GMT 19/20
lecture 6, 27/3/20

Covering Theorems (continued)

<u>Setting</u>

X metric space (+ usual ass.)

μ is a locally finite doubling measure on X

$\uparrow$

$\exists\, M$ s.t. $\mu(B(x,2r)) \leq M \mu(B(x,r))$ for every ball $B(x,r)$

(nothing changes if balls are closed, and even if you replace 2 by any fixed $\lambda > 1$)

Theorem 1 (Vitali's covering th.)

Let $E \subset X$ with $\mu(E) < +\infty$

$\mathcal{F}$ is a "fine cover" of E $\Bigg\{$ Let $\mathcal{F}$ be a family of <u>closed</u> balls s.t. $\forall x \in E$ $\forall \delta > 0$ $\exists B \in \mathcal{F}$ with radius $\leq \delta$ s.t. $x \in B$

Then $\forall \varepsilon > 0$ $\exists \mathcal{F}' \subset \mathcal{F}$ s.t. (countable)

- $\mathcal{F}'$ is disjoint and $\mathcal{F}'$ covers μ-a.e. of E
- $\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$

Theorem 2

Take E, $\mathcal{F}$ (countable) as before.

Then $\forall \varepsilon > 0$ $\exists \mathcal{F}' \subset \mathcal{F}$ s.t.

- $\mathcal{F}'$ covers E
- $\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$

<u>Lemma 1</u> Given $\mathcal{F}$ family of balls in X $\exists\, \mathcal{G} \subset \mathcal{F}$ disjoint s.t.
$$\bigcup_{B \in \mathcal{G}} \hat{B} \supset \bigcup_{B \in \mathcal{F}} B .$$

<u>Lemma 2</u> Take $\mathcal{F}$ and E as in theorem 1 then $\exists\, \mathcal{G} \subset \mathcal{F}$ disjoint s.t.
$$\mu\Big(E \cap \bigcup_{B \in \mathcal{G}} B\Big) \geq \frac{1}{2M^3}\,\mu(E)$$

M of doubling property

<u>Proof</u> Choose A open s.t. $A \supset E$ and $\mu(A \setminus E) \leq \frac{\mu(E)}{2M^3}$.

Let $\mathcal{F}' := \{ B \in \mathcal{F} \,/\, B \subset A, \text{rad.}(B) \leq 1 \}$

Apply lemma 0 to $\mathcal{F}'$ and obtain $\mathcal{G}$

Now $\quad \mathcal{F}$ fine cover of $E \Rightarrow \mathcal{F}'$ ~~fine~~ cover of E

- $\displaystyle \bigcup_{B\in\mathcal{G}} \hat{B} \supset \bigcup_{B\in\mathcal{F}'} B \supset E$

then
$$\mu\Big(\bigcup_{B\in\mathcal{G}} \hat{B}\Big) \geq \mu(E)$$
$$\wedge\!\!|$$
$$\sum_{B\in\mathcal{G}} \mu(\hat{B}) \leq M^3 \sum_{B\in\mathcal{G}} \mu(B)$$

that is
$$\sum_{B\in\mathcal{G}} \mu(B) \geq \frac{\mu(E)}{M^3}$$

then
$$\mu\Big(E \cap \bigcup_{B\in\mathcal{G}} B\Big) \geq \mu\Big(\bigcup_{B\in\mathcal{G}} B\Big) - \mu(A\setminus E)$$
$$\geq \frac{\mu(E)}{M^3} - \frac{\mu(E)}{2M^3} = \frac{\mu(E)}{2M^{3}}$$

$\square$

Proof of Th. 1

Fix A_0 open s.t. $A_0 \supset E$, $\mu(A_0) \le \mu(E) + \varepsilon$

Step 1

Let $\mathcal{F}_0 := \{ B \in \mathcal{F} \mid B \subset A_0 \}$

Choose $\mathcal{G}_0$ according to lemma 2

then $\mu\left(E \cap \bigcup_{B \in \mathcal{G}_0} B\right) \geq \frac{1}{2M^3} \mu(E)$

Choose $\mathcal{G}_0' \subset \mathcal{G}_0$ finite s.t.

$$\mu\left(E \cap \bigcup_{B \in \mathcal{G}_0'} B\right) \geq \frac{1}{3M^3} \mu(E)$$

thus

$$\boxed{\mu\Big(\underbrace{E \setminus \bigcup_{B \in \mathcal{G}_0'} B}_{=: E_1}\Big) \leq \left(1 - \frac{1}{3M^3}\right) \mu(E)}$$

Step 2 let $\mathcal{F}_1 := \left\{ B \in \mathcal{F} \text{ s.t. } B \subset A_0 \;\&\; B \cap \left(\bigcup_{B \in \mathcal{G}_0'} B \right) = \emptyset \right\}$

$\mathcal{F}_1$ is a fine cover of E_1 !!

(Here I need that $\bigcup_{B \in \mathcal{G}_0'} B$ is a closed set)

Apply lemma 2 to $\mathcal{F}_1$ and E_1 and obtain $\mathcal{G}_1'$ finite disjoint s.t.

$$\mu\left(E_1 \setminus \bigcup_{B \in \mathcal{G}_1'} B \right) \leq \left(1 - \frac{1}{3M^3}\right) \mu(E_1)$$
$$\leq \left(1 - \frac{1}{3M^3}\right)^2 \mu(E)$$

And we go on .....

In the end I get

$\mathcal{F} \supset \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ finite disjoint

Moreover $\mathcal{F}' := \bigcup_{n=0}^{\infty} \mathcal{G}_n'$

is also disjoint.

Moreover all $B \in \mathcal{F}'$ are cont. in A_0, so

$$\sum_{B \in \mathcal{F}'} \mu(B) = \mu\Big(\bigcup_{B \in \mathcal{F}'} B\Big) \le \mu(A_0) \le \mu(E) + \varepsilon$$

Moreover

$$\mu\Big(E \setminus \bigcup_{B \in \mathcal{F}'} B\Big) \le \cancel{\left(1 - \frac{1}{2M^3}\right)^n \mu(E) \quad \forall n}$$

$$\le 0$$

$\square$

Lemma 3 Let E_0 be s.t. $\mu(E_0)=0$ and let $\mathcal{F}$ be a fine cover of E_0. Then $\forall \varepsilon>0$ $\exists$ $\mathcal{F}' \subset \mathcal{F}$ ~~disjoint~~ countable s.t. $\circ$ $\mathcal{F}'$ covers E_0

$$\circ \sum_{B\in\mathcal{F}'} \mu(B) \leq \varepsilon$$

Proof Fix $\varepsilon'>0$ (to be chosen later)

take A open s.t. $A \supset E_0$, $\mu(A) \leq \varepsilon'$

I can assume that all $B \subset \mathcal{F}$ satisfy $B \subset A$ and $\mathrm{rad}(B) \leq 1$

let $\check{\mathcal{F}} := \{ \check{B} \mid B \in \mathcal{F} \}$

$B := B(x,r)$
$\Downarrow$
$\check{B} := B(x, \frac{r}{5})$

Apply lemma 0 and obtain

$\mathcal{G} \subset \check{\mathcal{F}}$, $\mathcal{G}$ disjoint, $\bigcup_{B\in\mathcal{G}} \hat{B} \supset E_0$

and let $\mathcal{F}' := \hat{\mathcal{G}} = \{ \hat{B} \mid B \in \mathcal{G} \}$

implies $\mathcal{G}$ countable

Then $\mathcal{F}' \subset \mathcal{F}$, $\bigcup_{B\in\mathcal{F}'} B \supset E_0$

Finally: $\sum_{B\in\mathcal{G}} \mu(B) = \mu(\bigcup_{B\in\mathcal{G}} B) \leq \mu(A) \leq \varepsilon$

thus

$$\sum_{B\in\mathcal{F}'} \mu(B) = \sum_{B\in\mathcal{G}} \mu(\hat{B}) \leq M^3 \sum_{B\in\mathcal{G}} \mu(B) \leq M^3\varepsilon'$$

I conclude by choosing $M^3\varepsilon' = \varepsilon$.

$\square$

Proof of th. 2

Put together Th. 1 & lemma 3.

Remarks

1) For Theor 1 to hold you don't need that μ is doubling, it suffices that μ is asymptotically doubling:

$$\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < +\infty \text{ for } \mu\text{-a.e. } x$$

2) Open balls VS closed balls?

IS NOT A REAL ISSUE

If $\mathcal{F}$ is a family of open balls B with $\mu(\partial B)=0$ then Th. 1 holds.

(Apply Th. 1 to $\tilde{\mathcal{F}} := \{\bar{B} \mid B \in \mathcal{F}\}$)

Rem $\forall x$ $\mu(\partial B(x,r))=0$ for all $r>0$ except count. many

Also Theorem 2 works for such $\mathcal{F}$.

3) These theorems work for more general sets than just balls

↪ Falconer's book.