GMT 19/20, lecture 10, 17/4/20

Recall that for $E \subset X$, $d \in (0,+\infty)$

$$\Theta^*_d(E,x) := \limsup_{r\to 0} \frac{\mathcal{H}^d(E\cap \overline{B(x,r)})}{\alpha_d\, r^d}$$

α_d = volume of unit ball in $\mathbb{R}^d$ if d is integer

defined for every $x\in X$.

Prop 0 If $\mathcal{H}^d(E)<+\infty$ then

$$\frac{1}{2^d} 1_E(x) \leq \Theta^*_d(E,x) \leq c_d \cdot 1_E(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x\in X$$

where $c_d := \begin{cases} 1 & \text{if } X=\mathbb{R}^n \\ 5^d & \text{otherwise} \end{cases}$

Given a (locally) finite measure μ on X the d-dim. upper density of μ at $x\in X$ is

$$\Theta^*_d(\mu,x) := \limsup_{r\to 0} \frac{\mu(\overline{B(x,r)})}{\alpha_d\, r^d}$$

Prop 1 If $\mu = \rho \cdot \mathcal{H}^d$ with $\rho \in L^1_{loc}(\mathcal{H}^d)$

then

$$\frac{1}{2^d}\rho(x) \leq \Theta^*_d(\mu, x) \leq c_d\, \rho(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x$$

as in Prop. 0

In particular

$$0 < \Theta^*_d(\mu,x) < +\infty \quad \text{for } \cancel{\mathcal{H}^d\text{-a.e. } x \text{ s.t. } \rho(x)>0}$$

μ-a.e. x

What about the converse of Prop 1?

Prop 2

Let μ be a (locally) finite meas. on X.

Assume that

$$\Theta^*_d(\mu, x) < +\infty \quad \mu\text{-a.e. } x$$

then

$$\mu \ll \mathcal{H}^d \quad \left(\mu(E) > 0 \Rightarrow \mathcal{H}^d(E) > 0\right)$$

Prop 3

Assume that

$$\Theta^*_d(\mu, x) > 0 \quad \mu\text{-a.e. } x$$

then there exists a Borel set E which is σ-finite wrt $\mathcal{H}^d$ and μ is supported on E (that is, $\mu(X \setminus E) = 0$)

Think of the case $d=0$.

It also shows that E cannot be ($\mathcal{H}^d$-)finite and $\text{supp}(\mu)$ ~~can~~ can be X

Theorem 4 Assume that

$$0 < \Theta^*_d(\mu, x) < +\infty \quad \mu\text{-a.e. } x$$

Then $\mu = \rho \cdot \mathcal{H}^d$ for some $\rho \in L^1_{loc}(\mathcal{H}^d)$

(immediate corollary of Propositions 2 and 3 and R.N. Th.)

Moreover $\frac{1}{C^d} \Theta^*_d \leq \rho(x) \leq 2^d \cdot \Theta^*_d \quad \mathcal{H}^d\text{-a.e.}$

Back to Cantor set C

$C = \bigcap_{i=0}^{\infty} C_i$ and $C_i := \bigcup_{j=1}^{2^i} I_{i,j}$ with

$I_{i,j}$ interval with length 3^{-i}

$I_{0,1} := [0,1]$

$I_{1,1} := [0,\frac{1}{3}]$ $I_{1,2} := [\frac{2}{3},1]$

....

$d := \frac{\log 2}{\log 3}$

Exercise $\mathcal{H}^d(C) > 0$.

Proof Construct $\mu \neq 0$ measure on C s.t. $\Theta^*_d(\mu, x) < +\infty$ for μ-a.e. $x \in C$.

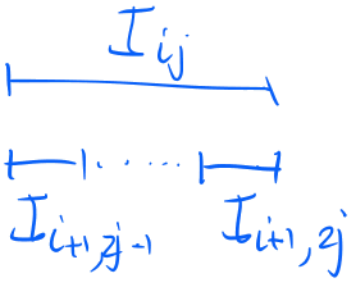
Then by Prop. 2 we know that $\mu \ll \mathcal{H}^d$. In particular $\mu(C) > 0 \Rightarrow \mathcal{H}^d(C) > 0$.

Natural candidate: Recall that C is homeomorphic to $\{0,1\}^{\mathbb{N}}$

A natural measure on $\{0,1\}^{\mathbb{N}}$ is the infinite product of probabilities $\mu \otimes \mu \otimes \cdots$ where $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Exercise (method to construct measures on C). Let $\mathcal{I} = \{I_{i,j}\}$, and let $\rho : \mathcal{I} \longrightarrow [0,+\infty)$ s.t.

$$\rho(I_{i,j}) = \rho(I_{i+1,2j-1}) + \rho(I_{i+1,2j})$$



Then ρ can be uniquely extended to a finite measure μ on C

(that is $\rho(I_{i,j}) = \mu(I_{i,j} \cap C)$)

Idea of proof Let μ be the meas. on C given by Carath. construction.

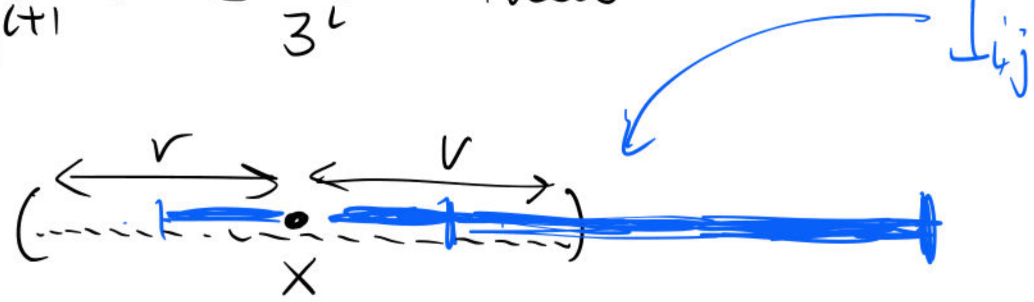
Let now μ be given by taking

$$\rho(I_{i,j}) := 2^{-i}$$

Let's estimate $\Theta^*_d(\mu, x)$!

Fix $x \in C$, $r > 0$ and take i

s.t. $\frac{1}{3^{i+1}} \leq r \leq \frac{1}{3^i}$ then



$$\overline{B(x,r)} = C \cap [x-r, x+r] \subset I_{i,j}$$

then $\mu(\overline{B(x,r)}) \leq 1/2^i$

$r^d \geq \frac{1}{2^{i+1}}$ and then $\frac{\mu(\overline{B(x,r)})}{r^d} \leq 2$

$$\Longrightarrow \quad \Theta^*_d(\mu, x) \leq 2 \quad \forall x \in C$$

$$\left(\frac{1}{2} \leq \Theta_{*d}(\mu, x) \leq \Theta^*_d(\mu, x) \leq 2 \right)$$

Self-similar fractals according to Hutchinson

We say that $E \subset \mathbb{R}^n$ is a self-similar fractal if $\exists\ \phi_1, \dots, \phi_N$ contractive similarities of $\mathbb{R}^n$ such that

$$E = \bigcup_{i=1}^{N} \phi_i(E)$$

$\phi_i : \mathbb{R}^n \to \mathbb{R}^n$, $\phi_i : x \mapsto x_i + \lambda_i R_i x$

where $x_i \in \mathbb{R}^n$, $\lambda_i \in (0,1)$, $R_i \in O(n)$.

Lemma: Assume $\{\phi_i(E)\}$ are disjoint. Assume that $0 < \mathcal{H}^d(E) < +\infty$ for some d $\left(\Rightarrow \dim_H(E) = d \right)$ then d is the unique solution of

$$(*) \qquad \sum_{i=1}^{N} \lambda_i^d = 1$$

For C = Cantor set, $\phi_1(x) := \frac{1}{3}x$; $\phi_2(x) := \frac{2}{3} + \frac{1}{3}x$

Proof Since $E = \dot{\bigcup} \phi_i(E)$ then

$$\mathcal{H}^d(E) = \cancel{\sum_{i=1}^N \mathcal{H}^d(\phi_i(E))} = \sum_{i=1}^N \lambda_i^d \mathcal{H}^d(E)$$

$$\Longrightarrow \quad 1 = \sum_{i=1}^N \lambda_i^d$$

(I do not prove that (*) has always a unique sol.)

$\square$

Let be given $\phi_1, \dots, \phi_N$ contractive similarities of $\mathbb{R}^n$ with scaling factor λ_i

Let d be the unique sol. of (*) $1 = \sum_{i=1}^N \lambda_i^d$

Theorem

(i) $\exists!$ C compact in $\mathbb{R}^n$ s.t.

$$C = \bigcup_{i=1}^N \phi_i(C)$$

(ii) $\mathcal{H}^d(C) < +\infty$

"OPEN SET CONDITION"

(iii) Assume that $\exists$ V open and bounded s.t. $\phi_i(V) \subset V$ $\forall i$ & $\{\phi_i(V)\}$ are disjoint

then $\mathcal{H}^d(C) > 0$.

Proof of (i)

Let $X := \{$ compact nonempty subsets of $\mathbb{R}^n \}$

Let $d_H :=$ Hausdorff dist. on X

Let $\Phi : X \longrightarrow X$ be given by

$$\Phi(E) := \bigcup_{i=1}^{N} \phi_i(E)$$

Then

- X is a complete metric space (well-known?)
- Φ is a contraction on X

Thus Φ has a unique fixed point C.

take $E, E' \in X$ then

$$d_H(\Phi(E), \Phi(E')) = d_H\left(\bigcup_i \phi_i(E) ; \bigcup_i \phi_i(E')\right)$$

$$(!) \longrightarrow \leq \sup_i d_H(\phi_i(E) ; \phi_i(E'))$$

$$= \underbrace{\sup_i \lambda_i}_{\lambda^*} d_H(E, E')$$

$$= \lambda^* d_H(E, E').$$