agrees } \mathcal{H}^d\text{-a.e. with a Borel one}}$

Proof

STEP 1 Formula (AF3) holds if $f \in C^1(\Omega)$ & $Jf(x) \neq 0 \ \forall x \in \Omega$ (i.e. $\mathrm{rank}(d_x f) = d$)

Cover Ω with open sets Ω_i s.t. f is a diffeom. on Ω_i. Split E as $E = \dot{\bigcup} E_i$ with $E_i \subset \Omega_i$

disjoint union

Apply (AF1) to $f_i := f \llcorner \Omega_i$, $\Sigma_i := f(\Omega_i)$ and $F_i := f_i(E_i) = f(E_i)$.

NOTE THAT $F_i = f(E_i) = f_i(E_i)$ is Borel

and $y \mapsto \#(f^{-1}(y) \cap E) = \sum_i \mathbb{1}_{F_i}(y)$

is also Borel

STEP 2 If $f \in C^1(\Omega)$ and $Jf \equiv 0$ on E then $\mathcal{H}^d(f(E)) = 0$ and in particular (AF3) holds.

$\forall \varepsilon > 0$ let $g_\varepsilon : \Omega \to \mathbb{R}^d \times \mathbb{R}^n$

$$g_\varepsilon(x) := (\varepsilon x, f(x))$$

let p projection of $\mathbb{R}^d \times \mathbb{R}^n$ onto $\mathbb{R}^n$

then $f = p \circ g_\varepsilon$

$$\mathcal{H}^d(f(E)) = \mathcal{H}^d(p(g_\varepsilon(E)))$$

$$\leq \mathcal{H}^d(g_\varepsilon(E)) = \int_E Jg_\varepsilon \, dx \xrightarrow[\varepsilon \to 0]{} 0$$

by Step 1 or even Th. 1

dominate convergence

$$Jg_\varepsilon^2 = \det(\nabla g_\varepsilon^t \nabla g_\varepsilon) =$$

$$\nabla g_\varepsilon = \left(\frac{\varepsilon I}{\nabla f}\right) \qquad = \det(\varepsilon^2 I + \nabla^t f \nabla f) \xrightarrow[\varepsilon \to 0]{} \det(\nabla^t f \nabla f) = (Jf)^2 = 0$$

STEP 3 (AF3) holds if $f \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$

(put together step 1 and step 2)

Step 4 If f is lip. and $|E| = 0$ then $\mathcal{H}^d(f(E)) = 0$ and in particular (AF3) holds.

Known!

Step 5 general case.

By the Lusin property of Lipschitz maps I can find a sequence $f_n \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$ and of compact sets K_n s.t.

$$f_n = f \text{ on } K_n \quad \& \quad |\Omega \setminus K_n| \to 0.$$

(and $Jf_n = Jf$ a.e. on K_n)

I split E as $E = \overset{\bullet}{\bigcup} E_n$ with

$E_1 \subset K_1,\ E_2 \subset K_2, \ldots \quad \& \ |E_0| = 0$

Then (AF3) holds for each E_n with $n > 0$ (Step 3) and also for E_0 (Step 4) . .....

$\square$

Cor 3 Let $f : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ Lipschitz and consider $h : \Omega \longrightarrow [0, +\infty]$ Borel then

(AV4) $$\int \Big(\sum_{x \in f^{-1}(y)} h(x) \Big) \, d\mathcal{H}^d(y) = \int_\Omega h(x) \, Jf(x) \, dx$$

(obtain (AV3) for $h = \mathbb{1}_E$)

RECTIFIABLE SETS

building bloc of everything...

<u>Definition</u> $E \subset X$ metric space is Borel
d-rectifiable if $E = \bigcup_{i=0}^{\infty} E_i$

s.t.

(i) $\mathcal{H}^d(E_0) = 0$

Borel

(ii) $E_i = f_i(F_i)$ with $F_i \subset \mathbb{R}^d$
and $f_i : F_i \to X$ is
Lipschitz

<u>Prop</u> If $X = \mathbb{R}^n$ then we can replace
(ii) with any the following (without
changing the class of rectif. sets)

(ii') $E_i \subset f_i(A_i)$, $A_i \subset \mathbb{R}^d$ open $f_i \in C^1(A_i)$

(ii'') $E_i \subset f_i(A_i)$, $A_i \subset \mathbb{R}^d$ open $f_i \in C^1(A_i)$
<u>regular parametriz.</u>

(iii) $E_i \subset \Sigma_i \leftarrow$ d-dim. surface of
class C^1 in $\mathbb{R}^n$

Piece of proof.

<u>Lemma</u> If $E = f(F)$ with $F \subset \mathbb{R}^d$

$f : F \to \mathbb{R}^n$ then $E = \bigcup E_n$

s.t. $\mathcal{H}^d(E_0) = 0$ & $E \subseteq f_n(A_n)$ with

$f \in C^1(A_n, \mathbb{R}^n)$.

open in $\mathbb{R}^d$

<u>Proof</u> Extend f to $\mathbb{R}^d$

+ use Lusin property to find

$f_n : \mathbb{R}^d \to \mathbb{R}^n$ of class C^1

and $K_n \subset \mathbb{R}^d$ s.t. $f_n = f$ on K_n

& $\bigcup_n K_n$ covers a.e. of $\mathbb{R}^d$

$E_n = f_n(\mathbb{R}^d)$

+ note that $E_0 = f(\underbrace{F \setminus \bigcup K_n}_{\text{null set}})$ is $\mathcal{H}^d$-null

Definition $E \subset X$ is d-purely unrectif. if $\mathcal{H}^d(E \cap f(F)) = 0$ for every $f : \underset{\mathbb{R}^d}{\underset{\cap}{F}} \longrightarrow X$ Lipschitz.

Even here, if $X = \mathbb{R}^n$ you can consider alternative def. for $f(F)$

$\boxed{\text{Remarks on rectifiable and p.unrect. sets}}$

- E d-rectif. $\Longrightarrow$ $\dim_H(E) \le d$.

  because it is $\mathcal{H}^d$ σ-finite.

- if $\mathcal{H}^d(E) = 0 \iff E$ is d-rectif. & d-p.unrect.