GMT 19/20, lecture 8, 3/4/20

Missing step in $\mathcal{H}^d = \mathcal{H}^d_\delta = \mathcal{L}^d$ on $\mathbb{R}^d$ $\forall\ \delta \in [0,+\infty]$

$\boxed{\text{Namely } \mathcal{H}^d_\infty \geq \mathcal{L}^d.}$ (*)

Lemma (Isodiametric Inequality)
For every $E \subset \mathbb{R}^d$ there holds

$$\frac{\alpha_d}{2^d}(\operatorname{diam} E)^d \geq \mathcal{L}^d(E)$$

($\alpha_d :=$ volume of unit ball in $\mathbb{R}^d$ $= \mathcal{L}^d$)

that is

"among all set with prescribed diameter the ball has largest volume"

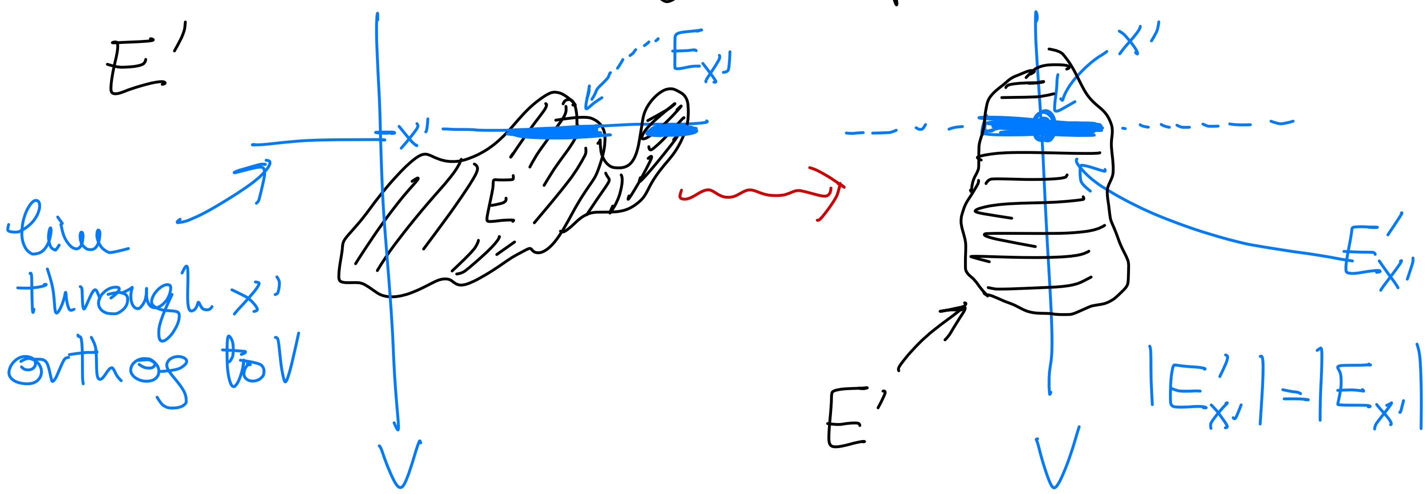
($= \mathcal{L}^d$)

Proof of (*). Fix $E \subset \mathbb{R}^d$, take $\{E_i\}$ countable cover of E. Then

$$\mathcal{H}^d_\infty(E) \leq \frac{\alpha_d}{2^d}\sum_i (\operatorname{diam} E_i)^d \geq \sum_i \mathcal{L}^d(E_i) \geq \mathcal{L}^d(E)$$

To prove Isodiam. Ineq. we need Steiner symmetrization:

Fix $E \subset \mathbb{R}^d$, Fix V hyperplane of $\mathbb{R}^d$, the Steiner symm. of E wrt V is E'



that is

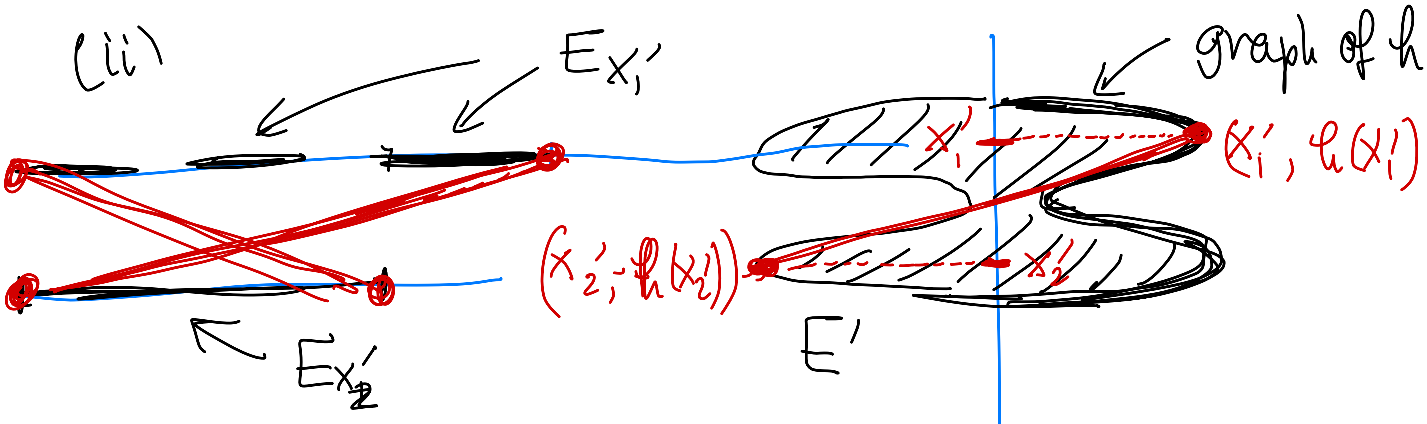
$$E' := \{ \underset{\substack{\cap \\ V \times V^\perp}}{(x', x'')} \mid |x''| \leq h(x') \} \quad \text{where } h(x') := \frac{1}{2} |E_{x'}|$$

Lemma (i) $\mathcal{L}^d(E') = \mathcal{L}^d(E)$

(ii) $\operatorname{diam}(E') \leq \operatorname{diam}(E)$

Proof (i) obvious by Fubini!

(ii)



Choose $x_1', x_2' \in p_V(E)$ s.t.

$$(\operatorname{diam}(E'))^2 = (d(x_1, x_2))^2$$

$$= |x_1' - x_2'|^2 + (h(x_1') + h(x_2'))^2$$

Note that

$$h_1(x_1') + h_2(x_2') \leq \frac{1}{2}\Big(\sup(\underset{E_1}{\underset{\shortparallel}{E_{x_1'}}}) - \inf(\underset{E_1}{\underset{\shortparallel}{E_{x_1'}}})\Big)$$

$$+ \frac{1}{2}\Big(\sup(\underset{E_2}{\underset{\shortparallel}{E_{x_2'}}}) - \inf(E_2)\Big)$$

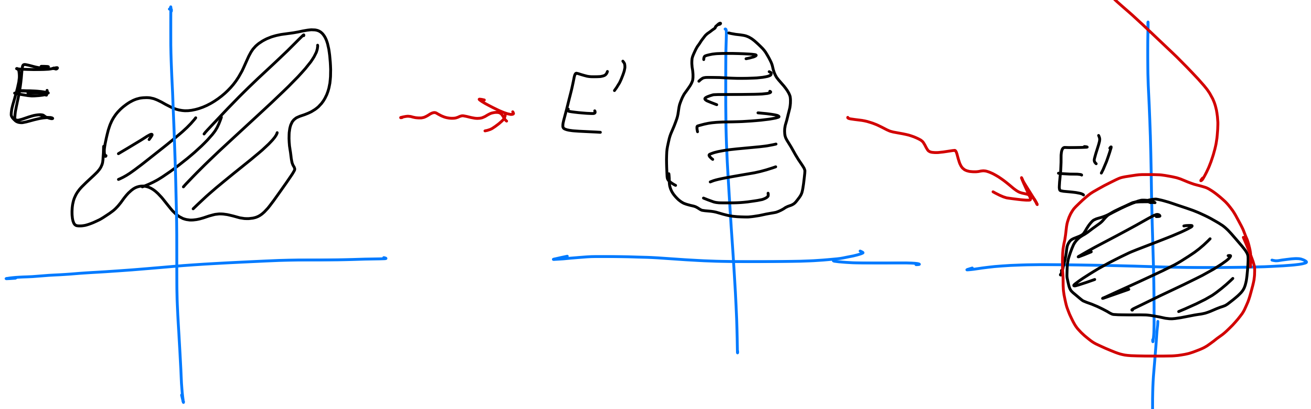
$$= \frac{1}{2}\left(\sup E_1 - \inf E_2\right) + \frac{1}{2}\left(\sup E_2 - \inf E_1\right)$$

$$\leq \max\left\{ \underline{(\sup E_1 - \inf E_2)} ; \underline{(\sup E_2 - \inf E_1)} \right\}$$

Proof of Isodiam. Ineq.

(for $d=2$)

$B = B\Big(0, \overbrace{\tfrac{\mathrm{diam}(E'')}{2}}^{r}\Big)$



E'' is symm. wrt origin $\Rightarrow E'' \subseteq B(0, r)$, $\quad r = \frac{\mathrm{diam}(E'')}{2}$

$$\mathrm{diam}(E) \geq \mathrm{diam}(B)$$

$$\pi\left(\frac{\mathrm{diam}\, E}{2}\right)^2 \geq \pi\left(\frac{\mathrm{diam}\, B}{2}\right)^2 = \mathrm{area}(B)$$

$$\geq \mathrm{area}(E'') = \mathrm{area}(E)$$

$\square$

2 | Results on densities of sets

<u>Theorem 1</u> E Borel set in X + μ locally finite measure + $\begin{cases} X = \mathbb{R}^d \\ \mu \text{ doubling} \end{cases}$

Then

$$\Theta_\mu(E,x) := \lim_{r\to 0} \frac{\mu(E\cap \overline{B(x,r)})}{\mu(\overline{B(x,r)})} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } x\in E \ (b) \\ 0 & \text{for } \mu\text{-a.e. } x\notin E \ (a) \end{cases}$$

<u>Rem</u>

$$\Theta^*_\mu(E,x) := \limsup_{r\to 0} \frac{\mu(E\cap \overline{B(x,r)})}{\mu(\overline{B(x,r)})} = \limsup_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))}$$

$$\Theta_{*\mu}(E,x) := \liminf_{r\to 0} \ \ldots\ldots$$

<u>Proof of Th. 1 (a)</u>

$\Theta_\mu(E,x) = 0$ for μ a.e. $x\notin E$

$\Uparrow$

$\Theta^*_\mu(E,x) = 0$ for μ a.e. $x\notin E$

$\Uparrow$

$$\boxed{\begin{gathered}\forall \delta>0 \text{ let } E_\delta := \{x\notin E \mid \Theta^*_\mu(E,x) > \delta\} \\ \text{then } \mu(E_\delta) = 0\end{gathered}}$$

Let $\mathcal{F} := \left\{ \overline{B(x,r)} \;\middle|\; x \in E_\delta \;\; \& \;\; \mu(E \cap \overline{B(x,r)}) \geq \delta\, \mu(\overline{B(x,r)}) \right\}$

$$\underset{\lambda(\overline{B(x,r)})}{\overset{\shortparallel}{\mu(E \cap \overline{B(x,r)})}}$$

— $\mathcal{F}$ is a Besicovitch cover of E_δ —

Let $\lambda := \mu \llcorner E$.

Then $\lambda(E_\delta) = 0$

Then $\forall \varepsilon > 0$ I can find $\mathcal{G} \subset \mathcal{F}$ s.t.

$\mathcal{G}$ covers E_δ & $\displaystyle\sum_{B \in \mathcal{G}} \lambda(B) \leq \cancel{\lambda(E_\delta)} + \varepsilon$, with $\lambda(E_\delta) = 0$

$$\varepsilon \geq \sum_{B \in \mathcal{G}} \lambda(B) \underset{\uparrow\, B \in \mathcal{F}}{\geq} \delta \sum_{B \in \mathcal{G}} \mu(B) \geq \delta\, \mu(E_\delta)$$

Since ε is arbitrary $0 \geq \delta\, \mu(E_\delta)$

that is $\mu(E_\delta) = 0$.

<u>Lemma</u> If μ is doubling then $\mu \llcorner F$ is asymptotically doubling

Proof of Th 1 (b)

Pass to E^c and use statement (a) and the fact that

$$\Theta^*_\mu(E^c,x) = 1 - \Theta_{*\mu}(E,x) \qquad \square$$

Th 2 $E \subset X$, $\mathcal{H}^d(E) < +\infty$ for some $d \in (0,+\infty)$

Then

(a) $$\Theta^*_d(E,x) := \limsup_{r\to 0} \frac{\mathcal{H}^d(E \cap \overline{B(x,r)})}{\alpha_d\, r^d} = 0$$

$\Theta_d(E,x)$

for $\mathcal{H}^d$-a.e. $x \notin E$

$\frac{\alpha_d}{2^d}$ is the renorm. constant in the def. of Hausdorff meas.

(b)
$$\frac{1}{2^d} \leq \Theta^*_d(E,x) \leq \begin{cases} 1 & \text{if } X = \mathbb{R}^d \\ 5^d & \text{if } X \text{ arbitrary} \end{cases}$$

for $\mathcal{H}^d$-a.e. $x \in E$

Rem $\exists\, E \subset \mathbb{R}^m$ s.t. $0 < \mathcal{H}^d(E) < +\infty$ & $\Theta_{*d}(E,x) = 0$ $\mathcal{H}^d$-a.e.

Proof of Th 2(c) [for $X = \mathbb{R}^n$]

$$\Theta^*_d(E,x) = 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \notin E$$

$$\Uparrow$$

$$\boxed{\begin{array}{c} \forall \varepsilon > 0 \text{ let } E_\varepsilon := \{ x \notin E \mid \Theta^*_d(E,x) > \varepsilon \} \\ \text{then } \mathcal{H}^d(E_\varepsilon) = 0 \end{array}}$$

Let $\mu := \mathcal{H}^d \llcorner E$. (finite measure!!)

$$\mathcal{F} := \left\{ \overline{B(x,r)} \;\middle|\; x \in E_\varepsilon \;\&\; \frac{\mathcal{H}^d(E \cap \overline{B(x,r)})}{\alpha_d r^d} \geq \varepsilon \right.$$
$$\left. \text{that is } \underline{\mu(\overline{B(x,r)}) \geq \varepsilon \alpha_d r^d} \right\}$$

- $\mathcal{F}$ is a Besicovitch cover of E_ε !
- since $E_\varepsilon \cap E = \emptyset \Rightarrow \mu(E_\varepsilon) = 0$

$\forall \delta > 0 \quad \exists \, \mathcal{G} \subset \mathcal{F}$ s.t. $\mathcal{G}$ covers E_ε

$$\& \quad \sum_{B \in \mathcal{G}} \mu(B) \leq \delta$$

$$\delta \geq \sum_{B \in \mathcal{G}} \mu(B) \geq \varepsilon \frac{\alpha_d}{2^d} \sum_{B \in \mathcal{G}} (\operatorname{diam} B)^d \geq \varepsilon \mathcal{H}^d_\infty(E_\varepsilon)$$

Since δ is arbitrary, $\mathcal{H}^d_\infty(E_\varepsilon)=0$

$\Rightarrow \mathcal{H}^d(E_\varepsilon)=0$.

<u>Lemma</u> $\mathcal{H}^d_\infty(F)=0 \Rightarrow \mathcal{H}^d(F)=0$